NON-ISOMORPHISM OF CATEGORIES OF ALGEBRAS

WALTER D. NEUMANN

We recall that if $\Delta = (K_i)_{i\varepsilon I}$ is a collection of sets, then an algebra $(A,(f_i)_{i\varepsilon I})$ of type Δ consists of a set A and a collection $(f_i)_{i\varepsilon I}$ of operations on A, such that for each $i\varepsilon I$ f_i is a K_i-ary operation; that is $f_i\colon A^{K_i} \longrightarrow A$. We shall use the same symbol for an algebra A and its underlying set.

We say that the types $\Delta = (K_i)_{i\varepsilon I}$ and $\Delta' = (L_j)_{j\varepsilon J}$ are <u>equivalent</u> if there is a bijection $\varphi\colon I \longrightarrow J$ such that for each i the cardinalities $|K_i|$ and $|L_{\varphi(i)}|$ are equal.

We denote by $\mathfrak{A}(\Delta)$ the category of all algebras of type Δ and all homomorphisms between them. Clearly if Δ and Δ' are equivalent types, then $\mathfrak{A}(\Delta)$ and $\mathfrak{A}(\Delta')$ are isomorphic categories. The purpose of this note is to prove the converse, thereby answering a question posed by A. PULTR.

This question was motivated by the result of Z. HEDRLÍN and A. PULTR [2] which states that even if Δ and Δ' are not equivalent, $\mathfrak{A}(\Delta)$ and $\mathfrak{A}(\Delta')$ are embeddable as full subcategories in each other, so long as Δ and Δ' are not too small (the sums of the cardinalities of the sets in Δ and Δ' should each exceed 1).

In §1 we recall the necessary properties of algebraic operations, and in §2 we prove the theorem.

§1. Operations.

Let $f\colon F^K \longrightarrow F$ be a K-ary operation on F (F,K any sets). The <u>support</u> of f (FELSCHER [1]) is defined by

$$\operatorname{supp}(f) = \{A \subseteq K \mid \text{for all } \alpha,\beta \in F^K,\ \alpha|A = \beta|A \Rightarrow f(\alpha) = f(\beta)\}.$$

That is $A \in \mathrm{supp}(f)$ means that the value of f on any $\alpha \in F^K$ is already determined by the restriction of α to A.

The <u>essential rank</u> of f is defined as $\min\{|A| \mid A \in \mathrm{supp}(f)\}$. If $\mathcal{O}\!\!\mathcal{L}$ is a primitive class (variety) of algebras, define its <u>rank</u> to be the supremum of the essential ranks of all $\mathcal{O}\!\!\mathcal{L}$-algebraic operations. Since the essential rank of an algebraic operation is always less than the dimension (SŁOMIŃSKI [3]) of $\mathcal{O}\!\!\mathcal{L}$, this supremum exists.

Now let f be any element of the free algebra $F(X,\mathcal{O}\!\!\mathcal{L})$ with basis X of the primitive class $\mathcal{O}\!\!\mathcal{L}$, and let A be any algebra in $\mathcal{O}\!\!\mathcal{L}$. One can define an X-ary operation $\hat{f}^A$ on A by $\hat{f}^A(\alpha) = \bar{\alpha}(f)$ for any $\alpha \in A^X$. Here $\bar{\alpha}$ is the homomorphic extension of α to a homomorphism of $F(X,\mathcal{O}\!\!\mathcal{L})$ to A. $\hat{f}^A$ is an algebraic operation on A; in fact $f \longmapsto \hat{f}^A$ is a surjective homomorphism of $F(X,\mathcal{O}\!\!\mathcal{L})$ onto the algebra $H^X(A)$ of all X-ary algebraic operations on A (see for instance [1]).

<u>§2. Non-isomorphism of categories.</u>

<u>THEOREM:</u> <u>If the categories $\mathcal{O}\!\!\mathcal{L}(\Delta)$ and $\mathcal{O}\!\!\mathcal{L}(\Delta^*)$ are isomorphic, then Δ and Δ^* are equivalent types.</u>

We shall prove this theorem in two steps. Let s denote the canonical underlying set functors on $\mathcal{O}\!\!\mathcal{L}(\Delta)$ and $\mathcal{O}\!\!\mathcal{L}(\Delta^*)$.

<u>LEMMA 1:</u> <u>If the concrete categories $(\mathcal{O}\!\!\mathcal{L}(\Delta),s)$ and $(\mathcal{O}\!\!\mathcal{L}(\Delta^*),s)$ are concretely isomorphic (that is isomorphic by a functor which preserves underlying sets), then Δ is equivalent to Δ^*.</u>

Proof: Let $\Delta = (K_i)_{i\in I}$. It suffices to show that the knowledge of the category $\mathcal{O}\!\!\mathcal{L}(\Delta)$ and its underlying set functor is sufficient to recover Δ up to equivalence. Now the definition of free algebra of $\mathfrak{A}(\Delta)$ over a basis involves only the underlying sets and homomorphisms, so we know the free algebras of $\mathfrak{A}(\Delta)$. Hence by the last paragraph of §1 we know the algebraic operations for $\mathfrak{A}(\Delta)$, so we can calculate the rank δ of $\mathfrak{A}(\Delta)$.

Let F be a free algebra of $\mathfrak{A}(\Delta)$ with basis X such that $|X| \geq \delta$. Let $Y = F - X$. Define a relation R on Y by $y_1 R y_2$ if and only if there is an endomorphism of F which maps y_1 onto y_2. Let S be the smallest equivalence relation on Y containing R, and let $J = Y/S$ be the set of equivalence classes of Y under S.

We now consider the meaning of this construction in terms of the actual algebraic structure of F. The set Y is the set of all $f_i(\alpha)$, $i \in I$, $\alpha \in F^{K_i}$, where the f_i are the defining operations of the class. Under an endomorphism of F an element $f_i(\alpha)$ cannot be mapped onto an element $f_j(\beta)$ with $i \neq j$. Further if $\alpha_0 \in F^{K_i}$ is injective with $\alpha_0(K_i) \subseteq X$, then every element of the form $f_i(\alpha)$, $\alpha \in F^{K_i}$, is the image of $f_i(\alpha_0)$ under suitable endomorphisms of F. Hence the equivalence classes under S are just the subsets of Y of the form $\varphi(i) = \{f_i(\alpha) \mid \alpha \in F^{K_i}\}$, and $\varphi : i \mapsto \varphi(i)$ is a bijective map from I to J.

It remains only to show that to each $\varphi(i) \in J$ we can recover $|K_i|$. We can choose a $y \in \varphi(i)$ with the property that every element of $\varphi(i)$ is the image of y under some endomorphism of F. We then have the corresponding X-ary algebraic operation $\hat{y} = \hat{y}^F$ in $H^X(F)$. We claim that $\hat{y}$ has essential rank $|K_i|$, completing the proof. Indeed y is of the form $f_i(\alpha_0)$ for some injective $\alpha_0 \in F^{K_i}$ with $\alpha_0(K_i) \subseteq X$. $\mathrm{Supp}(\hat{y})$ is just the set of all subsets of X which contain $\alpha_0(K_i)$, so the essential rank of $\hat{y}$ is $|\alpha_0(K_i)|$. Since α_0 is injective, $|\alpha_0(K_i)| = |K_i|$.

The second part of the proof of the theorem is given by the following lemma:

LEMMA 2: **If the categories $\alpha(\Delta)$ and $\alpha(\Delta')$ are isomorphic, then the concrete categories $(\alpha(\Delta),s)$ and $(\alpha(\Delta'),s)$ are concretely isomorphic.**

Proof: We first determine a free algebra P of rank 1 (that is basis cardinality 1) in $\alpha(\Delta)$. This can be done in many ways. For instance $P \in \alpha(\Delta)$ is a free algebra if and only if ~~it is a projective object of the category~~ every epimorphism to P has a section, and it furthermore has rank 1 if and only if every morphism $P \longrightarrow P$ is mono.

The functor $mor(P,-): \alpha(\Delta) \longrightarrow Set$ is a "new underlying set functor" which is naturally equivalent to the standard underlying set functor s on $\alpha(\Delta)$.

Let $T: \alpha(\Delta) \longrightarrow \alpha(\Delta')$ be an isomorphism. Then T(P) is a free algebra of rank 1 in $\alpha(\Delta')$, so we also have a "new underlying set functor" mor(T(P),-) on $\alpha(\Delta')$ which is naturally equivalent to the standard one.

If one identifies the sets mor(P,A) and mor(T(P),T(A)) by means of T for each $A \in \alpha(\Delta)$, then T commutes with these "new underlying set functors", and it is not difficult, using the properties of the standard underlying set functors, to deduce that $\alpha(\Delta)$ and $\alpha(\Delta')$ are also concretely isomorphic with respect to the standard underlying set functors.

Lemma 2 states that two categories of the form $\alpha(\Delta)$ are abstractly isomorphic if and only if they are concretely isomorphic, or in more algebraic terminology: rationally equivalent. This in fact holds for more general primitive classes of algebras; for instance a slight modification of the above proof shows that two Schreier primitive classes (subalgebras of free algebras are free) ~~with surjective epis and injective monos~~ whose free algebras of rank 1 are not isomorphic to any of higher rank are abstractly isomorphic as categories if and only if they are rationally equivalent. Some restriction on the classes considered is however necessary, for it is known that to any primitive class one can find primitive classes not rationally equivalent to the given one, such that the categories are isomorphic.

REFERENCES

[1] W. FELSCHER: Equational maps, Contributions to Mathematical Logic, Amsterdam 1967.

[2] Z. HEDRLÍN and A. PULTR: On full embeddings of categories of algebras, Illinois J. Math. 10,3 (1966) 392-406.

[3] J. SŁOMIŃSKI: The theory of abstract algebras with infinitary operations, Rozprawy Mat. 18 (1959).

Mathematisches Institut der Universität Bonn.