A SHORT PROOF THAT POSITIVE GENERATION IMPLIES THE HANNA NEUMANN CONJECTURE

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The Strengthened Hanna Neumann Conjecture [3] posits that if U and V are finitely generated subgroups of a free group F then

$$\sum_{x\in T}\chi_0(U\cap x^{-1}Vx)\le\chi_0(U)\chi_0(V)\,,$$

where $\chi_0(H)=\max(0,\operatorname{rank} H-1)$ and T is a set of double coset representatives for $V\backslash F/U$. This is proved in [1] and [2] when U is positively generated (i.e., generated by elements of the subsemigroup generated by a basis of F). In a February 2003 CUNY Group Theory Seminar I pointed out a simple proof. It was suggested then and again recently that I record this in print.

Since any free group embeds in the free group of rank 2, we may assume $F=\langle a,b\rangle$. By embedding F in itself using the map $a\mapsto a^2$, $b\mapsto ab$, we may further assume that U is generated by positive words in the elements a^2 and ab. Then the core graph $G_0(U)$ has just two types of valence 3 vertices (since this is true of $G_0(\langle a^2,ab\rangle)$) and they occur in exactly equal numbers (since they have (incoming, outgoing) valence $(2,1)$ and $(1,2)$ respectively and the strings of $G_0(U)$ are directed). Proposition 3.1 of [3] says that a counterexample to the generalized HN conjecture must have over half the valence 3 vertices of both $G_0(U)$ and $G_0(V)$ all of the same type, so U can't belong to a counterexample.

References

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[2] J. Meakin, P. Weil. Subgroups of free groups: a contribution to the Hanna Neumann conjecture. Proceedings of the Conference on Geometric and Combinatorial Group Theory, Part I (Haifa, 2000). Geom. Dedicata 94 (2002), 33–43.

[3] Walter D. Neumann. On intersection of finitely generated subgroups of free groups. *Groups – Canberra 1989*. Lecture Notes in Mathematics **1456**, 161–170 (Springer Verlag, Berlin, Heidelberg, New York, 1990).

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