ON INTERSECTIONS OF FINITELY GENERATED SUBGROUPS OF FREE GROUPS

WALTER D. NEUMANN

Let U and V be non-trivial finitely generated subgroups of ranks u and v respectively in a free group F and let $N = U \cap V$, of rank n. In [**N**] Hanna Neumann, improving on a result of Howson [**H**], proved the inequality

$$n - 1 \leqslant 2(u-1)(v-1),$$

and asked if the factor 2 can be dropped. If one translates her approach (which is a slight modification of Howson's) to graph-theoretic terms, it easily shows that the answer is often "yes"—in fact, for most U the answer is "yes" for all V.

According to Gersten [**G**], the above problem has come to be known as the "Hanna Neumann Conjecture." Using ideas of immersions of graphs originating from Stallings ([**St**]), Gersten solved the problem in some special cases (his approach is close to the one of Howson and Hanna Neumann, but seems weaker in practice). I am grateful to Alan Reid for bringing Gersten's paper to my attention, and also to Peter Neumann for leading me to other literature. In particular, [**I**] gives the same graph-theoretical translation of Hanna Neumann's proof[1], and [**Ni**] and [**Se**] use similar methods to prove Burns' bound [**B**]:

$$n - 1 \leqslant 2(u-1)(v-1) - \min(u-1,\, v-1),$$

which is the best general bound known so far. We give a version of their proof in the final section.

Hanna Neumann's question can be strengthened to ask about the sum of $\operatorname{rank} N - 1$ as N runs through a set of representatives of conjugacy classes of non-trivial intersections $N = y^{-1}\,Uy \cap z^{-1}\,Vz$, and as we shall describe, the bounds that we can give, including Burns' bound, remain the same.

The support of the NSF and of the Center for Mathematical Analysis in Canberra is gratefully acknowledged.

[1] as does D. E. Cohen, who also calls Hanna Neumann's question a conjecture, in his 1989 text, dedicated in part to Hanna Neumann's memory, "Combinatorial Group Theory: a topological approach" (London Math. Soc. Student Texts 14, 1989, Prop. 8.35, p. 294, and p. ii). I am grateful to O. Kegel for this reference.

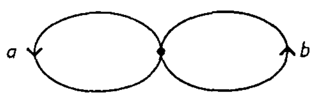
1. Hanna Neumann's proof

We shall write for short

$$\chi_0(H) = \max(0,\ \operatorname{rank} H - 1)$$

if H is a finitely generated free group.

We can assume with no loss of generality that F is of rank 2, generated by elements a and b, say, and that $N = U \cap V$ is non-trivial. Let G denote the labelled figure eight graph



whose fundamental group is F. For any subgroup H of F, let $G(H)$ denote the covering of G with fundamental group H. The vertex set of $G(H)$ can be identified with the set $H\backslash F$ of right H-cosets, in which case the a-labelled edges are the pairs (Hg, Hga) and the b-labelled edges are the pairs $(Hg,\ Hgb)$. If H is non-trivial, let $G_0(H)$ denote the *spine* of $G(H)$, that is, the minimal deformation retract of $G(H)$. (It is obtained by cutting off all maximal branches of $G(H)$, where a *branch* is a contractible subgraph of $G(H)$ which meets the rest of $G(H)$ only at one end of one edge: alternatively, it is the union of the supports of all reduced circuits of $G(H)$, a *reduced circuit* being a closed path that is not homotopic to a shorter closed path.) After choosing a base point, $G_0(H)$ has fundamental group a conjugate of H. If H has finite rank then $G_0(H)$ is finite and, moreover,

$$2\chi_0(H) = \sum_{p \in \operatorname{vert} G_0(H)} \partial(p) - 2, \tag{1}$$

where $\partial(p)$ is the valency (number of incidences of edges) at vertex p. Indeed, as an equation for minus twice the euler characteristic of a finite graph, this equation is well known and is easily proved by induction.

The graph $G(N)$ is a mutual covering of the graphs $G(U)$ and $G(V)$. The projection maps $G(N) \to G(U)$ and $G(N) \to G(V)$ map the spine $G_0(N)$ into the spines $G_0(U)$ and $G_0(V)$ respectively. Let $\pi_U \colon G_0(N) \to G_0(U)$ and $\pi_V \colon G_0(N) \to G_0(V)$ denote these maps. Note that the map of vertex sets $\operatorname{vert} G(N) \to \operatorname{vert} G(U) \times \operatorname{vert} G(V)$ is injective, so the same holds for $\pi = (\pi_U, \pi_V)\colon\ \operatorname{vert} G_0(N) \to \operatorname{vert} G_0(U) \times \operatorname{vert} G_0(V)$. For any $p \in \operatorname{vert} G_0(N)$, we clearly have

$$\begin{aligned} 0 \leqslant \partial(p) - 2 &\leqslant \min\big(\partial(\pi_U(p)) - 2,\ \partial(\pi_V(p)) - 2\big) \\ &\leqslant \big(\partial(\pi_U(p)) - 2\big)\big(\partial(\pi_V(p)) - 2\big). \end{aligned} \tag{2}$$

Thus, by (1) and (2) and the injectivity of π,

$$\begin{aligned}
2\chi_0(N) &= \sum_{p\in\operatorname{vert} G_0(N)} \partial(p) - 2 \\
&\leqslant \sum_{p\in\operatorname{vert} G_0(N)} \big(\partial(\pi_U(p)) - 2\big)\big(\partial(\pi_V(p)) - 2\big) \\
&\leqslant \sum_{(q,r)\in\operatorname{vert} G_0(U)\times\operatorname{vert} G_0(V)} \big(\partial(q) - 2\big)\big(\partial(r) - 2\big) \\
&= \Big(\sum_{q\in\operatorname{vert} G_0(U)} (\partial(q) - 2)\Big)\Big(\sum_{r\in\operatorname{vert} G_0(V)} (\partial(r) - 2)\Big) \\
&= 2\chi_0(U)\,2\chi_0(V),
\end{aligned}$$

which is the desired inequality. $\square$

2. Improving the proof, I.

Instead of just asking about $\operatorname{rank}(U\cap V)$, one can ask about the ranks of all intersections $y^{-1}Uy\cap z^{-1}Vz$. Any such intersection is conjugate to one of the form $U\cap x^{-1}Vx$. Moreover, if y is in the double coset VxU, then $U\cap x^{-1}Vx$ and $U\cap y^{-1}Vy$ are conjugate. Thus we need only let x run through a set S of double coset representatives for $V\backslash F/U$. Let T be the subset of $x\in S$ with $U\cap x^{-1}Vx$ nontrivial. Denote

$$\chi_F(U,V) = \sum_{x\in T} \chi_0(U\cap x^{-1}Vx).$$

The size of T and $\chi_F(U,V)$ depend only on the conjugacy classes of U and V. Hanna Neumann's inequality can be strengthened as follows (see also [**I2**]).

Proposition 2.1. *T is finite and* $\chi_F(U,V)\leqslant 2\chi_0(U)\chi_0(V)$.

Proof. We can again assume F has rank 2 (since embedding F into a larger free group at worst increases the size of T and $\chi_F(U,V)$). So let $G(U)$ and $G(V)$ be as before. Let $G(U)\times G(V)$ denote the graph with vertex set $\operatorname{vert} G(U)\times\operatorname{vert} G(V)$ and with an a-labelled edge from (p,q) to (p',q') if and only if $G(U)$ and $G(V)$ have a-labelled edges from p to p' and q to q' respectively; similarly for b-labelled edges. We claim that the components of $G(U)\times G(V)$ are just the graphs $G(N)$ as N runs through the groups $U\cap x^{-1}Vx$, $x\in S$. Given this claim, if we denote by $G_0(U,V)$ the disjoint union of the $G_0(N)$ as N ranges over the intersections $U\cap x^{-1}Vx$ with $x\in T$ then $G_0(U,V)$ is the union of the spines of the non-contractible components of

$G(U) \times G(V)$, so it is a subgraph of $G_0(U) \times G_0(V)$. In particular, it is finite, so T is finite. Also, by equation (1),

$$(3) \qquad 2\chi_F(U,V) = \sum_{p \in \operatorname{vert} G_0(U,V)} (\partial(p) - 2),$$

and applying the computation of section 1 to this proves the Proposition.

To see the claim, recall that we can identify the vertices of $G(U)$ and $G(V)$ with cosets of U and V in F. A component of $G(U) \times G(V)$ containing the vertex (Uy, Vz) contains the vertex (U, Vzy^{-1}), so every component contains a vertex of the form (U, Vx). The fundamental group of this component consists of all $z \in F$ with $Uz = U$ and $Vxz = Vx$; that is, $z \in U \cap x^{-1}Vx$. Moreover, another vertex (U, Vy) will be in the same component if and only if there is a $z \in F$ with $Uz = U$ and $Vxz = Vy$. That is, $Vxz = Vy$ for some $z \in U$, in other words, y is in the double coset VxU. ∎

Note that although T is finite, easy examples (e.g., $U = V = \langle a^n \rangle$) show that its size cannot be bounded in terms of $\chi_0(U)$ and $\chi_0(V)$. But probably the number of conjugacy classes of non-trivial subgroups $U \cap x^{-1}Vx$ can be so bounded. Only those of rank 1 are at issue, since those of rank $\geqslant 2$ number at most $\chi_F(U,V)$.

The above Proposition suggests the following:

QUESTION 2.2 (STRENGTHENED H. NEUMANN QUESTION).

Is $\chi_F(U,V) \leqslant \chi_0(U)\chi_0(V)$?

We shall say "$HN\{F; U, V\}$ holds" if this question has positive answer for $\{U, V\}$. It is easy to see that it holds if either U or V has finite index in F. In fact, in this case

$$\chi_F(U,V) = \chi_0(U)\chi_0(V)/\chi_0(F).$$

This is implied by the stronger result:

PROPOSITION 2.3. *If U_1 has finite index d in U then* $\chi_F(U_1, V) = d\chi_F(U,V)$.

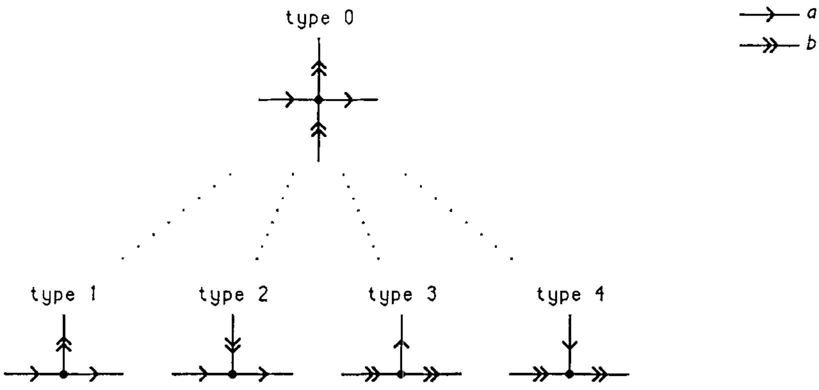
PROOF. $G_0(U_1, V)$ is a d-fold covering of $G_0(U,V)$. ∎

In particular, if U_1 and V_1 have finite index in U and V respectively, then $HN\{F; U, V\}$ holds if and only if $HN\{F; U_1, V_1\}$ holds.

3. IMPROVING THE PROOF, II.

The above proof only used the valencies of vertices of the graphs; by taking account of the form of the vertices we can do better.

Only *nodes* (vertices of valency $\partial(p) \geqslant 3$) contribute in formulae (1) and (3). There are five forms that a node can take, and they can be listed and named as in the following poset.



For any non-trivial subgroup H of finite rank m in F, denote the number of type i nodes of $G_0(H)$ by $k_i(H)$ for $i = 0, \ldots, 4$, so $k(H) = \sum_{i=1}^{4} k_i(H)$ is the total number of valency 3 nodes. Then equation (1) can be re-written

$$2\chi_0(H) = 2k_0(H) + k(H). \tag{4}$$

On the other hand, a vertex p of $G_0(U, V)$ is a node of type i only if the image vertices $\pi_U(p) \in \operatorname{vert} G_0(U)$ and $\pi_V(p) \in \operatorname{vert} G_0(V)$ are each nodes of type i or type 0. Thus, if we use equation (3) to compute $2\chi_F(U, V)$, then $p \in \operatorname{vert} G_0(U, V)$ contributes at most 2, 1, 1, or 0 according as $\pi_U(p)$ and $\pi_V(p)$ are both nodes of type 0, one of type 0 and the other of type $i \neq 0$, both of the same type $i \neq 0$, or none of the above. Thus

$$2\chi_F(U, V) \leqslant 2k_0(U)k_0(V) + k_0(U)k(V) + k_0(V)k(U) + \sum_{i=1}^{4} k_i(U)k_i(V). \tag{5}$$

By (4) with $H = U$, (4) with $H = V$, and (5),

$$\begin{aligned} 4\chi_0(U)\chi_0(V) - 4\chi_F(U, V) \geqslant{}& \Big(2k_0(U) + k(U)\Big)\Big(2k_0(V) + k(V)\Big) \\ &- 2\Big(2k_0(U)k_0(V) + k_0(U)k(V) + k_0(V)k(U) + \sum_{i=1}^{4} k_i(U)k_i(V)\Big). \end{aligned}$$

This simplifies to

$$\begin{aligned} 4[\chi_0(U)\chi_0(V) - \chi_F(U, V)] &\geqslant k(U)k(V) - 2\sum_{i=1}^{4} k_i(U)k_i(V) \\ &= \sum_{i=1}^{4}(k(U) - 2k_i(U))k_i(V), \end{aligned} \tag{6}$$

Suppose this is negative. Then for some i

$$(7) \qquad k(U) - 2k_i(U) < 0.$$

By symmetry, for some j

$$(8) \qquad k(V) - 2k_j(V) < 0.$$

We claim $j = i$. To see this note that the set of 4-tuples $(k_1(V), \dots, k_4(V))$ of non-negative reals satisfying (8) for some j has four components, one for each value of j. The set of 4-tuples making (6) negative is convex, hence contained in just one of the components determined by (8), but it contains the 4-tuple $(0, \dots, k_i(V) = k(V), \dots, 0)$ which is in the i-th one. We have proven:

PROPOSITION 3.1. *A counter-example to $HN\{F; U, V\}$ (Question 2.2) would have to have over half the valency 3 nodes of $G_0(U)$ and over half the valency three nodes of $G_0(V)$ all of type i for some $i = 1, 2, 3,$ or 4.* □

In particular, if at most half the valency 3 nodes of $G_0(U)$ are of each type then $HN\{F; U, V\}$ holds for any V.

For "random" U the probability that over half the valency 3 nodes of $G_0(U)$ have a given type i is at most $1/16$ and is asymptotic to

$$\frac{1}{\sqrt{2\pi k(U)}}\left(\frac{3}{4}\right)^{k(U)/2}$$

as $k(U) \to \infty$. (Note that for U of fixed rank u, $k(U) = 2u - 2$ with probability 1, in the sense that among all such U with $G_0(U)$ of bounded size, the proportion with $k(U) = 2u - 2$ approaches 1 as the bound on size increases.)

4. Remarks on rank 2

Suppose U has rank 2, that is, $\chi_0(U) = 1$. Then the only possibilities are

(i) $k_0(U) = 1$, $k_1(U) = \cdots = k_4(U) = 0$,

(ii) $k_0(U) = 0$, exactly two of $k_1(U), \dots, k_4(U)$ equal 1 and the other two equal 0,

(iii) $k_0(U) = 0$, some $k_i(U)$ equals 2 and the others equal 0.

By Proposition 3.1, only in the last case could U be part of a counterexample to Question 2.2. (For example, the examples of [G; Prop. 6.12] are of type (ii).) All "small" examples of type (iii) can be changed to one of the first two types by applying an automorphism of the free group F. However, we have more than just automorphisms of F at our disposal, as we now describe.

Write $U \prec F$ if U is a finitely generated subgroup of the free group F and $HN\{F; U, V\}$ holds for all finitely generated $V < F$. The following proposition implies that the set of U of rank $\leqslant 2$ satisfying $U \prec F$ is a sub-semilattice of the semilattice of all finite rank subgroups of F.

PROPOSITION 4.1. (i) *If* U_1 *has rank 2 and* $U_2 \prec U_1 \prec F$ *then* $U_2 \prec F$.
(ii) *If* U_1 *and* U_2 *have rank 2 then* $U_1 \cap U_2$ *has rank* $\leqslant 2$*; if, moreover,* $U_1 \prec F$ *and* $U_2 \prec F$ *then* $U_1 \cap U_2 \prec F$.

LEMMA 4.2. *Suppose* U_1 *and* U_2 *are subgroups of* F *of ranks* u_1 *and* u_2 *respectively. Suppose* $U_1 \prec F$ *and for each subgroup* V *of* U_1, $HN\{G; U_2, V\}$ *holds for some* G *containing* U_1 *and* U_2. *Then*

$$\chi_F(U_1 \cap U_2, V) \leqslant \chi_0(U_1)\chi_0(U_2)\chi_0(V).$$

PROOF OF PROPOSITION. The first statement of part (ii) of the Proposition is Burns' bound quoted in the Introduction and otherwise the Proposition is immediate from the Lemma. ∎

PROOF OF LEMMA. Let S be a set of double coset representatives for $V\backslash F/U_1$. For each x in S, choose a set of double coset representatives for $(U_1 \cap x^{-1}Vx)\backslash U_1/(U_2 \cap U_1)$ and call it S_x, say. Since $y^{-1}U_1 y = U_1$ for each $y \in S_x$, and as S_x is part of a set of double coset representatives for $(U_1 \cap x^{-1}Vx)\backslash G/U_2$,

$$\chi_G(U_2, U_1 \cap x^{-1}Vx) \geqslant \sum_{y \in S_x} \chi_0(U_2 \cap U_1 \cap y^{-1}x^{-1}Vxy). \tag{9}$$

Further, any double coset $Vz(U_2 \cap U_1)$ inside VxU_1 can be written in the form $Vxy(U_2 \cap U_1)$ with $y \in U_1$, and therefore also with $y \in S_x$. Thus each double coset in $V\backslash F/(U_2 \cap U_1)$ has at least one representative in $S' = \{\, xy \mid x \in S,\ y \in S_x \,\}$. Now, using (9) and $HN\{G; U_2, U_1 \cap x^{-1}Vx\}$ and $HN\{F; U_1, V\}$, we get

$$\begin{aligned}
\chi_F(U_1 \cap U_2, V) &\leqslant \sum_{xy \in S'} \chi_0(U_2 \cap U_1 \cap y^{-1}x^{-1}Vxy) \\
&= \sum_{x \in S} \sum_{y \in S_x} \chi_0(U_2 \cap U_1 \cap y^{-1}x^{-1}Vxy) \\
&\leqslant \sum_{x \in S} \chi_G(U_2, U_1 \cap x^{-1}Vx) \\
&\leqslant \sum_{x \in S} \chi_0(U_2)\chi_0(U_1 \cap x^{-1}Vx) \\
&= \chi_0(U_2)\chi_F(U_1, V) \\
&\leqslant \chi_0(U_2)\chi_0(U_1)\chi_0(V).
\end{aligned}$$
∎

Using automorphisms of F and Proposition 4.1 one can easily show that if U has rank 2 and $G_0(U)$ is not too large (certainly up to seven edges, but one can probably go quite a bit further) then $HN\{F;U,V\}$ holds for any V. However, there exist U of rank 2 for which $G_0(U)$ is of type (iii) whatever basis of F one chooses and U is contained in no larger rank 2 proper subgroup of F, so these techniques do not suffice to resolve the question.

The strengthened form of Burns' bound (Proposition 5.1 below) implies that $HN\{F;U,V\}$ holds if both U and V have rank 2. In particular, $U\cap x^{-1}Vx$ can have rank at most 2, and has rank 2 for at most one double coset VxU. This has some trivial but amusing consequences, whose proofs we leave to the reader.

PROPOSITION 4.3. (i) *If U has rank 2 and $V\subseteq U$ then $U\cap x^{-1}Vx$ has rank $\geqslant 2$ only if $x\in U$. In particular, $HN\{F;U,V\}$ holds.*
(ii) *If U and V are finitely generated subgroups of a rank 2 subgroup G of the free group F then $\chi_G(U,V)=\chi_F(U,V)$.* □

5. IMPROVING THE PROOF, III.

We discuss a further strengthening of the approach of section 3.

Let G be one of the graphs $G_0(U)$, $G_0(V)$ or $G_0(U,V)$ under discussion. Rather than assigning to a node q of G one of the five types discussed in section 3, we can consider the "type" of q to be the isomorphism class of the pair $(\tilde G,\tilde q)$ consisting of the universal cover $\tilde G$ together with a lift of the point q. Thus a "type" is an infinite contractible labelled graph with no vertices of valency 1 and with a chosen node as "base-point" (with additional properties that are not relevant to us here). Partially order such types by embeddability in each other and call two types "comparable" if they have a common lower bound in the poset of types.

Observe that the type of a node p of $G_0(U,V)$ embeds in the types of $q=\pi_U(p)$ and $r=\pi_V(p)$, so q and r are comparable. The same derivation as for equation (6) of section 3 gives

$$4[\chi_0(U)\chi_0(V)-\chi_F(U,V)]\geqslant\sum_{q,r}c(q,r), \tag{10}$$

where the sum is over the valency 3 nodes q of $G_0(U)$ and r of $G_0(V)$ and $c(q,r)=-1$ or 1 according as q and r are comparable or not.

We describe why Burns' inequality holds in the strengthened form

PROPOSITION 5.1. $\chi_F(U,V)\leqslant 2\chi_0(U)\chi_0(V)-\min(\chi_0(U),\chi_0(V))$.

PROOF. We follow Servatius' proof [**Se**], which proceeds essentially as follows:

(a) Form a bipartite graph Ω with vertex set the union of the sets of valency 3 nodes of $G_0(U)$ and $G_0(V)$ and with an edge connecting a node q of $G_0(U)$ to a node

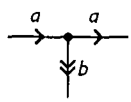
r of $G_0(V)$ if and only if q and r are comparable. An easy calculation shows that if this graph is disconnected then (10) implies the result.

(b) By embedding F in itself using the embedding with graph

all our graphs become covers of this graph, so there are no vertices of valency 4.

(c) If no vertex has valency 4 then the bipartite graph Ω of (a) is disconnected.

Nickolas [**Ni**] gives a simpler proof of (c), which we paraphrase. Consider a minimal counterexample to (c) (least number of vertices of $G_0(U) \cup G_0(V)$). We construct a smaller one to get a contradiction. All the nodes of $G_0(U) \cup G_0(V)$ must have the same type in the sense of section 2. By renaming we may assume it is



If a chain of two or more b's occurs in either $G_0(U)$ or $G_0(V)$, then replacing every chain of b's by a single b gives a smaller counterexample to (c), so no such chain of b's occurs. If a chain bab^{-1} occurs anywhere, then replacing each such chain by a single b gives a smaller counterexample (the graphs are still reduced since they had no chain bb), so no such chain occurs. But now a chain ba must occur somewhere, and replacing each occurrence of ba by b gives a smaller counterexample. □

Following Nickolas, we can use the same argument to get the minor result:

PROPOSITION 5.2. *Suppose* $\operatorname{rank} U \geqslant 2$. *Then* $\chi_F(U,U) \leqslant 2\chi_0(U)^2 - 2\chi_0(U) + 1$, *and if* $V \subseteq U$ *then* $\chi_F(U,V) \leqslant 2\chi_0(U)\chi_0(V) - \chi_0(V)$.

PROOF. As above, we may assume $G_0(U)$ has only valency 3 nodes. The Servatius-Nickolas argument applied to $G_0(U)$ alone shows that these nodes can be partitioned into two non-empty mutually incomparable subsets S_1 and S_2. Equation (10) implies the first inequality, and also the second on noting that the type of any node of $G_0(V)$ is bounded above by the type of some node of $G_0(U)$, so it is incomparable with all nodes of S_1 or all nodes of S_2.

It is worth mentioning that (10) is actually an equality if $G_0(U)$ and $G_0(V)$ have no valency 4 nodes. More generally, if valency 4 nodes occur then

$$4[\chi_0(U)\chi_0(V) - \chi_F(U,V)] = \sum_{q,r} c(q,r),$$

where the sum is now over all nodes q of $G_0(U)$ and r of $G_0(V)$ and $c(q,r)$ is defined by the last column of the following table whose first column is $\{\partial(q),\ \partial(r)\}$ and whose second is the valency of the greatest lower bound type of the types of q and r (or 0 if no such type exists).

$\{3,3\}$	0	1
$\{3,3\}$	3	-1
$\{3,4\}$	0	2
$\{3,4\}$	3	0
$\{4,4\}$	0	4
$\{4,4\}$	3	2
$\{4,4\}$	4	0

It is not clear how useful this is. For instance, one might hope to answer Question 2.2 affirmatively for $\operatorname{rank} U = 2$ by showing that if $G_0(U)$ has two valency 3 nodes then each valency 3 node of $G_0(V)$ is comparable with at most one of them, but simple examples show that this can fail.

References

[B] Robert G. Burns, 'On the intersection of finitely generated subgroups of a free group', *Math. Z.* **119** (1971), 121–130.

[G] S.M. Gersten, 'Intersections of finitely generated subgroups of free groups and resolutions of graphs', *Invent. Math.* **71** (1983), 567–591.

[H] A.G. Howson, 'On the intersection of finitely generated free groups', *J. London Math. Soc.* **29** (1954), 428–434.

[I] Wilfried Imrich, 'On finitely generated subgroups of free groups', *Arch. Math.* **28** (1977), 21–24.

[I2] Wilfried Imrich, 'Subgroup theorems and graphs', in *Combinatorial Mathematics V*, Melbourne 1976, ed. by C.H.C. Little, Lecture Notes in Math. **622**, pp. 1–27 (Springer-Verlag, Berlin Heidelberg New York, 1977).

[N] Hanna Neumann, 'On the intersection of finitely generated free groups', *Publ. Math. Debrecen* **4** (1955–1956), 186–169. 'Addendum', *ibid.* **5** (1957–1958), p. 128.

[Ni] Peter Nickolas, 'Intersections of finitely generated free groups', *Bull. Austral. Math. Soc.* **34** (1985), 339–348.

[Se] Brigitte Servatius, 'A short proof of a theorem of Burns', *Math. Z.* **184** (1983), 133–137.

[St] John R. Stallings, 'Topology of finite graphs', *Invent. Math.* **71** (1983), 551–565.

Department of Mathematics
Ohio State University
COLUMBUS OH 43210
USA