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der Universität Bonn

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Walter D. Neumann

S^1-Actions and the α-Invariant of their Involutions

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S^1-ACTIONS AND THE α-INVARIANT OF THEIR INVOLUTIONS

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Referent: Professor Dr. F. Hirzebruch
Korreferent: Priv.-Doz. Dr. K. Jänich

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Introduction

If a compact lie group G acts differentiably on a differentiable manifold X, then the orbit space X/G need not be a manifold, and the question arises as to when it is a manifold. In the first chapter we answer this question for G cyclic or equal to the circle group S^1, in terms of the slice representations of the isotropy subgroups of the action. This is then applied to obtain some results on the orbit space $V(a) = V(a_o, a_1, \ldots, a_n)$ of the well known S^1-action on the Brieskorn - Hirzebruch manifold

$$\Sigma(a) = \{z \in \mathbb{C}^{n+1} \mid \sum z_i^{a_i} = 0 \ , \ \sum |z_i|^2 = 1\} .$$

We show that $V(a)$ is a manifold if and only if a has the form

$$a = (dt_o s_1 s_2 \cdots s_n, \ \ldots \ , ds_o s_1 \cdots s_{n-1} t_n)$$

with d, s_i, t_i positive integers ; t_i pairwise coprime ; s_i pairwise coprime ; and $(s_i, t_i) = 1$ for each i (Theorem 3.5).

By Brieskorn and Van de Ven [3], $V(a)$ is in a natural way a complex projective variety. It turns out that this complex structure is already non-singular if $V(a)$ is topologically a manifold. Some biholomorphic equivalences between the $V(a)$ are found, and in particular if $V(a)$ is a manifold, then $V(a)$ is equivalent to $V(a')$, where a' is obtained from a by replacing each t_i above by $t'_i = 1$.

Chapters II and III of the thesis consider only closed orientable 3-dimensional S^1-manifolds with no fixpoints. The $\Sigma(a_o, a_1, a_2)$ are of this type, and we show how they fit into the Seifert - Raymond classification ([11], [14], [15]) of all closed 3-dimensional S^1-manifolds. Namely

$$\Sigma(a_o,a_1,a_2) \cong \{b;(o,g,0,0);ds_o(t_o,\beta_o'),ds_1(t_1,\beta_1'),ds_2(t_2,\beta_2')\},$$

where d, s_i, t_i are as above, and the β_i', b, and g are calculated as in corollary 9.2. By $ds_i(t_i,\beta_i')$ is meant of course $(t_i,\beta_i'), \dots$, (t_i,β_i') ds_i times, and to make the above fit Raymond's notation, the pairs (t_i,β_i') with $t_i = 1$ should be disregarded.

Brieskorn [2] has calculated the integer homology of $\Sigma(a_o, \dots, a_n)$ in terms of a certain group ring. Theoretically this enables one to calculate the normal form of these homology groups in any given case, but in practice this seems very difficult. The results of chapter II enable us to do this for $\Sigma(a_o,a_1,a_2)$.

In chapters II and III we in fact use a slightly altered form of the Seifert - Raymond classification, as this simplifies much formulation and calculation. The connection of our notation with Raymond's is given by corollary 7.3.

Let X be a closed oriented differentiable (4k-1) - manifold and $J : X \to X$ an orientation preserving free involution on X . Hirzebruch [6] has defined an invariant $\alpha(X,J)$ by using a special case of the Atiyah - Bott - Singer fixpoint theorem. If the disjoint union mX $(m \geq 1)$ of m copies of X bounds a compact oriented differentiable manifold N in such a way that J can be extended to an involution J_1 on N , which may have fixpoints, then

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$$\alpha(X,J) = \frac{1}{m}(\tau(N,J_1) - \tau(\mathrm{Fix}\,J_1 \cdot \mathrm{Fix}\,J_1)) .$$

Here $\tau(N,J_1)$ is the signature of the quadratic form $x \cdot J_1 y$ on $H_{2k}(N,\mathbb{R})$, and $\tau(\mathrm{Fix}\,J_1 \cdot \mathrm{Fix}\,J_1)$ is the signature of the oriented self intersection cobordism class $\mathrm{Fix}\,J_1 \circ \mathrm{Fix}\,J_1 \in \Omega_*$. By Burdick's results on the bordism theory of $\mathbb{Z}_2$-actions, N and J_1 exist with $m = 2$ and $\mathrm{Fix}\,J_1 = \emptyset$.

Hirzebruch and Jänich [7] have proved an alternative definition of $\alpha(X,J)$, which coincides with the definition of the Browder-Livesay invariant if X is a homotopy sphere. In chapter III we use this alternative definition to calculate $\alpha(X,J)$ when X is a 3-dimensional S^1-manifold and J is the involution contained in the S^1-action, if this involution is free. In Raymond's notation the result is : if $X \cong \{b;(o,g,0,0);(\alpha_1,\beta_1), \dots ,(\alpha_n,\beta_n)\}$, then (see theorem 12.1)

$$\alpha(X,J) = b + \sum_{i=1}^{n}(c(\alpha_i,\beta_i) + 1) - \mathrm{sign}(b + \sum_{i=1}^{n}\frac{\beta_i}{\alpha_i}) .$$

Here $c(\alpha,\beta)$ is a function defined on pairs (α,β) of coprime integers with α positive and odd, which may be defined in terms of continued fractions (§12). $c(\alpha,\beta)$ can also be defined axiomatically by the properties given in lemma 13.2 or for $\beta > 0$, as the signature of a certain matrix (§15), or as $\alpha(L(\beta,\alpha),J)$, where J is the free involution on the lens space $L(\beta,\alpha)$ such that the orbit space is $L(2\beta,\alpha)$ (theorem 17.2).

The final section of chapter III discusses some weak connections with the work of Bredon and Wood [1] on embedding non-orientable surfaces in closed orientable 3-manifolds.

I would like to take the opportunity to thank my supervisor, Professor F. Hirzebruch, and also Dr. Klaus Jänich for many helpful discussions and suggestions. I also thank the University of Adelaide for financially supporting my stay in Bonn.

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Chapter I: The orbit space of an S^1-manifold

§1. Preliminaries

1.1. The language of slice diagrams (Jänich [9]) will be useful in the following, so we recall the essential points. For more detail see §4 of [9].

Let G be a compact lie group, U a closed subgroup, and V a differentiable U-manifold. Consider the fibre bundle $G \times_U V$ over G/U with fibre V, associated to the principal U fibre bundle $G \to G/U$. Recall that $G \times_U V$ is $G \times V$ factored by the equivalence relation: $(g,x) \sim (gu, u^{-1}x)$ for $u \in U$. $G \times_U V$ is a G-manifold under the action $g'[g,x] = [g'g,x]$, $g' \in G$, $[g,x] \in G \times_U V$. If V is a real vector space with U-action given by a representation $\sigma : U \to GL(V)$, we denote V also by σ and write $G \times_U \sigma$ for $G \times_U V$, σ/U for V/U, and so on.

Let σ and σ' be real representations of closed subgroups U and U' of G. Then $G \times_U \sigma$ and $G \times_{U'} \sigma'$ are equivariantly diffeomorphic G-manifolds if and only if there is a $g \in G$ with $U' = gUg^{-1}$, and such that the representations σ and $\sigma' \circ (g \ldots g^{-1})$ of U are equivalent. The pairs (U,σ) and (U',σ') are said to represent the same <u>slice type</u> $[U,\sigma]_G$ (briefly $[U,\sigma]$) for G, if this holds.

If X is a G-manifold and $x \in X$, let $V_x = T_xX/T_xGx$ be the normal space to the orbit Gx at x, and let $\sigma_x : G_x \to GL(V_x)$ be the slice representation of the isotropy subgroup G_x. $[G_x, \sigma_x]_G$ is called the <u>slice type at the point x</u>. Slice type is constant along orbits of X, for if $g \in G$ then $G_{gx} = gG_xg^{-1}$ and

$\sigma_{gx} \sim \sigma_x \circ (g^{-1} \cdots g)$, so

(1.1) $$[G_{gx}, \sigma_{gx}]_G = [G_x, \sigma_x]_G .$$

$[G_x, \sigma_x]$ determines the local structure of X at x completely, for the slice theorem states (see for instance p3 of [9])

<u>Theorem:</u> There is a G-equivariant diffeomorphism of $G \times_{G_x} \sigma_x$ onto a G-invariant open neighbourhood of the orbit Gx , which maps the 0-section G/G_x onto the orbit Gx .

<u>1.2.</u> One defines a partial order on the set of all slice types for G by: $[U,\sigma] \leq [U',\sigma']$ means $[U',\sigma']$ is a slice type of the G-manifold $G \times_U \sigma$. The partially ordered set of all slice types of a G-manifold X is called the <u>slice diagram</u> $\Delta(G,X)$ (briefly $\Delta(X)$) of X . If X/G is connected, then $\Delta(X)$ has a unique largest element $[H,\theta]$, the <u>principal type</u>, which is characterised by the fact that θ is a trivial representation ([9] p24).

If X is compact then $\Delta(X)$ is finite ([9] p22).

<u>1.3.</u> The following remarks often simplify the calculation of slice types. Let U be a closed subgroup of G , $\sigma : U \to GL(V)$ a real representation of U . We want to investigate the slice types of $G \times_U \sigma$, that is the slice types which are greater than or equal to $[U,\sigma]$. For $[g,v] \in G \times_U \sigma$, let $[U_v, \sigma_v]_U$ be the slice type of the U-manifold V at the point v . We claim

(1.2) $$[G_{[g,v]}, \sigma_{[g,v]}]_G = [U_v, \sigma_v]_G .$$

By (1.1), we need only show this for $g=1$. Clearly $G_{[1,v]} = U_v$.

Further, the map $\pi : G \times_U \sigma \to G/U$ defined by $\pi[g,w] = gU$, which makes $G \times_U \sigma$ to a fibre bundle over G/U with fibre V , when restricted to the orbit $G[1,v]$ gives a fibre bundle $G[1,v] \to G/U$ with typical fibre Uv . Splitting the tangent bundles of $G \times_U \sigma$ and $G[1,v]$ as the sum of "component parallel to base" plus "component along the fibres" gives

$$T_{[1,v]} G \times_U \sigma \cong T_{1U} G/U \oplus T_v V ,$$
$$T_{[1,v]} G[1,v] \cong T_{1U} G/U \oplus T_v Uv .$$

Hence the normal spaces $T_{[1,v]} G \times_U \sigma / T_{[1,v]} G[1,v]$ and $T_v V / T_v Uv$ are canonically isomorphic. This isomorphism is U_v-equivariant, so $\sigma_{[1,v]} \sim \sigma_v$, proving (1.2).

Observe that if v is not a fixpoint of U , then σ_v has at least one more trivial component than σ , corresponding to the line through o and v in V . Hence if $[U,\sigma] \leq [U',\sigma']$ then the dimension of the trivial component of σ is strictly less than that of σ'. This can be used to give bounds on the legnths of chains in a slice diagram $\Delta(X)$, but in any case it follows that all chains are finite. Thus every element of $\Delta(X)$ is greater than or equal to a minimal element or <u>foot</u> of $\Delta(X)$. Since $\Delta(X)$ contains with each slice type also all greater slice types for G , we have

<u>Lemma:</u> $\Delta(X)$ is the set of all slice types for G which are greater than or equal to a foot of $\Delta(X)$.

<u>1.4.</u> Recall that if U is a closed subgroup of the compact lie group G and X is a G-manifold then the orbit bundle $X_{(U)}$ is defined as

$$X_{(U)} := \{x \in X \mid G_x \text{ is conjugate to } U \text{ in } G\}.$$

$X_{(U)}$ is an invariant submanifold, and has a natural structure of a fibre bundle over $X_{(U)}/G$ with the orbits as fibres (see for instance [9] pp5-7).

If $[U,\sigma]$ is a slice type of X, then the proof (loc. cit.) that $X_{(U)}$ is an invariant submanifold also yields that

$$X_{[U,\sigma]} := \{x \in X \mid [G_x, \sigma_x] = [U,\sigma]\}$$

is an invariant open closed submanifold of $X_{(U)}$, which we shall also call an <u>orbit bundle</u>. In fact $X_{[U,\sigma]} \subseteq X$ is given locally via the slice theorem by $G \times_{G_x} \theta_x \subseteq G \times_{G_x} \sigma_x$, where θ_x is the trivial component of σ_x. In particular it follows that

$$\dim X_{[U,\sigma]}/G = \dim \theta ,$$

where θ is the trivial component of σ.

Now let Y be an invariant submanifold of X. The slice representation σ_x^Y at x in Y is a subrepresentation of σ_x. Suppose the codimension k of $Y \cap X_{[U,\sigma]}$ in $X_{[U,\sigma]}$ at x is equal to the codimension of Y in X at x. Then using the above remarks it follows that

$$\sigma_x \sim \sigma_x^Y \oplus \theta ,$$

where θ is a trivial representation of dimension k. This simple fact will be useful later. (Remark: clearly the above situation holds if Y is transversal to $X_{[U,\sigma]}$ at x, however it follows easily from the geometry of the situation that the converse is also true.)

§2. The local structure of X/G

If X is a differentiable G-manifold, we would like to know when the orbit space X/G is a manifold. We answer this here for G cyclic and G equal to the circle group S^1 in terms of the slice diagram of X.

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<u>Definition:</u> If $[U,\sigma]$ is a slice type for G, we say $[U,\sigma]$ <u>has QM</u> if the quotient σ/U is topologically a manifold.

<u>Proposition 2.1.</u> If X is a G-manifold then X/G is a manifold if and only if every element of $\Delta(X)$ has QM. This holds if and only if every foot of $\Delta(X)$ has QM.

<u>Proof.</u> Let $\pi: X \to X/G$ be the orbit map. If the slice type at $x \in X$ is $[G_x, \sigma_x]$, then X looks locally like $G \times_{G_x} \sigma_x$ at x, so X/G looks like $(G \times_{G_x} \sigma_x)/G = \sigma_x/G_x$ in a neighbourhood of $\pi(x)$. Thus the first statement follows. It is now clear that if a slice type has QM, then so do all greater slice types, so the second statement follows also. ∥

For the rest of this section we assume $G = S^1$ or $\mathbb{Z}_n$ $(n \geq 2)$. Further we assume G acts effectively on X and that X/G is connected; this is no real restriction as one can always reduce to this case. It follows that for any $[U,\sigma] \in \Delta(X)$, σ is a faithful representation (for if $H = \ker \sigma$, then by the slice theorem, since G is commutative, H acts trivially on an open neighbourhood of an orbit, and since X/G is connected it follows that H acts trivially on all of X).

<u>Notation:</u> The 2-dimensional representation of S^1 defined by $(e^{2\pi i t}, z) \mapsto e^{2\pi i p t} z$ for $z \in \mathbb{C} = \mathbb{R}^2$ will be denoted by σ_p. The trivial representation of dimension n is denoted by n. Thus for example $\sigma_p \oplus \sigma_q \oplus 1$ is the representation

$$e^{2\pi it} \longmapsto \begin{pmatrix} \cos 2\pi pt & -\sin 2\pi pt & 0 & 0 & 0 \\ \sin 2\pi pt & \cos 2\pi pt & 0 & 0 & 0 \\ 0 & 0 & \cos 2\pi qt & -\sin 2\pi qt & 0 \\ 0 & 0 & \sin 2\pi qt & \cos 2\pi qt & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \in GL(\mathbb{R}^5)$$

If σ is a representation of S^1, we denote the representation of $\mathbb{Z}_n$ obtained by restriction also by σ.

The representations σ_p give all irreducible representations of S^1 and of $\mathbb{Z}_n$, n odd. For n even there is also the 1-dimensional representation τ of $\mathbb{Z}_n$, defined by $\tau(g) = -1 \in O(1)$, where g is a generator of $\mathbb{Z}_n$. Note that $\tau \oplus \tau = \sigma_{n/2}$.

It follows that every representation of S^1 is equivalent to one of the form $\sigma_{p_1} \oplus \sigma_{p_2} \oplus \dots \oplus \sigma_{p_r} \oplus k$, and every representation of $\mathbb{Z}_n$ is equivalent to one of the form $\sigma_{p_1} \oplus \dots \oplus \sigma_{p_r} \oplus k$ or $\tau \oplus \sigma_{p_1} \oplus \dots \oplus \sigma_{p_r} \oplus k$, where the latter case can of course only occur for n even. The representation of S^1 is faithful if and only if $\gcd(p_1,\dots,p_r) = 1$, and the representation of $\mathbb{Z}_n$ is faithful if and only if $\gcd(p_1, \dots ,p_r,n) = 1$ or $\gcd(p_1, \dots ,p_r,n/2) = 1$ respectively.

__Theorem 2.2.__ (i). Let $\gcd(p_1, \dots ,p_r,n) = 1$ and define for $i=1,\dots,r$

$\bar{p}_i = \gcd(p_1, \dots ,\hat{p}_i, \dots ,p_r,n)$. Then for $r \geq 1$

$[\mathbb{Z}_n, \sigma_{p_1} \oplus \dots \oplus \sigma_{p_r} \oplus k]$ has QM $\Leftrightarrow$ $\bar{p}_1\bar{p}_2 \cdots \bar{p}_r = n$.

(ii). Let $\gcd(p_1, \dots ,p_r,m) = 1$ and define for $i=1,\dots,r$

$\bar{p}_i = \gcd(p_1, \dots ,\hat{p}_i, \dots ,p_r,m)$. Then for $r \geq 0$

$[\mathbb{Z}_{2m}, \tau \oplus \sigma_{p_1} \oplus \dots \oplus \sigma_{p_r} \oplus k]$ has QM $\Leftrightarrow$ $r = 0$, or m is odd, the p_i are even, and $\bar{p}_1\bar{p}_2 \cdots \bar{p}_r = m$.

(iii). Let $\gcd(p_1, \dots ,p_r) = 1$. Then

$[S^1, \sigma_{p_1} \oplus \dots \oplus \sigma_{p_r} \oplus k]$ has QM $\Leftrightarrow$ $r \leq 2$.

We first need the following lemma.

__Lemma 2.3.__ Let σ be a representation in $\mathbb{R}^l$ of the compact lie group G, $S\sigma = S^{l-1}$ an invariant sphere in $\mathbb{R}^l$. Then for any k, if $(\sigma \oplus k)/G$ is a manifold, then $S\sigma/G$ is a $\mathbb{Z}$-homology sphere.

__Proof:__ $(\sigma \oplus k)/G = \mathbb{R}^l/G \times \mathbb{R}^k$. $\mathbb{R}^l/G$ is homeomorphic to the open cone $C(S\sigma/G)$ over $S\sigma/G$, so $(\sigma \oplus k)/G \cong C(S\sigma/G) \times \mathbb{R}^k$. $S\sigma/G$ admits a finite triangulation ([16]). Thus the result follows by lemma 2 of Mostert [10], which states that for any finite complex N, if $CN \times \mathbb{R}^k$ is a manifold, then N is a $\mathbb{Z}$-homology sphere. ||

__Proof of theorem 2.2.__ Suppose the cyclic group $\mathbb{Z}_p$ acts effectively by σ_q on $\mathbb{C}$. Then the map $z \mapsto z^p$ from $\mathbb{C}$ to $\mathbb{C}$ induces a homeomorphism

$$\varphi_p : \mathbb{C}/Z_p \xrightarrow{\cong} \mathbb{C}.$$

It would be more exact to write $\varphi_{p,q}$, but our notation leads to no ambiguity.

__Proof of (i).__ Let $\mathbb{C}_i$, $1 \leq i \leq r$, be r copies of $\mathbb{C}$. $\mathbb{Z}_n$ acts on $V := \mathbb{C}_1 \times \dots \times \mathbb{C}_r \times \mathbb{R}^k$ by $\sigma_{p_1} \oplus \dots \oplus \sigma_{p_r} \oplus k$.

Note that the $\bar{p}_i$ are pairwise coprime and divide n. Hence $\bar{p}_1\bar{p}_2 \cdots \bar{p}_r =: \bar{p}$ divides n and $\mathbb{Z}_{\bar{p}_1} \times \dots \times \mathbb{Z}_{\bar{p}_r} = \mathbb{Z}_{\bar{p}} \subseteq \mathbb{Z}_n$. We consider first the action of this subgroup on V. $\mathbb{Z}_{\bar{p}_i}$ acts trivially on $\mathbb{C}_j$ for $i \neq j$ and acts by σ_{p_i} on $\mathbb{C}_i$. Hence $V/\mathbb{Z}_{\bar{p}} = \mathbb{C}_1/\mathbb{Z}_{\bar{p}_1} \times \dots \times \mathbb{C}_r/\mathbb{Z}_{\bar{p}_r} \times \mathbb{R}^k \cong \mathbb{C}'_1 \times \mathbb{C}'_2 \times \dots \times \mathbb{C}'_r \times \mathbb{R}^k$, where the $\mathbb{C}'_i$ are r copies of $\mathbb{C}$, and the homeomorphism $\mathbb{C}_i/Z_{\bar{p}_i} \cong \mathbb{C}'_i$ is $\varphi_{\bar{p}_i}$. In particular if $\bar{p} = n$ then $V/\mathbb{Z}_n = V/\mathbb{Z}_{\bar{p}} \cong \mathbb{R}^{2r+k}$, proving the sufficiency of the condition.

The converse is by induction on r. If $r=1$, then $(\mathbb{C}\times\mathbb{R}^k)/\mathbb{Z}_n = \mathbb{C}/\mathbb{Z}_n\times\mathbb{R}^k = \mathbb{C}\times\mathbb{R}^k$. Since $\bar p_1 = n$ for $r=1$, statement (i) holds in this case.

Suppose $r>1$ and $[Z_n, \sigma_{p_1}\oplus\dots\oplus\sigma_{p_r}\oplus k]$ has QM. Then all greater slice types have QM, and by §1.3 these may be considered as slice types of the $\mathbb{Z}_n$-manifold $V=\mathbb{C}_1\times\dots\times\mathbb{C}_r\times\mathbb{R}^k$. The slice type at $(1,0,\dots,0)\in V$ is $[\mathbb{Z}_{q_1}, 2\oplus\sigma_{p_2}\oplus\dots\oplus\sigma_{p_r}\oplus k]$, where $q_1=\gcd(p_1,n$
Since $\gcd(p_2,p_3,\dots,\hat p_j,\dots,p_r,q_1) = \gcd(p_2,\dots,\hat p_j,\dots,p_r,p_1,n) = \bar p_j$, induction hypothesis implies that $\bar p_2\bar p_3\cdots\bar p_r = q_1$. That is $\bar p/\bar p_1 = q_1$. Similarly $\bar p/\bar p_i = q_i$ for each i, where $q_i=\gcd(p_i,n)$.

The $\mathbb{Z}_n$-action on V induces a $\mathbb{Z}_n/\mathbb{Z}_{\bar p} = \mathbb{Z}_{n/\bar p}$-action on $V/\mathbb{Z}_{\bar p}$ and since $V/\mathbb{Z}_{\bar p}\cong\mathbb{C}_1'\times\dots\times\mathbb{C}_r'\times\mathbb{R}^k$, we have an induced $\mathbb{Z}_{n/\bar p}$-action on $\mathbb{C}_1'\times\dots\times\mathbb{C}_r'\times\mathbb{R}^k$. A simple calculation shows that this action is given by $\sigma_{p_1'}\oplus\dots\oplus\sigma_{p_r'}\oplus k$, where $p_i' = p_i\bar p_i/\bar p$ for each i. But $p_i' = p_i\bar p_i/\bar p = p_i/q_i$, which is coprime to n/q_i, as $q_i = \gcd(p_i,n)$. Hence p_i' is coprime to $(n/q_i)/\bar p_i = n/\bar p$ for each i. It follows that the $\mathbb{Z}_{n/\bar p}$-action is free on any invariant sphere $S\cong S^{2r-1}$ in $\mathbb{C}_1'\times\dots\times\mathbb{C}_r'$. Thus for such an S, $S/\mathbb{Z}_{n/\bar p}$ has fundamental group $\mathbb{Z}_{n/\bar p}$ (recall that $r>1$), so $H_1(S/\mathbb{Z}_{n/\bar p},\mathbb{Z}) = \mathbb{Z}_{n/\bar p}$. But $V/\mathbb{Z}_n = (V/\mathbb{Z}_{\bar p})/\mathbb{Z}_{n/\bar p}\cong(\mathbb{C}_1'\times\dots\times\mathbb{C}_r'\times\mathbb{R}^k)/\mathbb{Z}_{n/\bar p}$, and by assumption this is a manifold. By lemma 2.3 $S/\mathbb{Z}_{n/\bar p}$ is a $\mathbb{Z}$-homology sphere, so $n=\bar p$.

This completes the proof of (i). The proofs of (ii) and (iii) are very similar, so we omit some of the detail.

Proof of (ii). $\mathbb{Z}_{2m}$ acts on $V := \mathbb{R}\times\mathbb{C}_1\times\dots\times\mathbb{C}_r\times\mathbb{R}^k$ by $\tau\oplus\sigma_{p_1}\oplus\dots\oplus\sigma_{p_r}\oplus k$. If the conditions of the theorem are satisfied, then

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$\mathbb{Z}_{2m} = \mathbb{Z}_2\times\mathbb{Z}_{\bar p_1}\times\dots\times\mathbb{Z}_{\bar p_r}$, and similarly to the proof of sufficiency for part (i), $V/\mathbb{Z}_{2m} = (\mathbb{R}/\mathbb{Z}_2)\times(\mathbb{C}_1/\mathbb{Z}_{\bar p_1})\times\dots\times(\mathbb{C}_r/\mathbb{Z}_{\bar p_r})\times\mathbb{R}^k$, which is a manifold (with boundary).

Conversely suppose $[\mathbb{Z}_{2m},\tau\oplus\sigma_{p_1}\oplus\dots\oplus\sigma_{p_r}\oplus k]$ has QM. The slice type of the $\mathbb{Z}_{2m}$-manifold V in the point $(1,0,\dots,0)\in V$ is $[\mathbb{Z}_m, 1\oplus\sigma_{p_1}\oplus\dots\oplus\sigma_{p_r}\oplus k]$. Since this must have QM, $\bar p_1\bar p_2\cdots\bar p_r = m$, by part (i).

It follows that $\mathbb{Z}_{\bar p_1}\times\dots\times\mathbb{Z}_{\bar p_r} = \mathbb{Z}_m\subset\mathbb{Z}_{2m}$ and $V/\mathbb{Z}_m = \mathbb{R}\times(\mathbb{C}_1/\mathbb{Z}_{\bar p_1})\times\dots\times(\mathbb{C}_r/\mathbb{Z}_{\bar p_r})\times\mathbb{R}^k\cong\mathbb{R}\times\mathbb{C}_1'\times\dots\times\mathbb{C}_r'\times\mathbb{R}^k$, where, as before, the $\mathbb{C}_i'$ are r copies of $\mathbb{C}$.

Now $V/\mathbb{Z}_{2m} = (V/\mathbb{Z}_m)/\mathbb{Z}_2\cong(\mathbb{R}\times\mathbb{C}_1'\times\dots\times\mathbb{C}_r'\times\mathbb{R}^k)/\mathbb{Z}_2$, where one checks that $\mathbb{Z}_2$ acts by $\tau\oplus\sigma_{p_1'}\oplus\dots\oplus\sigma_{p_r'}\oplus k$, with $p_i' = p_i\bar p_i/m$ for each i. Without loss of generality suppose p_i' is odd for $1\le i\le s$ and even for $s+1\le i\le r$, with $0\le s\le r$. Then $\tau\oplus\sigma_{p_1'}\oplus\dots\oplus\sigma_{p_r'}\oplus k = \tau\oplus(\tau\oplus\tau)\oplus\dots\oplus(\tau\oplus\tau)\oplus 2\oplus\dots\oplus 2\oplus k$ with s summands $\tau\oplus\tau$ and $r-s$ summands 2. Taking an invariant sphere $S\cong S^{2s}$ in $\tau\oplus\dots\oplus\tau$, one has $S/\mathbb{Z}_2\cong\mathbb{P}_{2s}(\mathbb{R})$, which is not a $\mathbb{Z}$-homology sphere unless $s=0$. Hence by lemma 2.3, $s=0$, so all the p_i' are even. Since $p_i = p_i'(m/\bar p_i)$, all the p_i are even. The condition $\gcd(p_1,\dots,p_r,m) = 1$ now yields that m is odd.

Proof of (iii). Let $\bar p_i = \gcd(p_1,\dots,\hat p_i,\dots,p_n)$ for $1\le i\le r$. S^1 acts on $V=\mathbb{C}_1\times\dots\times\mathbb{C}_r\times\mathbb{R}^k$ by $\sigma_{p_1}\oplus\dots\oplus\sigma_{p_r}\oplus k$. The slice type in the point $(1,0,\dots,0)$ is $[\mathbb{Z}_{p_1}, 1\oplus\sigma_{p_2}\oplus\dots\oplus\sigma_{p_r}\oplus k]$, so by part (i) we must have $p_1 = \bar p_2\bar p_3\cdots\bar p_r$ for QM. That is $p_1 = \bar p/\bar p_1$, where $\bar p = \bar p_1\bar p_2\cdots\bar p_r$. Similarly $p_i = \bar p/\bar p_i$ for each i.

$\mathbb{Z}_{\bar p_1}\times\dots\times\mathbb{Z}_{\bar p_r} = Z_{\bar p}\subset S^1$, and as before $Z_{\bar p_i}$ acts trivially on every $\mathbb{C}_j$ with $j \neq i$. Hence $V/\mathbb{Z}_{\bar{p}} = (\mathbb{C}_1/\mathbb{Z}_{\bar{p}_1}) \times \dots \times (\mathbb{C}_r/\mathbb{Z}_{\bar{p}_r}) \times \mathbb{R}^k \cong \mathbb{C}'_1 \times \dots \times \mathbb{C}'_r \times \mathbb{R}^k$, where the $\mathbb{C}'_i$ are r copies of $\mathbb{C}$. The induced action of $S^1/\mathbb{Z}_{\bar{p}} \cong S^1$ on $\mathbb{C}'_1 \times \dots \times \mathbb{C}'_r \times \mathbb{R}^k$ is by $\sigma_{p'_1} \oplus \dots \oplus \sigma_{p'_r} \oplus k$, where $p'_i = p_i \bar{p}_i/\bar{p} = 1$ for each i. Hence for an invariant sphere $S \cong S^{2r-1}$ in $\mathbb{C}'_1 \times \dots \times \mathbb{C}'_r$, $S/(S^1/\mathbb{Z}_{\bar{p}}) = \mathbb{P}_{r-1}(\mathbb{C})$ (complex projective space of $\mathbb{C}$-dimension $r-1$). This is a $\mathbb{Z}$-homology sphere if and only if $r \leq 2$, and for $r \leq 2$ it is even a topological sphere. The result now follows by lemma 2.3, since $V/S^1 \cong (\mathbb{C}'_1 \times \dots \times \mathbb{C}'_r \times \mathbb{R}^k)/(S^1/\mathbb{Z}_{\bar{p}})$. ||

In the above proof one always has a natural differentiable structure on the quotient σ/U. Further σ/U is a manifold with non-empty boundary only in case (ii). Hence one has the corollary to the above proof:

Corollary 2.4. If X is a differentiable G-manifold with $G = S^1$ or G cyclic, and X/G is a topological manifold, then X/G has a natural structure of a differentiable manifold such that the orbit map $X \longrightarrow X/G$ is differentiable. Further X/G has non-empty boundary if and only if slice types of type (ii) of theorem 2.2 occur.

In particular if $G = S^1$ and X is orientable and X/G is a manifold, then X/G has no boundary, for if $[U,\sigma]$ is a slice type of type (ii) of theorem 2.2, then $S^1 \times_U \sigma$ is non-orientable.

An easily deducible consequence of theorem 2.2 is that for $G = S^1$ and G cyclic, the property that a G-manifold have a manifold as orbit space is inherited by invariant submanifolds. It seems likely that this holds much more generally.

§3. Application to Brieskorn-Hirzebruch manifolds

Let $a = (a_0, \dots, a_n)$ be an $(n+1)$-tuple of positive integers $(n \geq 2)$.

$$X(a) = \{ z \in \mathbb{C}^{n+1} \mid z_0^{a_0} + \dots + z_n^{a_n} = 0 \}$$

is a non-singular variety in $\mathbb{C}^{n+1}$ if some $a_i = 1$, otherwise it has just one singular point $0 = (0,\dots,0)$. Let

$$\Sigma(a) = X(a) \cap S^{2n+1},$$

where S^{2n+1} is the sphere $\sum |z_i|^2 = 1$ in $\mathbb{C}^{n+1}$. $\Sigma(a)$ is in a natural way a closed connected oriented differentiable $(2n-1)$-manifold. These manifolds $\Sigma(a)$ have been studied extensively by Brieskorn, Hirzebruch, Milnor, and others (see for instance [2],[6],[8]).

Let

(3.1) $\qquad a'_i = \mathrm{lcm}(a_0, \dots, a_n)/a_i \quad , \quad (i=0,\dots,n)$.

There is a well known effective S^1-action on $\Sigma(a)$ given by

$$e^{2\pi i t}(z_0, \dots, z_n) = (e^{2\pi i a'_0 t} z_0, \dots, e^{2\pi i a'_n t} z_n).$$

Let $V(a)$ denote the orbit space $\Sigma(a)/S^1$.

Question. When is $V(a) = \Sigma(a)/S^1$ a manifold?

To answer this question we must first calculate the slice diagram $\Delta(S^1, \Sigma(a))$. Observe that S^1 acts on $\mathbb{C}^{n+1}$ by the representation $\sigma = \sigma_{a'_0} \oplus \dots \oplus \sigma_{a'_n}$, and that S^{2n+1} and $\Sigma(a)$ are invariant submanifolds of $\mathbb{C}^{n+1}$ under this action. The given action on $\Sigma(a)$ is just the restriction to $\Sigma(a)$ of this action on $\mathbb{C}^{n+1}$.

The slice diagram $\Delta(S^1, \mathbb{C}^{n+1}) = \Delta(S^1, \sigma)$ is just the set of all S^1 slice types which are greater than or equal to $[S^1, \sigma]$.

<u>Lemma 3.1.</u> $\Delta(S^1,\Sigma(a)) = \{[U,\tau] \mid [U,\tau\oplus 3]\in\Delta(S^1,\sigma)\}$.

The feet of $\Delta(S^1,\Sigma(a))$ are the slice types

$$[U_{i\,j},\sigma_{i\,j}] = [\mathbb{Z}_{\gcd(a_i',a_j')},\sigma_{a_o'}\oplus\cdots\oplus\hat{\sigma}_{a_i'}\oplus\cdots\oplus\hat{\sigma}_{a_j'}\oplus\cdots\oplus\sigma_{a_n'}]\,,$$

with $0\leq i<j\leq n$. The corresponding orbit bundles are

$$\Sigma(a)_{[U_{i\,j},\sigma_{i\,j}]} = \{(0,\dots,0,z_i,0,\dots,0,z_j,0,\dots,0)\in\Sigma(a)\}.$$

<u>Proof.</u> The foot $[S^1,\sigma]$ of $\Delta(S^1,\sigma)$ has orbit bundle $\{0\}$, which does not intersect $\Sigma(a)$. Any other slice type $[U,\tau]\in\Delta(S^1,\sigma)$ has the form

$$[U,\tau] = [\mathbb{Z}_{\gcd(a_o',\dots,a_k')},\sigma_{a_{k+1}'}\oplus\cdots\oplus\sigma_{a_n'}\oplus(2k-1)]\,,$$

with $0\leq k\leq n$, or can be obtained from such a slice type by permuting the indices. The corresponding orbit bundle in $\mathbb{C}^{n+1}$ is

$$\mathbb{C}^{n+1}_{[U,\tau]} = \{(z_o,\dots,z_k,0,\dots,0)\in\mathbb{C}^{n+1} \mid z_i\neq 0 \text{ for } 0\leq i\leq k\}.$$

For $k=0$ this has empty intersection with $\Sigma(a)$. For $k>0$ $\mathbb{C}^{n+1}_{[U,\tau]}\cap\Sigma(a)$ is an open subspace of $\{(z_o,\dots,z_k,0,\dots,0)\in\Sigma(a)\}$, and the latter is diffeomorphic to $\Sigma(a_o,\dots,a_k)$, which has dimension $2k-1$. Hence for $k>0$ $\mathbb{C}^{n+1}_{[U,\tau]}\cap\Sigma(a)$ has codimension 3 in $\mathbb{C}^{n+1}_{[U,\tau]}$. This is equal to the codimension of $\Sigma(a)$ in $\mathbb{C}^{n+1}$. The lemma now follows by part 1.4 of §1. ||

Define

(3.2) $\quad t_i = \gcd(a_o',\dots,\hat{a}_i',\dots,a_n')\,,\quad (i=0,\dots,n).$

Then by theorem 2.2 the foot $[U_{i\,j},\sigma_{i\,j}]$ of $\Delta(S^1,\Sigma(a))$ has QM if and only if

(3.3) $\quad \prod_{k\neq i,j} t_k = \gcd(a_i',a_j')\,.$

Hence by theorem 2.1 :

<u>Lemma 3.2.</u> $V(a)$ is a manifold if and only if (3.3) holds for all $i,j\in\{0,\dots,n\}$ with $i\neq j$. ||

<u>An alternative description of $V(a)$</u>

The additive group $\mathbb{C}$ acts on $X(a)-\{0\}$ by

$$t(z_o,\dots,z_n) = (e^{t/a_o}z_o,\dots,e^{t/a_n}z_n)\,.$$

Each orbit of this action intersects $\Sigma(a)\subset X(a)-\{0\}$ in an orbit of the S^1-action on $\Sigma(a)$, so the inclusion $\Sigma(a)\subset X(a)-\{0\}$ induces a homeomorphism

$$V(a) = \Sigma(a)/S^1 \longrightarrow (X(a)-\{0\})/\mathbb{C}$$

of the orbit spaces. We identify $V(a)$ with $(X(a)-\{0\})/\mathbb{C}$ by this map.

Brieskorn and Van de Ven [3] have remarked that $V(a)$ carries a natural complex structure such that the orbit map $X(a)-\{0\}\rightarrow V(a)$ is holomorphic, and that $V(a)$ is a projective algebraic complex space with this structure. They give the following necessary and sufficient condition that this complex structure be non-singular:

(3.4) For every subset $\{i_o,\dots,i_k\}$ of $\{0,\dots,n\}$ with $k\geq 1$

$$\frac{\operatorname{lcm}(a_o,\dots,a_n)}{\operatorname{lcm}(a_{i_o},\dots,a_{i_k})} = \prod_{i\notin\{i_o,\dots,i_k\}} \frac{\operatorname{lcm}(a_o,\dots,a_n)}{\operatorname{lcm}(a_o,\dots,\hat{a}_i,\dots,a_n)}$$

<u>Proposition 3.3.</u> $V(a)$ is a manifold if and only if the complex structure on $V(a)$ is non-singular.

<u>Proof.</u> For convenient reference we mention here the trivial number theoretic lemma:

Lemma 3.4. If $(c_o, \dots, c_k)$ is a tuple of divisors of an integer m, then $\gcd(m/c_o, \dots, m/c_k) = m/\operatorname{lcm}(c_o, \dots, c_k)$. ||

It follows that (3.4) can be written as:

For every subset $\{i_o, \dots, i_k\}$ of $\{0, \dots, n\}$ with $k \geq 1$

(3.5) $$\gcd(a'_{i_o}, \dots, a'_{i_k}) = \prod_{i \notin \{i_o, \dots, i_k\}} t_i .$$

Since $\gcd(a'_o, \dots, a'_n) = 1$, the t_i are pairwise coprime. It follows that (3.5) already holds if one only requires it to hold for two element subsets of $\{0, \dots, n\}$ $(k = 1)$. By lemma 3.2, this proves the proposition. ||

The following theorem gives a general method for constructing tuples $(a_o, \dots, a_n)$ such that $V(a)$ is a manifold.

Theorem 3.5. $V(a_o, \dots, a_n)$ is a manifold if and only if there exist positive integers d, s_i, t_i, $(0 \leq i \leq n)$ satisfying:

(i). the t_i are pairwise coprime,

(ii). the s_i are pairwise coprime,

(iii). $\gcd(s_i, t_i) = 1$ $(0 \leq i \leq n)$,

(iv). $a_i = d s_o s_1 \cdots s_{i-1} t_i s_{i+1} \cdots s_n$ $(0 \leq i \leq n)$.

Note that the conditions (i) to (iv) imply that

(3.6) $$d = \gcd(a_o, \dots, a_n) ,$$

(3.7) $$s_i = \tfrac{1}{d}\gcd(a_o, \dots, \hat{a}_i, \dots, a_n) ,$$

(3.8) $$a'_i = t_o t_1 \cdots t_{i-1} s_i t_{i+1} \cdots t_n ,$$

and that the t_i are as in (3.2). In particular d, s_i, t_i, $(0 \leq i \leq n)$ are uniquely defined by the conditions.

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Proof of theorem 3.5. If conditions (i) to (iv) hold then (3.3) follows from (3.8) and the fact that $s_i t_j$ is coprime to $s_j t_i$ for $i \neq j$. Hence by lemma 3.2, $V(a)$ is then a manifold.

Suppose conversely that (3.3) holds for all i, j with $0 \leq i < j \leq n$. Define t_i as in (3.2) and define s_i by

(3.9) $$s_i = a'_i / \prod_{k \neq i} t_k .$$

(3.3) implies that for $0 \leq i < j \leq n$, $a'_i / \prod_{k \neq i,j} t_k$ is coprime to $a'_j / \prod_{k \neq i,j} t_k$; that is $s_i t_j$ is coprime to $s_j t_i$. This implies (i), (ii), and (iii). Further (3.9) implies (3.8), so using (i), (ii), and (iii) one has

(3.10) $$\operatorname{lcm}(a'_o, \dots, a'_n) = \prod_{i=0}^{n} t_i \prod_{i=0}^{n} s_i .$$

Lemma 3.4 gives $\gcd(a_o, \dots, a_n) = \operatorname{lcm}(a_o, \dots, a_n)/\operatorname{lcm}(a'_o, \dots, a'_n)$, whence follows $\operatorname{lcm}(a_o, \dots, a_n) = d \prod_{i=0} t_i \prod_{i=0} s_i$. (3.1) and (3.8) now give (iv). ||

§4. Some properties of the $V(a)$

In this section we give a simple sufficient condition for $V(a_o, \dots, a_n)$ and $V(b_o, \dots, b_n)$ to be biholomorphically equivalent. It would be of interest to know more about the complex spaces $V(a)$, however we only obtain complete results in very special cases.

Let $(b_o, \dots, b_n)$ and $(d_o, \dots, d_n)$ be $(n+1)$-tuples of positive integers $(n \geq 2)$, and let

(4.1) $$(a_o, \dots, a_n) = (b_o d_o, \dots, b_n d_n) .$$

Let a_i' and t_i $(0 \leq i \leq n)$ be defined as in (3.1) and (3.2).

The holomorphic map

$$\varphi : X(a_o, \ldots, a_n) \longrightarrow X(b_o, \ldots, b_n)$$

defined by

$$\varphi(z_o, \ldots, z_n) = (z_o^{d_o}, \ldots, z_n^{d_n})$$

is $\mathbb{C}$-equivariant, and hence induces a map

$$\Psi : V(a_o, \ldots, a_n) \longrightarrow V(b_o, \ldots, b_n)$$

of the orbit spaces.

<u>Theorem 4.1.</u> Ψ is biholomorphic if and only if d_i divides t_i for each i.

<u>Proof.</u> Note that Ψ is a ramified covering, so it is biholomorphic if and only if it is bijective.

Let G_d be the group $\mathbb{Z}_{d_o} \times \ldots \times \mathbb{Z}_{d_n}$, and let w_i be a generator of $\mathbb{Z}_{d_i}$ for each i. G_d acts on $X(a)$ by

$$(w_o^{k_o}, \ldots, w_n^{k_n})(z_o, \ldots, z_n) = (e^{2\pi i k_o/d_o} z_o, \ldots, e^{2\pi i k_n/d_n} z_n) .$$

The orbits of this action are just the fibres of the map φ defined above, so φ induces a bijective map $X(a)/G_d \longrightarrow X(b)$. One thus has a commutative diagram

$$\begin{array}{ccc} X(a) & \xrightarrow{\varphi} & X(b) \\ & \searrow & \nearrow_{\cong} \\ & X(a)/G_d & \end{array}$$

The G_d-action on $X(a)$ is $\mathbb{C}$-equivariant, so it induces a G_d-action on $V(a) = X(a) - \{0\}/\mathbb{C}$. The above diagram induces a commutative diagram

$$\begin{array}{ccc} V(a) & \xrightarrow{\Psi} & V(b) \\ & \searrow & \nearrow_{\Psi'} \\ & V(a)/G_d & \end{array}$$

Thus Ψ is bijective if and only if the G_d-action on $V(a)$ is trivial.

Let $b_i' = \text{lcm}(b_o, \ldots, b_n)/b_i$ for each i. A simple calculation shows that an element of G_d acts trivially on $V(a)$ if and only if it is in the cyclic subgroup generated by $(w_o^{b_o'}, \ldots, w_n^{b_n'}) \in G_d$, and this subgroup has order $\text{lcm}(a_o, \ldots, a_n)/\text{lcm}(b_o, \ldots, b_n)$. Hence, since G_d has order $\prod_{i=0}^{n} d_i$,

(4.2) $\quad \text{lcm}(a_o, \ldots, a_n)/\text{lcm}(b_o, \ldots, b_n)$ divides $\prod_{i=0}^{n} d_i$,

and G_d acts trivially on $V(a)$ if and only if

(4.3) $\quad \text{lcm}(a_o, \ldots, a_n)/\text{lcm}(b_o, \ldots, b_n) = \prod_{i=0}^{n} d_i$.

Suppose (4.3) holds. Then $\text{lcm}(a_o, \ldots, a_n) = \text{lcm}(b_o, \ldots, b_n)\prod_{i=0}^{n} d_i$, so $a_j' = \text{lcm}(b_o, \ldots, b_n)\prod_{i=0}^{n} d_i / b_j d_j = b_j' \prod_{i \neq j} d_i$. Hence d_i divides a_j' if $j \neq i$, so it divides $t_i = \text{lcm}_{j \neq i}\, a_j'$.

Conversely suppose d_i divides t_i for each i. Since the t_i are pairwise coprime and divide a_j' for $i \neq j$, $\prod_{i \neq j} t_i$ divides a_j'. Hence $\prod_{i \neq j} d_i$ divides a_j'. But $a_j' = \text{lcm}(a_o, \ldots, a_n)/b_j d_j$, so $d_j \prod_{i \neq j} d_i$ divides $\text{lcm}(a_o, \ldots, a_n)/b_j$. That is $\prod_{i=0}^{n} d_i$ divides $\text{lcm}(a_o, \ldots, a_n)/b_j$ for each j, so it divides $\gcd_{0 \leq j \leq n}(\text{lcm}(a_o, \ldots, a_n)/b_j)$. By lemma 3.4, the latter is equal to $\text{lcm}(a_o, \ldots, a_n)/\text{lcm}(b_o, \ldots, b_n)$. With (4.2) this gives (4.3). ||

Suppose $(a_o, \dots, a_n)$ satisfies the condition of theorem 3.5; that is $V(a_o, \dots, a_n)$ is a manifold. Taking $d_i = t_i$ in the previous theorem gives

<u>Corollary 4.2.</u> Up to biholomorphic equivalence the complex manifold $V(a)$ of theorem 3.5 depends only on the values of d and the s_i $(0 \le i \le n)$, and not on the values of the t_i.

Suppose $V(a)$ is a manifold, and the notation is as in theorem 3.5. Then

<u>Example 4.3.</u> $V(t_o, s_o t_1, \dots, s_o t_n) \cong \mathbb{P}_{n-1}(\mathbb{C})$ (complex projective space)

<u>Example 4.4.</u> $V(dt_o, dt_1, \dots, dt_n) \cong H_d$ (the hypersurface of order d in $\mathbb{P}_n(\mathbb{C})$ defined by the homogeneous equation $z_o^d + \dots + z_n^d = 0$).

<u>Proof.</u> By corollary 4.2 we may assume $t_o = t_1 = \dots = t_n = 1$. The above biholomorphic equivalences are induced by the maps $\Sigma(1, s_o, \dots, s_o) \longrightarrow \mathbb{P}_{n-1}(\mathbb{C})$ defined by $(z_o, \dots, z_n) \longmapsto \langle z_1, \dots, z_n \rangle$, and $\Sigma(d, d, \dots d) \longrightarrow H_d \subset \mathbb{P}_n(\mathbb{C})$ defined by $(z_o, \dots, z_n) \longmapsto \langle z_o, \dots, z_n \rangle$. ||

<u>Remark.</u> If $\Sigma(a)$ is a $\mathbb{Q}$-homology sphere and $V(a)$ is a manifold, then $V(a) = \mathbb{P}_{n-1}(\mathbb{C})$, or n is even and $V(a) = H_2$ (complex quadric).

Indeed, using the calculations in Hirzebruch - Mayer [8] (§8.1 and theorem 13.3) one can show that these conditions lead (after suitable permutation of indices) to example 4.3, or to example 4.4 with n even and $d = 2$.

If $\Sigma(a)$ is a homotopy sphere and $V(a)$ is a manifold, then similar considerations show $V(a) = \mathbb{P}_{n-1}(\mathbb{C})$ (see [3]).

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<u>Chapter II: 3-dimensional S^1-manifolds</u>

<u>§5. Introduction</u>

In this chapter we discuss the Seifert - Raymond classification [11], [14], [15], of 3-dimensional S^1-manifolds, and show how the S^1-manifolds $\Sigma(a_o, a_1, a_2)$ fit into this classification. We are only interested in <u>oriented closed 3-dimensional differentiable effective S^1-manifolds with no fixpoints</u>. There are then no special exceptional orbits (orbits with slice type $[\mathbb{Z}_2, \tau \oplus 1]$), and the orbit space is a closed oriented surface. This can easily be seen directly, and also follows immediately from the Raymond classification.

Orlik and Raymond [11] construct a set of "standard actions" and shows that each 3-dimensional S^1-manifold is equivariantly homeomorphic to precisely one such standard action. To simplify later calculations we extend Raymond's list of standard actions. We must then say to which of Raymond's standard actions any action from the extended list is equivalent. This is done by corollary 7.3.

Since it is necessary to keep a careful track of orientations, we use the following conventions.

<u>Orientation conventions.</u> If M is an oriented manifold, give the boundary ∂M the orientation which, when followed by an inward normal to ∂M, gives the orientation of M (i.e. M is locally orientation preservingly diffeomorphic to an open subset of $\mathbb{R}_+^n = \{ x \in \mathbb{R}^n \mid x_n \ge 0 \}$, where $\mathbb{R}^n$ and $\mathbb{R}^{n-1} \subset \mathbb{R}^n$ have their usual orientations).

If M is an oriented S^1-manifold and M/S^1 is an orientable manifold, orient M/S^1 so that the orientation of M/S^1 followed by the natural orientation of the orbits gives the orientation of M.

We stress that this fixing of orientation conventions is for practical purposes only; the results are independent of the orientation conventions used in their derivation.

§6. The standard actions

Let $\mathfrak{X} = ((g), (\alpha_1, \beta_1), \ldots, (\alpha_n, \beta_n))$ be a tuple of integers with $g \geq 0$, $\alpha_i > 0$ for each i, and α_i coprime to β_i for each i. We permit β_i to be positive, negative, or (if $\alpha_i = 1$) zero.

We shall construct a 3-dimensional S^1-manifold $X(\mathfrak{X})$, whose orbit structure is determined by the (α_i, β_i) with $\alpha_i \neq 1$, and whose orbit space is an oriented surface of genus g.

For $j = 1, \ldots, n$ let ν_j be the integer with $0 < \nu_j \leq \alpha_j$ and

(6.1) $\quad \beta_j \nu_j \equiv 1 \mod \alpha_j$.

Let

(6.2) $\quad \rho_j = \dfrac{(\beta_j \nu_j - 1)}{\alpha_j}$

Let T_j be a copy of the solid torus $D^2 \times S^1$, parametrised by $(re^{i\theta}, e^{i\psi})$ with $0 \leq r \leq 1$. Define an S^1-action on T_j by

(6.3) $\quad z(re^{i\theta}, e^{i\psi}) = (z^{\nu_j} re^{i\theta}, z^{\alpha_j} e^{i\psi}), \quad (|z| = 1)$.

If $\alpha_j \neq 1$, the centre circle $O_j = \{o\} \times S^1$ is an exceptional orbit with slice type $[Z_{\alpha_j}, \sigma_{\nu_j}]$. All other orbits are principal orbits.

Let Q_j be the curve in the boundary of T_j given by

(6.4) $\quad Q_j = \{(e^{i\rho_j\theta}, e^{i\beta_j\theta}) \mid 0 \leq \theta < 2\pi\} \subset \partial T_j$.

Q_j is a section to the S^1-action in ∂T_j. This is easily checked, but it also follows from the discussion of the map φ_j defined below.

Let X^* be an oriented surface of genus g. Choose n points $x_1^*, \ldots, x_n^*$ in X^* and remove the interior of a small closed disc neighbourhood D_j^* of each of these points. Call the resulting surface with boundary X_o^*. Let

$$X_o = X_o^* \times S^1$$

with the obvious S^1-action. Let

$$R = X_o^* \times \{1\} \subset X_o .$$

R is a section to the S^1-action on X_o.

The boundary of X_o^* consists of n circles $S_1^*, \ldots, S_n^*$, so the boundary of X_o consists of the n tori $S_1^* \times S^1, \ldots, S_n^* \times S^1$. In each of these boundary components we have the section $S_j^* \times \{1\} = (S_j^* \times S^1) \cap R$ to the S^1-action.

We sew the solid torus T_j equivariantly into X_o by matching the orbits in T_j with the orbits in $S_j^* \times S^1$ and matching Q_j with $S_j^* \times \{1\}$. If $S_j^* \times S^1$ is parametrised by $(e^{i\theta}, e^{i\psi})$, where increasing θ orients S_j^* as a boundary component of X_o^*, then a suitable identification map

$$\varphi_j : S_j^* \times S^1 \to \partial T_j$$

is given by

$$(e^{i\theta}, e^{i\psi}) \mapsto (e^{i(\rho_j\theta + \nu_j\psi)}, e^{i(\beta_j\theta + \alpha_j\psi)}) .$$

Indeed, this map is clearly equivariant, and maps $S_j^* \times \{1\}$ onto Q_j.

It is bijective and orientation reversing since $\begin{vmatrix} \rho_j & \nu_j \\ \beta_j & \alpha_j \end{vmatrix} = -1$ by (6.2).

The manifold

$$X(\ast) = X_o \underset{\varphi_1}{\cup} T_1 \underset{\varphi_2}{\cup} T_2 \cup \dots \underset{\varphi_n}{\cup} T_n$$

so obtained is the desired standard action.

Let M_j be the boundary of a slice D_j in T_j; say $D_j = \{(re^{i\theta}, e^{i\psi_o}) \mid 0 \le r \le 1,\ 0 \le \theta < 2\pi\}$ (ψ_o fixed) and $M_j = \{(e^{i\theta}, e^{i\psi_o}) \mid 0 \le \theta < 2\pi\}$. Orient M_j by increasing θ, that is as the boundary of D_j, where D_j has the orientation which, when followed by the natural orientation of an orbit through D_j, gives the orientation of $X(\ast)$. Let H be any orbit in ∂T_j, oriented by the circle group. Orient Q_j by decreasing θ in (6.4), that is as a boundary component of $-R$. One then has the homology relation in ∂T_j

(6.5) $\quad M_j \sim \alpha_j Q_j + \beta_j H$.

It follows that if $\ast = ((g),(1,b),(\alpha_1,\beta_1), \dots ,(\alpha_n,\beta_n))$ with $0 < \beta_j < \alpha_j$ for each j, then our construction of $X(\ast)$ coincides with the construction ([11] §2) of Raymond's standard action $\{b;(o,g,0,0);(\alpha_1,\beta_1), \dots ,(\alpha_n,\beta_n)\}$. Thus

<u>Lemma 6.1.</u> The standard action $\{b;(o,g,0,0);(\alpha_1,\beta_1), \dots ,(\alpha_n,\beta_n)\}$ of Raymond [11] is the standard action $X((g),(1,b),(\alpha_1,\beta_1), \dots ,(\alpha_n,\beta_n))$ constructed above. ||

<u>Remark.</u> In the above construction we specified very precisely what Q_j should look like. However it suffices to specify only, as Raymond (loc. cit.) does, that Q_j be a section to the S^1-action in ∂T_j and satisfy the homology relation (6.5). In fact if Q_j is as above, and Q_j' is any other section which also satisfies (6.5), then it is not hard to show that there is an equivariant homeomorphism of T_j onto itself which maps Q_j onto Q_j' (see proof of Hilfsatz VI in [15]). If Q_j' is smooth, there is even a diffeomorphism.

<u>§7. The equivariant classification</u>

Let X be any 3-dimensional closed oriented differentiable effective S^1-manifold without fixpoints. Let $O_1, \dots ,O_n$ $(n \ge 1)$ be a non-empty set of orbits in X which includes all the exceptional orbits of X. Let $T_1, \dots ,T_n$ be small closed invariant tubular neighbourhoods of these orbits, and let

$$X_o = X - (\mathring{T}_1 \cup \dots \cup \mathring{T}_n)$$

be obtained by removing the interior of each T_i from X.

X_o is a free 3-dimensional S^1-manifold with non-empty boundary, so considered as a principal S^1 fibre bundle over the orbit space X_o^*, X_o is the product bundle (principal S^1-bundles over X_o^* are classified by their characteristic class $c_1 \in H^2(X_o^*, \mathbb{Z})$. X_o^*, as a surface with non-empty boundary, has $H^2(X_o^*,\mathbb{Z}) = 0$). Thus there exists a section $R \subset X_o$ to the S^1-action. R is oriented by the condition that the natural homeomorphism $R \to X_o^*$ be orientation preserving. It is not necessary to require that R be smooth.

Let $Q_j = R \cap \partial T_j$ for each j, oriented as a boundary component of $-R$. Let D_j be a slice in T_j, and orient $M_j = \partial D_j \subset \partial T_j$ as the boundary of D_j . Let H be an orbit in ∂T_j , with the natural orientation. Since Q_j is a section to the S^1-action in ∂T_j , it is a conjugate curve to H , so the homology classes of Q_j and H freely generate $H_1(\partial T_j, \mathbb{Z})$. We hence have a homology relation

(7.1) $$M_j \sim \alpha_j Q_j + \beta_j H$$

in ∂T_j with uniquely defined coefficients.

Definition. (α_j, β_j) is the Seifert invariant pair of the orbit O_j with respect to the partial section R in X .

An alternative description of the Seifert invariant

Let B_j be any curve in ∂T_j which is homologous in T_j to the orbit O_j . Then Q_j, H and M_j, B_j are both ordered pairs of conjugate curves in ∂T_j and both determine the same orientation of ∂T_j . Hence

(7.2) $$B_j \sim -\nu_j Q_j - \rho_j H$$

for some ν_j and ρ_j with

(7.3) $$\begin{vmatrix} \alpha_j & \beta_j \\ -\nu_j & -\rho_j \end{vmatrix} = +1 \quad .$$

(By replacing B_j by $B_j + sM_j$ for some s , one may assume $0 < \nu_j \leq \alpha_j$. Then ν_j and ρ_j have the same meaning as in §6.)

Solving (7.1) and (7.2) for Q_j and H gives

(7.4) $$Q_j \sim -\rho_j M_j - \beta_j B_j \quad ,$$

(7.5) $$H \sim \nu_j M_j + \alpha_j B_j \quad .$$

Now $M_j \sim 0$ and $B_j \sim O_j$ in T_j , so in T_j

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(7.6) $$Q_j \sim -\beta_j O_j \quad ,$$

(7.7) $$H \sim \alpha_j O_j \quad .$$

(7.6) and (7.7) can be used as a definition of α_j and β_j . Geometrically (7.7) says that the isotropy subgroup of the orbit O_j is $\mathbb{Z}_{\alpha_j}$, and (7.6) says that Q_j winds β_j times in the reverse direction around the tube T_j .

Theorem 7.1. Let X, R, $O_1, \dots, O_n$ be as at the beginning of this section, and let $\divideontimes = ((g),(\alpha_1,\beta_1), \dots ,(\alpha_n,\beta_n))$, where g is the genus of the orbit space X^*, and t[illegible] (α_i,β_i) are the Seifert invariants of the orbits O_i with respect to R . Then X is equivariantly diffeomorphic to the standard action $X(\divideontimes)$.

Proof. The proof that X is equivariantly homeomorphic to $X(\divideontimes)$ is the same proof as for the Seifert-Raymond classification theorem ([15] Satz 5, [14] Corollary 2b, [11] Theorem 2). By theorem 6 of [14] , two differentiable 3-dimensional S^1-manifolds which are equivariantly homeomorphic are equivariantly diffeomorphic. Hence X is equivariantly diffeomorphic to $X(\divideontimes)$. ||

We now investigate under what conditions two standard actions $X(\divideontimes)$ and $X(\divideontimes')$ are equivariantly diffeomorphic. For brevity define

Definition. $\divideontimes \sim \divideontimes'$ ($\divideontimes$ is equivalent to $\divideontimes'$) means $X(\divideontimes)$ is equivariantly diffeomorphic to $X(\divideontimes')$. More generally define

$$((\alpha_1,\beta_1), \dots ,(\alpha_k,\beta_k)) \sim ((\alpha_1',\beta_1'), \dots ,(\alpha_l',\beta_l'))$$

if always

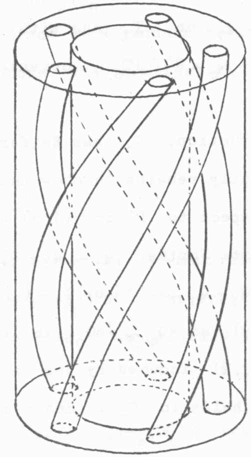
$$((g),(\alpha_1,\beta_1),\ \dots\ ,(\alpha_k,\beta_k),(\alpha_{k+1},\beta_{k+1}),\ \dots\ ,(\alpha_n,\beta_n)) \sim$$
$$\sim ((g),(\alpha'_1,\beta'_1),\ \dots\ ,(\alpha'_l,\beta'_l),(\alpha_{k+1},\beta_{k+1}),\ \dots\ ,(\alpha_n,\beta_n))$$

Lemma 7.2. $((1,b),(\alpha,\beta)) \sim (\alpha,\beta+b\alpha)$ for any $b \in \mathbb{Z}$.

Proof. Let X be an oriented 3-dimensional S^1-manifold, and $R \subset X$ a section to the S^1-action outside the interiors of a set of small invariant tubular neighbourhoods $T_1, \dots, T_n$ of orbits $O_1, \dots, O_n$ of X, as described at the beginning of this section. Assume O_1 has Seifert invariant $(\alpha_1,\beta_1) = (\alpha,\beta+b\alpha)$ with respect to R. It suffices to show that we can find a principal orbit O_o near O_1, small invariant tubular neighbourhoods T_o and T'_1 of O_o and O_1, and a section R' to the S^1-action on $X - (\mathring{T}_o \cup \mathring{T}'_1 \cup \mathring{T}_2 \cup \dots \cup \mathring{T}_n)$ such that R' and R coincide outside a small neighbourhood of O_1, and the Seifert invariants of O_o and O_1 with respect to R' are $(1,b)$ and (α,β) respectively.

By §6 we may assume that T_1 is parametrised by $(re^{i\theta},e^{i\psi})$, $0 \le r \le 1$, with S^1-action $z(re^{i\theta},e^{i\psi}) = (z^\nu re^{i\theta},z^\alpha e^{i\psi})$, where $(\beta+b\alpha)\nu \equiv 1 \bmod \alpha$. Let $T'_1 = \{(re^{i\theta},e^{i\psi}) \in T_1 \mid 0 \le r \le \frac{1}{2}\}$. Let O_o be the orbit through the point $(\frac{3}{4},1) \in T_1$, and let T_o be the closed tubular neighbourhood of radius $\frac{1}{8}$ around O_o.

The diagram on the following page shows T_1, T'_1, and T_o for $\alpha = 5$, $\nu = 2$. The top and bottom of the cylinder are to be identified.



Let $Q_1 = R \cap \partial T_1$, oriented as a boundary component of R. By (7.6) the homology relation

(7.8) $\quad Q_1 \sim -(\beta+b\alpha)O_1$

holds in T_1, since $\beta_1 = \beta + b\alpha$. Also, since O_o is a principal orbit, and any two principal orbits in T_1 are homologous, (7.7) yields

(7.9) $\quad O_o \sim \alpha O_1$.

Let Q'_1 be a section to the S^1-action in $\partial T'_1$ which runs β times in the reverse direction around T'_1. That is, the relation

(7.10) $\quad Q'_1 \sim -\beta O_1$

holds in T'_1, and hence also in T_1.

Let S be a section to the S^1-action in $T_1 - (\mathring{T}_o \cup \mathring{T}'_1)$ which coincides with Q_1 on ∂T_1 and with Q'_1 on $\partial T'_1$, and put $R' = R \cup S$. Let $Q_o = R' \cap \partial T_o$, oriented as a boundary component of $-R'$.

By construction, O_1 has Seifert invariant (α, β) with respect to R', so it only remains to show that the Seifert invariant (α_o, β_o) of O_o with respect to R' is $(1,b)$. $\alpha_o = 1$ is clear, since O_o is a principal orbit. Further $Q_o \sim Q_1 - Q'_1$ in T_1, since $Q_1 - Q'_1 - Q_o = \partial S \sim 0$. But $Q_1 - Q'_1 \sim -(\beta + b\alpha)O_1 - (-\beta O_1) = -b\alpha O_1 \sim -bO_o$ by (7.8), (7.10), and (7.9), so $Q_o \sim -bO_o$ in T_1. Since the homomorphism $H_1(T_o,\mathbb{Z}) \longrightarrow H_1(T_1,\mathbb{Z})$ induced by the inclusion $T_o \subset T_1$ is injective, $Q_o \sim -bO_o$ also holds in T_o. Hence by (7.6) $\beta_o = b$. ||

Corollary 7.3. Let $\mathfrak{X} = ((g),(\alpha_1,\beta_1+b_1\alpha_1), \dots ,(\alpha_n,\beta_n+b_n\alpha_n),(1,b_{n+1}), \dots ,(1,b_{n+m}))$ with $0<\beta_j<\alpha_j$ for each $j=1,\dots,n$.
Then $X(\mathfrak{X})$ is equivariantly diffeomorphic to Raymond's standard action
$\{b;(o,g,0,0);(\alpha_1,\beta_1), \dots ,(\alpha_n,\beta_n)\}$ with
$b = b_1 + b_2 + \dots + b_{n+m}$.

Proof. Repeated application of lemma 7.2 gives that $\mathfrak{X} \sim ((g),(1,b_1), \dots ,(1,b_n),(1,b_{n+1}), \dots ,(1,b_{n+m}),(\alpha_1,\beta_1), \dots ,(\alpha_n,\beta_n)) \sim ((g),(1,b_1+b_2+\dots+b_{n+m}),(\alpha_1,\beta_1), \dots ,(\alpha_n,\beta_n))$, so the corollary now follows by lemma 6.1. ||

Corollary 7.4. If $X(\mathfrak{X}) \cong X(\mathfrak{X}')$, then $\mathfrak{X}$ can be transformed into $\mathfrak{X}'$ by transformations of the type given in lemma 7.2.

Proof. By the proof of corollary 7.3, both $\mathfrak{X}$ and $\mathfrak{X}'$ can be transformed into the standard form

$$((g),(1,b),(\alpha_1,\beta_1), \dots ,(\alpha_n,\beta_n))$$

with $0<\beta_j<\alpha_j$ for each j. Since by Raymond's classification theorem this standard form is unique, the corollary follows. ||

§8. Quotients by cyclic groups

Let X be a 3-dimensional effective S^1-manifold, and consider the restricted action of a subgroup $\mathbb{Z}_p \subset S^1$. The orbit space $X/\mathbb{Z}_p$ is again in a natural way an effective S^1-manifold. The following theorem will be useful later.

Theorem 8.1. If $X \cong X((g),(\alpha_1,\beta_1), \dots ,(\alpha_n,\beta_n))$ then $X/\mathbb{Z}_p \cong X((g),(\alpha'_1,\beta'_1), \dots ,(\alpha'_n,\beta'_n))$ where

$(\alpha'_j,\beta'_j) = \left(\frac{\alpha_j}{(\alpha_j,p)}, \frac{p\beta_j}{(\alpha_j,p)}\right)$ for each j. In particular if p is coprime to α_j then $(\alpha'_j,\beta'_j) = (\alpha_j,p\beta_j)$.

Proof. (Notation as at the beginning of §7). Denote the orbit map $X \longrightarrow X/\mathbb{Z}_p$ by π and denote $\pi X = X/\mathbb{Z}_p$ by X', πR by R', πO_j by O'_j, and so on. R' is a section to the S^1-action on $X' - (\mathring{T}'_1 \cup \dots \cup \mathring{T}'_n)$, and π maps R bijectively onto R'. Let (α'_j,β'_j) be the Seifert invariant of O'_j with respect to R' for each j.

One sees easily that the order of the isotropy subgroup at the orbit O'_j is $\frac{\alpha_j}{(\alpha_j,p)}$, so $\alpha'_j = \frac{\alpha_j}{(\alpha_j,p)}$ for each j. (7.6) and (7.7) give the following homology relations in T_j and T'_j:

(8.1) $H \sim \alpha_j O_j$,

(8.2) $H' \sim \frac{\alpha_j}{(\alpha_j, p)} O'_j$,

(8.3) $Q_j \sim -\beta_j O_j$,

(8.4) $Q'_j \sim -\beta'_j O'_j$.

Further, since π maps Q_j bijectively onto Q'_j , and maps H as a p-fold covering onto H',

(8.5) $\pi_* Q_j = Q'_j$,

(8.6) $\pi_* H = pH'$,

where π_* is the homology map induced by π.

Applying π_* to (8.1) and comparing with (8.2) gives

(8.7) $\pi_* O_j = \frac{p}{(\alpha_j, p)} O'_j$.

Now applying π_* to (8.3) and comparing with (8.4) gives

$$\beta'_j = \frac{p\beta_j}{(\alpha_j, p)} ,$$

as was to be proved. ||

Together with corollary 7.3, theorem 8.1 gives an algorithm for computing the Seifert-Raymond classification of $X/\mathbb{Z}_p$ from that of X.

Similarly, but much more trivially

(8.8) $-X((g),(\alpha_1,\beta_1), \ldots ,(\alpha_n,\beta_n)) = X((g),(\alpha_1,-\beta_1), \ldots ,(\alpha_n,-\beta_n))$,

(compare Seifert [15], Satz 6).

§9. The classification of $\Sigma(a_o,a_1,a_2)$

Let (a_o,a_1,a_2) be a triple of positive integers. Orient $\Sigma(a_o,a_1,a_2)$ as the boundary of the manifold $X^1(a) = \{(z_o,z_1,z_2) \in \mathbb{C}^3 \mid z_o^{a_o} + z_1^{a_1} + z_2^{a_2} = 0 ,\ 0 < \sum_j |z_j|^2 \leq 1 \}$. $X^1(a)$ has a canonical orientation induced by the complex structure in its interior. One can check that this orientation of $\Sigma(a)$ is compatible with the orientation of $V(a) = \Sigma(a)/S^1$ induced by the complex structure on $V(a)$ (modulo the orientation conventions of §5).

We recall the notation and facts of §3:

Notation:

(9.1) $d = \gcd(a_o,a_1,a_2)$,

(9.2) $a'_i = \mathrm{lcm}(a_o,a_1,a_2)/a_i$, $(i=0,1,2)$,

(9.3) $t_i = \gcd(a'_j,a'_k)$, $(\{i,j,k\}=\{0,1,2\})$,

(9.4) $s_i = \frac{1}{d}\gcd(a_j,a_k)$, $(\{i,j,k\}=\{0,1,2\})$.

Facts:

(9.5) $a_o = dt_os_1s_2$, $a_1 = ds_ot_1s_2$, $a_2 = ds_os_1t_2$,

(9.6) $a'_o = s_ot_1t_2$, $a'_1 = t_os_1t_2$, $a'_2 = t_ot_1s_2$,

(9.7) $\gcd(a'_o,a'_1,a'_2) = 1$.

The S^1-action on $\Sigma(a_o,a_1,a_2)$ is given by

$$e^{2\pi it}(z_o,z_1,z_2) = (e^{2\pi ia'_ot}z_o, e^{2\pi ia'_1t}z_1, e^{2\pi ia'_2t}z_2) .$$

The facts (9.5) and (9.6) follow by theorem 3.5 and subsequent comments, since $V(a)$ is a manifold; alternatively by elementary number theory.

By (9.7) we can find integers β_o, β_1, β_2 with

(9.8) $\beta_o a'_o + \beta_1 a'_1 + \beta_2 a'_2 = 1$.

Theorem 9.1. $\Sigma(a_o,a_1,a_2) \cong X((g), ds_o(t_o,\beta_o), ds_1(t_1,\beta_1), ds_2(t_2,\beta_2))$, where the β_i are as in (9.8), and

$$g = \frac{1}{2}(d^2 s_o s_1 s_2 - d(s_o + s_1 + s_2)) + 1 .$$

Here $ds_i(t_i,\beta_i)$ of course means $(t_i,\beta_i), \dots ,(t_i,\beta_i)$ ds_i times.

Corollary 9.2. Let $0 \leq \beta'_j < t_j$ with $\beta'_j a'_j \equiv 1 \bmod t_j$ for $j=0,1,2$. Let $g = \frac{1}{2}(d^2 s_o s_1 s_2 - d(s_o+s_1+s_2)) + 1$. Then in Raymond's notation ([11], [14])

1). If $t_o \neq 1$, $t_1 \neq 1$, $t_2 \neq 1$, then

$\Sigma(a) \cong \{b;(o,g,0,0); ds_o(t_o,\beta'_o), ds_1(t_1,\beta'_1), ds_2(t_2,\beta'_2)\}$,

with $b = d(1 - \beta'_o a'_o - \beta'_1 a'_1 - \beta'_2 a'_2)/t_o t_1 t_2$.

2). If $t_o \neq 1$, $t_1 \neq 1$, $t_2 = 1$, then

$\Sigma(a) \cong \{b;(o,g,0,0); ds_o(t_o,\beta'_o), ds_1(t_1,\beta'_1)\}$,

with $b = d(1 - \beta'_o a'_o - \beta'_1 a'_1)/t_o t_1$.

3). If $t_o \neq 1$, $t_1 = t_2 = 1$, then

$\Sigma(a) \cong \{b;(o,g,0,0); ds_o(t_o,\beta'_o)\}$,

with $b = d(1 - \beta'_o a'_o)/t_o$.

4). If $t_o = t_1 = t_2 = 1$, then

$\Sigma(a) \cong \{d;(o,g,0,0);\}$.

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Proof of corollary 9.2 (Notation as above). From (9.8) and (9.6) it follows that $\beta_j a'_j \equiv 1 \bmod t_j$ for each j . Hence $\beta_j \equiv \beta'_j \bmod t_j$ for each j , say $\beta_j = \beta'_j + k_j t_j$. Observe that $ds_o k_o + ds_1 k_1 + ds_2 k_2 = ds_o(\beta_o - \beta'_o)/t_o + ds_1(\beta_1 - \beta'_1)/t_1 + ds_2(\beta_2 - \beta'_2)/t_2 = d(s_o t_1 t_2(\beta_o - \beta'_o) + s_1 t_o t_2(\beta_1 - \beta'_1) + s_2 t_o t_1(\beta_2 - \beta'_2))/t_o t_1 t_2 = d(a'_o(\beta_o - \beta'_o) + a'_1(\beta_1 - \beta'_1) + a'_2(\beta_2 - \beta'_2))/t_o t_1 t_2 = d(1 - a'_o \beta'_o - a'_1 \beta'_1 - a'_2 \beta'_2)/t_o t_1 t_2$. Corollary 9.2 now follows from theorem 9.1 by corollary 7.3, and by observing that if $t_j = 1$, then $\beta'_j = 0$. ||

Proof of theorem 9.1.

Part I. Construction of a partial section.

The exceptional orbits of $\Sigma = \Sigma(a_o,a_1,a_2)$ are among the orbits with $z_o = 0$ or $z_1 = 0$ or $z_2 = 0$ (see lemma 3.1). For $z_o = 0$ the orbits have isotropy subgroup $\mathbb{Z}_{t_o}$. There are ds_o such orbits, for one verifies easily that two points $(0,z_1,z_2)$ and $(0,w_1,w_2)$ of Σ are on the same orbit if and only if $z_1^{t_1}/z_2^{t_2} = w_1^{t_1}/w_2^{t_2}$, and that the possible values of $z_1^{t_1}/z_2^{t_2}$ with $(0,z_1,z_2) \in \Sigma$ are just the ds_o-th roots of -1. . More generally for each j there are precisely ds_j orbits with $z_j = 0$, and they have isotropy subgroup $\mathbb{Z}_{t_j}$.

Let $\varepsilon > 0$ be small, and let

$$\Sigma_1 = \{(z_o,z_1,z_2) \in \Sigma \mid \ |z_o| \leq \varepsilon \ \text{ or } \ |z_1| \leq \varepsilon \ \text{ or } \ |z_2| \leq \varepsilon \} .$$

Σ_1 is the union of small closed invariant tubular neighbourhoods of the orbits mentioned above. Let

$$\Sigma_o = \Sigma - \Sigma_1 .$$

That is $\Sigma_o = \{(r_o e^{i\theta_o}, r_1 e^{i\theta_1}, r_2 e^{i\theta_2}) \in \Sigma \mid r_o \geq \varepsilon , r_1 \geq \varepsilon , r_2 \geq \varepsilon \}$.

Let

$$R = \{(r_0 e^{i\theta_0}, r_1 e^{i\theta_1}, r_2 e^{i\theta_2}) \in \Sigma_0 \mid \sum_0^2 \beta_j \theta_j \equiv 0 \mod 2\pi \},$$

where the β_j are as in (9.8). We claim that R is a section to the S^1-action in Σ_0.

Indeed, let $x = (r_0 e^{i\theta_0}, r_1 e^{i\theta_1}, r_2 e^{i\theta_2}) \in \Sigma_0$. Then the orbit through x is

$$S^1 x = \{(r_0 e^{i(\theta_0 + a_0'\psi)}, r_1 e^{i(\theta_1 + a_1'\psi)}, r_2 e^{i(\theta_2 + a_2'\psi)}) \mid 0 \le \psi < 2\pi \},$$

so

$$S^1 x \cap R = \{(r_0 e^{i(\theta_0 + a_0'\psi)}, r_1 e^{i(\theta_1 + a_1'\psi)}, r_2 e^{i(\theta_2 + a_2'\psi)}) \mid \sum_0^2 \beta_j(\theta_j + a_j'\psi) \equiv 0 \mod 2\pi \}.$$

Now $\sum_0^2 \beta_j(\theta_j + a_j'\psi) = \sum_0^2 \beta_j\theta_j + (\sum_0^2 \beta_j a_j')\psi = \sum_0^2 \beta_j \theta_j + \psi$ by (9.8), so $\sum_0^2 \beta_j(\theta_j + a_j'\psi) \equiv 0 \mod 2\pi$ if and only if $\psi \equiv -\sum_0^2 \beta_j\theta_j \mod 2\pi$. Hence $S^1x \cap R$ contains one and only one point, as was to be shown. In fact R is clearly even a smooth section.

Part II. Calculation of the Seifert invariants.

Let $x = (0, r_1 e^{i\theta_1}, r_2 e^{i\theta_2}) \in \Sigma$. The orbit through x is

$$S^1 x = \{(0, r_1 e^{i(\theta_1 + a_1'\psi)}, r_2 e^{i(\theta_2 + a_2'\psi)}) \mid 0 \le \psi < 2\pi \}.$$

As ψ runs from 0 to 2π, $(0, r_1 e^{i(\theta_1 + a_1'\psi)}, r_2 e^{i(\theta_2 + a_2'\psi)})$ runs around S^1x t_0 times, so

$$S^1 x = \{(0, r_1 e^{i(\theta_1 + a_1'\psi/t_0)}, r_2 e^{i(\theta_2 + a_2'\psi/t_0)}) \mid 0 \le \psi < 2\pi \}$$
$$= \{(0, r_1 e^{i(\theta_1 + s_1 t_2 \psi)}, r_2 e^{i(\theta_2 + t_1 s_2 \psi)}) \mid 0 \le \psi < 2\pi \},$$

parametrised as a circle by $0 \le \psi < 2\pi$. By replacing x by a different point on the orbit if necessary we may assume

(9.9) $\quad \beta_1\theta_1 + \beta_2\theta_2 \equiv 0 \mod 2\pi$.

Let T_x be the component of Σ_1 which contains S^1x. In first order approximation T_x is given by

$$T_x \doteq \{(\varepsilon r e^{i\theta}, r_1 e^{i(\theta_1 + s_1 t_2 \psi)}, r_2 e^{i(\theta_2 + t_1 s_2 \psi)}) \mid 0 \le \psi < 2\pi, \; 0 \le \theta < 2\pi, \; 0 \le r < 1 \}.$$

The map $\varphi: T_x \to D^2 \times S^1$ given by

$$\varphi(\varepsilon r e^{i\theta}, r_1 e^{i(\theta_1 + s_1 t_2 \psi)}, r_2 e^{i(\theta_2 + t_1 s_2 \psi)}) = (r e^{i\theta}, e^{i\psi})$$

is a diffeomorphism, and one can check that it preserves orientation.

Let $Q_x = T_x \cap R$. Then Q_x is given by

$$Q_x \doteq \{(\varepsilon r e^{i\theta}, r_1 e^{i(\theta_1 + s_1 t_2 \psi)}, r_2 e^{i(\theta_2 + t_1 s_2 \psi)}) \mid \beta_0\theta + \beta_1(\theta_1 + s_1 t_2 \psi) + \beta_2(\theta_2 + t_1 s_2 \psi) \equiv 0 \mod 2\pi \}.$$

If one defines $\rho_0 = -\beta_1 s_1 t_2 - \beta_2 t_1 s_2$, then by (9.9) $\beta_0\theta + \beta_1(\theta_1 + s_1 t_2\psi) + \beta_2(\theta_2 + t_1 s_2 \psi) \equiv 0 \mod 2\pi$ can be written as $\beta_0\theta - \rho_0\psi \equiv 0 \mod 2\pi$. Hence

$$\varphi Q_x = \{(e^{i\theta}, e^{i\psi}) \mid \beta_0\theta - \rho_0\psi \equiv 0 \mod 2\pi \},$$

which can also be written

(9.10) $\quad \varphi Q_x = \{(e^{i\rho_0\theta}, e^{i\beta_0\theta}) \mid 0 \le \theta < 2\pi \}$.

The orbit H through the point $(\varepsilon, r_1 e^{i\theta_1}, r_2 e^{i\theta_2})$ is

$$H = \{(\varepsilon e^{ia_0'\psi}, r_1 e^{i(\theta_1 + a_1'\psi)}, r_2 e^{i(\theta_2 + a_2'\psi)}) \mid 0 \le \psi < 2\pi \}$$
$$= \{(\varepsilon e^{ia_0'\psi}, r_1 e^{i(\theta_1 + s_1 t_2 t_0 \psi)}, r_2 e^{i(\theta_2 + s_2 t_1 t_0 \psi)}) \mid 0 \le \psi < 2\pi \},$$

which is mapped by φ onto

(9.11) $\quad \varphi H = \{(e^{ia_0'\psi}, e^{it_0\psi}) \mid 0 \le \psi < 2\pi \}$.

Note that $\beta_0 a_0' - \rho_0 t_0 = \beta_0 a_0' + (\beta_1 s_1 t_2 + \beta_2 t_1 s_2) t_0 = \beta_0 a_0' + \beta_1 a_1' + \beta_2 a_2'$, so by (9.8)

(9.12) $\beta_o a_o' - \rho_o t_o = 1$.

Let M be the boundary of a slice in T_x, say φM equal to the curve $(e^{i\theta}, e^{i\psi_o})$, ψ_o fixed, in $\partial(D^2 \times S^1)$. Then (9.10), (9.11), and (9.12) imply the following homology relation in ∂T_x :

$$M \sim \pm t_o Q_x + \beta_o H ,$$

where the sign depends on the orientation of Q_x. Comparing with (7.1) shows that the sign should be positive, and that the Seifert invariant of the orbit $S^1 x$ with respect to R is (t_o, β_o). Similarly also for $j=1,2$, the Seifert invariant of each of the ds_j orbits with $z_j = 0$ is (t_j, β_j) with respect to R.

Theorem 7.1 now shows that

(9.13) $\Sigma \cong X((g), ds_o(t_o, \beta_o), ds_1(t_1, \beta_1), ds_2(t_2, \beta_2))$,

where only the genus g of $\Sigma/S^1 = V(a_o, a_1, a_2)$ remains to be determined.

<u>Part III.</u> Determination of genus $V(a_o, a_1, a_2)$.

We calculate the homology of Σ in two different ways. Comparing will give the value of g .

<u>§10. Homology of certain manifolds</u>

In this section all homology is with integer coefficients.

<u>Homology of $\Sigma(a_o, \ldots, a_n)$, $n \geq 2$.</u>

We summarise some of the results of Brieskorn [2]. Brieskorn there assumes that $a_i > 1$ for each i, but what we need is valid without this restriction.

Let J_a be the group ring of the group $G_a = Z_{a_o} \times \ldots \times Z_{a_n}$, and I_a the annulator ideal of the element $(1-w_o)(1-w_1)\cdots(1-w_n) \in J_a$ (w_i a generator of $\mathbb{Z}_{a_i}$). As a $\mathbb{Z}$-module J_a/I_a is free of rank $\prod_i (a_i - 1)$.

Let $w = w_o w_1 \cdots w_n$ and let

$$1-w : J_a/I_a \rightarrow J_a/I_a$$

be multiplication with $1-w$. Then ([2] §2)

$$H_i(\Sigma(a)) = 0 \quad \text{for } i \neq 0, n-1, n, 2n-1 ;$$

$$H_{n-1}(\Sigma(a)) = \text{cokern } 1-w ;$$

$$H_n(\Sigma(a)) = \text{kern } 1-w .$$

In the proof of lemma 4 of [2], Brieskorn shows that the eigenvalues of the endomorphism $1-w$ of J_a/I_a are the numbers

$$1 - \zeta_o^{i_o} \zeta_1^{i_1} \cdots \zeta_n^{i_n} , \quad 0 < i_k < a_k ,$$

where $\zeta_k = e^{2\pi i/a_k}$. A simple counting argument shows that precisely $\varkappa$ of these eigenvalues are equal to zero, where

$$\varkappa = \frac{a_o \cdots a_n}{\text{lcm}(a_o, \ldots, a_n)} - \sum_i \frac{a_o \cdots \hat{a}_i \cdots a_n}{\text{lcm}(a_o, \ldots, \hat{a}_i, \ldots, a_n)} +$$
$$\sum_{i<j} \frac{a_o \cdots \hat{a}_i \cdots \hat{a}_j \cdots a_n}{\text{lcm}(a_o, \ldots, a_i, \ldots, a_j, \ldots, a_n)} + \ldots + (-1)^n \sum_i \frac{a_i}{a_i} +$$
$$(-1)^{n+1} .$$

Hence

$$H_{n-1}(\Sigma(a)) = \varkappa\mathbb{Z} \oplus \text{Torsion}$$

$$H_n(\Sigma(a)) = \varkappa\mathbb{Z} ,$$

where $\varkappa\mathbb{Z}$ means the sum of $\varkappa$ copies of $\mathbb{Z}$.

For $n=2$ we have in the notation of (9.1) to (9.4) that $a_o a_1 a_2/\mathrm{lcm}(a_o,a_1,a_2) = d^2 s_o s_1 s_2$ and $a_j a_k/\mathrm{lcm}(a_j,a_k) = ds_i$ $(\{i,j,k\} = \{0,1,2\})$, so

$$\varkappa = d^2 s_o s_1 s_2 - (ds_o + ds_1 + ds_2) + 3 - 1 .$$

That is

(10.1) $$H_1(\Sigma(a_o,a_1,a_2)) = (d^2 s_o s_1 s_2 - d(s_o+s_1+s_2) + 2)\,\mathbb{Z} \oplus \text{Torsion} .$$

Homology of $X((g),(\alpha_1,\beta_1), \ldots ,(\alpha_n,\beta_n))$.

Let $X = X((g),(\alpha_1,\beta_1), \ldots ,(\alpha_n,\beta_n))$. An elementary application of the Van Kampen theorem to the construction (§6) of X gives that the fundamental group of X has a presentation

(10.2) $$\pi_1(X) = \langle a_i,b_i,q_j,h \mid \pi_*,[a_i,h],[b_i,h],[q_j,h],q_j^{\alpha_j} h_j^{\beta_j} \rangle$$

where $i=1,2,\ldots,g$; $j=1,\ldots,n$; and $\pi_* = q_1 \cdots q_n [a_1,b_1] \cdots [a_g,b_g]$ (see also Seifert [15] §10). Here the letters are chosen to suggest the geometric interpretation of the generators. The a_i and b_i come from the fundamental group of the surface $X^* = X/S^1$.

Abelianising gives that $H_1(X)$ has the abelian presentation

$$H_1(X) = \langle a_i,b_i,q_j,h \mid q_1 + \cdots + q_n = 0,\ \alpha_j q_j + \beta_j h = 0 \rangle .$$

(This could also be obtained directly by the Mayer-Vietoris sequence).

Thus

(10.3) $$H_1(X) = 2g\mathbb{Z} + H_1'$$

where

(10.4) $$H_1' = \langle q_j,h \mid q_1 + \cdots + q_n = 0,\ \alpha_j q_j + \beta_j h = 0 \rangle .$$

The relation matrix of H_1' is

$$A = \begin{pmatrix} 1 & 1 & \cdots & 1 & 0 \\ \alpha_1 & 0 & \cdots & 0 & \beta_1 \\ 0 & \alpha_2 & \cdots & 0 & \beta_2 \\ \vdots & & & & \vdots \\ 0 & 0 & \cdots & \alpha_n & \beta_n \end{pmatrix} ,$$

which has determinant (expanding by the first row)

(10.5) $$|A| = (-1)^{n-1} \sum_i \beta_i \alpha_1 \alpha_2 \cdots \hat{\alpha}_i \cdots \alpha_n .$$

Now let $X = \Sigma(a_o,a_1,a_2)$. By (9.13)

$$X \cong X((g), ds_o(t_o,\beta_o), ds_1(t_1,\beta_1), ds_2(t_2,\beta_2)) ,$$

so in this case

(10.6) $$A = \begin{pmatrix} 1 & 1 & \cdots & & \cdots & & \cdots & 1 & 0 \\ t_o & 0 & & & & & & 0 & \beta_o \\ 0 & t_o & & & & & & 0 & \beta_o \\ & & \ddots & & & & & & \vdots \\ & & & t_o & \cdots & & \cdots & 0 & \beta_o \\ & & & & t_1 & \cdots & \cdots & 0 & \beta_1 \\ \vdots & & & & & \ddots & & & \vdots \\ & & & & & t_1 & \cdots & 0 & \beta_1 \\ & & & & & & t_2 & \cdots 0 & \beta_2 \\ & & & & & & & \ddots & \vdots \\ 0 & 0 & & \cdots & & & & t_2 & \beta_2 \end{pmatrix} ,$$

and by (10.5), (9.2), and (9.8)

$$\begin{aligned} |A| &= \pm(ds_o(\beta_o t_o^{ds_o-1} t_1^{ds_1} t_2^{ds_2}) + ds_1(\beta_1 t_o^{ds_o} t_1^{ds_1-1} t_2^{ds_2}) + \\ &\quad ds_2(\beta_2 t_o^{ds_o} t_1^{ds_1} t_2^{ds_2-1})) \\ &= \pm d t_o^{ds_o-1} t_1^{ds_1-1} t_2^{ds_2-1}(s_o\beta_o t_1 t_2 + s_1\beta_1 t_o t_2 + s_2\beta_2 t_o t_1) \\ &= \pm d t_o^{ds_o-1} t_1^{ds_1-1} t_2^{ds_2-1}(\beta_o a_o' + \beta_1 a_1' + \beta_2 a_2') \\ &= \pm d t_o^{ds_o-1} t_1^{ds_1-1} t_2^{ds_2-1} . \end{aligned}$$

This is not equal to zero, so H_1' is a torsion group, so by (10.3)

$$H_1(\Sigma(a_o,a_1,a_2)) = 2g\mathbb{Z} \oplus \text{Torsion} \quad .$$

Comparing with (10.1) gives

$$g = \frac{1}{2}(d^2 s_o s_1 s_2 - d(s_o+s_1+s_2) + 2) \quad ,$$

as was to be shown. This completes the proof of theorem 9.1. ||

Remark. By reducing the matrix (10.6) to diagonal form one can calculate $H_1(\Sigma(a_o,a_1,a_2))$ explicitely in normal form. Assuming without loss of generality that $ds_o \geq ds_1 \geq ds_2$, the result is:

	$H_1(\Sigma(a_o,a_1,a_2),\mathbb{Z})$
$ds_o = ds_1 = ds_2 = 1$	$2g\mathbb{Z}$
$ds_o > ds_1 = ds_2 = 1$	$2g\mathbb{Z} \oplus (ds_o - 1)\mathbb{Z}_{t_o}$
$ds_o \geq ds_1 > ds_2 = 1$	$2g\mathbb{Z} \oplus (ds_o - ds_1)\mathbb{Z}_{t_o} \oplus (ds_1 - 1)\mathbb{Z}_{t_o t_1}$
$ds_o \geq ds_1 \geq ds_2 > 1$	$2g\mathbb{Z} \oplus (ds_o - ds_1)\mathbb{Z}_{t_o} \oplus (ds_1 - ds_2)\mathbb{Z}_{t_o t_1} \oplus (ds_2 - 2)\mathbb{Z}_{t_o t_1 t_2} \oplus \mathbb{Z}_{t_o t_1 t_2 d}$.

Institut für Angewandte Mathematik
der Universität Bonn

Chapter III : Involutions and the Hirzebruch invariant

§11. Introduction

Let X be a closed oriented differentiable $(4k-1)$-dimensional manifold, and $J : X \to X$ an orientation preserving fixpoint free differentiable involution on X . Hirzebruch [6] has defined an invariant $\alpha(X,J)$ using a special case of the Atiyah - Bott - Singer fixed point theorem. In [7] Hirzebruch and Jänich prove an alternative definition, which we shall use here. It coincides with the definition of the Browder - Livesay invariant [5] in the case that X is a homotopy sphere.

Let W be a characteristic submanifold for (X,J) . That is, W is the boundary ∂A of a compact submanifold A of X with $A \cup JA = X$ and $A \cap JA = W$. It is well known that characteristic submanifolds always exist. W is a $(2k-2)$-dimensional invariant submanifold of X , and J is orientation reversing on W , since $JW = J\partial A = \partial JA = -\partial A = -W$.

Let

(11.1) $\mathcal{L} = \mathrm{kern}(i_* : H_{2k-1}(W,\mathbb{Q}) \to H_{2k-1}(A,\mathbb{Q}))$,

where $i : W \subset A$ is the inclusion. Define a quadratic form f on $\mathcal{L}$ by

(11.2) $f : (x,y) \mapsto x \circ Jy$.

f is symmetric, since $y \circ Jx = -J(y \circ Jx) = -Jy \circ x = -(-1)^{2k-1} x \circ Jy = x \circ Jy$. Denote by $\tau(f)$ the signature of f (number of positive minus number of negative eigenvalues of f); it depends only on X and J , and not on the choice of W (Hirzebruch and Jänich loc.cit.) .

Definition. $\tau(f)$ is called the Hirzebruch invariant of (X,J), and is denoted by $\alpha(X,J)$.

Observe that $\alpha(-X,J) = -\alpha(X,J)$.

Let X be an oriented closed differentiable 3-dimensional S^1-manifold. Then $-1 \in S^1$ is an orientation preserving involution on X, and is fixpoint free if and only if the S^1-action has no fixpoints and no isotropy subgroups of even order. That is

$$X \cong X((g),(\alpha_1,\beta_1), \ldots ,(\alpha_n,\beta_n))$$

with α_j odd for each j. To avoid confusion later, we denote this involution $-1 \in S^1$ by J. In this chapter we shall calculate $\alpha(X,J)$

§12. The main theorem

If (α,β) is a pair of coprime integers with $\alpha > 0$, we define an integer $c(\alpha,\beta)$ as follows.

If $\beta \neq 0$, then $\frac{\alpha}{\beta}$ has a unique continued fraction representation of the form

$$\frac{\alpha}{\beta} = \pm k_0 + \cfrac{(-1)^{\varepsilon_1}}{l_1 + \cfrac{1}{k_1 + \cfrac{(-1)^{\varepsilon_2}}{l_2 + \cdots + \cfrac{(-1)^{\varepsilon_s}}{l_s + \cfrac{1}{k_s}}}}} \tag{12.1}$$

with $k_0 \geq 0$; $k_i > 0$ and $l_i > 0$ $(i=1,\ldots,s)$; and k_i even for $i=0,\ldots,s-1$. k_s need not be even.

Definition. If $\beta \neq 0$ and $\frac{\alpha}{\beta}$ is as in (12.1), then

$$c(\alpha,\beta) := (-1)^{\varepsilon_1} l_1 + (-1)^{\varepsilon_1+\varepsilon_2} l_2 + \ldots + (-1)^{\varepsilon_1+\cdots+\varepsilon_s} l_s . \tag{12.2}$$

If $\beta = 0$, then $\alpha = 1$, and define

$$c(1,0) := 0 . \tag{12.3}$$

Recall that if x is a real number, $\operatorname{sign} x$ is defined by

$$\operatorname{sign} x = \begin{cases} 0 & \text{if } x=0 \\ \frac{x}{|x|} & \text{if } x \neq 0 . \end{cases}$$

Theorem 12.1. Let $X = X((g),(\alpha_1,\beta_1), \ldots ,(\alpha_n,\beta_n))$ with all α_i odd, and let J be the involution contained in the S^1-action on X. Then

$$\alpha(X,J) = \sum_{j=1}^{n} (c(\alpha_j,\beta_j) + \operatorname{sign} \beta_j) - \operatorname{sign} \sum_{j=1}^{n} \frac{\beta_j}{\alpha_j} .$$

The next 3 sections will be devoted to proving this theorem. We first note a corollary.

If (a_0,a_1,a_2) is a triple of positive integers, let d, a_j', t_j, and s_j be defined as in (9.1) to (9.4). The involution J contained in the S^1-action on $\Sigma(a_0,a_1,a_2)$ is fixpoint free if and only if t_0, t_1, and t_2 are odd. Observe that in this case $(-1)^{a_j'} = (-1)^{s_j}$, so the involution J is given by

$$J : (z_0,z_1,z_2) \mapsto ((-1)^{s_0} z_0,(-1)^{s_1} z_1,(-1)^{s_2} z_2) . \tag{12.4}$$

Let β_0, β_1, β_2 be chosen to satisfy (9.8).

Corollary 12.2. Let (a_o,a_1,a_2) be a triple of positive integers such that t_o, t_1, and t_2 are odd. Then

$$\alpha(\Sigma(a_o,a_1,a_2),J) = \sum_{o}^{2} ds_j(c(t_j,\beta_j) + \operatorname{sign}\beta_j) - 1 \quad .$$

Proof. This follows immediately from theorems 9.1 and 12.1, when one notes that the term $\sum_{j=1}^{n}\frac{\beta_j}{\alpha_j}$ occuring in theorem 12.1 is in this case

$$ds_o\frac{\beta_o}{t_o} + ds_1\frac{\beta_1}{t_1} + ds_2\frac{\beta_2}{t_2} = \frac{d}{t_o t_1 t_2}(s_o t_1 t_2 \beta_o + s_1 t_o t_2 \beta_1 + s_2 t_o t_1 \beta_2) =$$

$$\frac{d}{t_o t_1 t_2}(a_o'\beta_o + a_1'\beta_1 + a_2'\beta_2) = \frac{d}{t_o t_1 t_2} \quad ,$$ which has signum +1. ||

Remark. The involution J considered here is not always the same as the involution $T_a : (z_o,z_1,z_2) \mapsto (-z_o,-z_1,-z_2)$ (defined on $\Sigma(a_o,a_1,a_2)$ whenever $a_o \equiv a_1 \equiv a_2 \mod 2$), considered by Hirzebruch and Jänich in [7]. See also §16.

§13. Reduction of the main theorem. Properties of $c(\alpha,\beta)$

Since Seifert pairs of the form (1,0) can by lemma 7.2 be dropped from any tuple $\mathfrak{X} = ((g),(\alpha_1,\beta_1), \dots ,(\alpha_n,\beta_n))$ without changing the manifold $X(\mathfrak{X})$, and such pairs also contribute nothing to the formula for $\alpha(X,J)$ in theorem 12.1, we loose no generality by making the following general assumption

Assumption: In any Seifert pair (α,β) we assume $\beta \neq 0$.

In this section we show how one can deduce theorem 12.1 from the following apparently weaker lemma. The lemma will be proved in the following 2 sections.

Lemma 13.1. There exists an integer valued function c defined on pairs (α,β) of coprime integers with $\beta \geq -1$, $\beta \neq 0$, $\alpha > 0$, α odd, with the following properties:

(i). If $X = X((g),(\alpha_1,\beta_1), \dots ,(\alpha_n,\beta_n))$ with $\beta_j \geq -1$, $\alpha_j > 0$, α_j odd for all j, and $(\alpha_j,\beta_j) = (1,1)$ for some j; then

$$\alpha(X,J) = \sum_{j=1}^{n}(c(\alpha_j,\beta_j) + \operatorname{sign}\beta_j) - \operatorname{sign}\sum_{j=1}^{n}\frac{\beta_j}{\alpha_j} \quad .$$

(ii). $c(\alpha,\pm 1) = 0$.

(iii). $c(2k\beta+\alpha,\beta) = c(\alpha,\beta)$ for $k \in \mathbb{Z}$, $2k\beta+\alpha > 0$.

Proof of theorem 12.1 modulo lemma 13.1.

Let c be the function of lemma 13.1. Extend the definition of c to allow $\beta < -1$ as follows: if (α,β) is a pair of coprime integers with $\beta < -1$, $\alpha > 0$, α odd, let b be the smallest integer such that $\beta + b\alpha > 0$ and define

(13.1) $\quad c(\alpha,\beta) := c(\alpha,\beta+b\alpha) - b + 2$.

We first prove the formula of theorem 12.1. Let $X = X((g),(\alpha_1,\beta_1), \dots ,(\alpha_n,\beta_n))$ with all α_j odd. Without loss of generality $\beta_j < -1$ for $j=1,\dots,m$ and $\beta_j \geq -1$ for $j=m+1,\dots,n$. For each $j=1,\dots,m$ let b_j be the smallest integer with $\beta_j + b_j\alpha_j > 0$. Then by repeated application of lemma 7.2

$$X \cong X((g),(b+1)(1,-1),(1,1),(\alpha_1,\beta_1+b_1\alpha_1), \dots ,(\alpha_m,\beta_m+b_m\alpha_m), (\alpha_{m+1},\beta_{m+1}), \dots ,(\alpha_n,\beta_n)) ,$$

where $b = b_1 + \dots + b_m$, and $(b+1)(1,-1)$ means $(1,-1),\dots,(1,-1)$ b+1 times.

We can now apply lemma 13.1 (i), giving

$$\alpha(X,J) = (b+1)(c(1,-1) + \operatorname{sign} -1) + c(1,1) + \operatorname{sign} 1 +$$

(13.2) $$\sum_{j=1}^{m}(c(\alpha_j,\beta_j+b_j\alpha_j) + \operatorname{sign}(\beta_j+b_j\alpha_j)) + \sum_{j=m+1}^{n}(c(\alpha_j,\beta_j) + \operatorname{sign}\beta_j) - \operatorname{sign}\Big(-(b+1) + 1 + \sum_{1}^{m}\frac{\beta_j+b_j\alpha_j}{\alpha_j} + \sum_{m+1}^{n}\frac{\beta_j}{\alpha_j}\Big) \quad .$$

Using (13.1), part (ii) of lemma 13.1, and the facts that $\operatorname{sign}\beta_j = -1$ and $\operatorname{sign}(\beta_j+b_j\alpha_j) = 1$ for $j=1,\dots,m$, (13.2) simplifies to

$$\alpha(X,J) = -b + \sum_{1}^{m}(c(\alpha_j,\beta_j) + b_j + \operatorname{sign}\beta_j) + \sum_{m+1}^{n}(c(\alpha_j,\beta_j) + \operatorname{sign}\beta_j) - \operatorname{sign}\Big(-b + \sum_{1}^{m}\big(\frac{\beta_j}{\alpha_j} + b_j\big) + \sum_{m+1}^{n}\frac{\beta_j}{\alpha_j}\Big)$$

(13.3) $$= \sum_{j=1}^{n}(c(\alpha_j,\beta_j) + \operatorname{sign}\beta_j) - \operatorname{sign}\sum_{j=1}^{n}\frac{\beta_j}{\alpha_j} \; ,$$

proving the formula of theorem 12.1.

It hence only remains to prove that $c(\alpha,\beta)$ is defined as in §12. We need the following lemma.

Lemma 13.2. Properties of $c(\alpha,\beta)$. Assume $\alpha > 0$; $k,l \in \mathbb{Z}$.

(i). $c(\alpha,\pm 1) = 0$,

(ii). $c(\alpha,-\beta) = -c(\alpha,\beta)$,

(iii). $c(\alpha,\beta+l\alpha) = c(\alpha,\beta) + l$ if $\operatorname{sign}\beta = \operatorname{sign}(\beta+l\alpha)$,

(iv). $c(\alpha,\beta+l\alpha) = c(\alpha,\beta) + l - 2$ if $\beta + l\alpha > 0 > \beta$,

(v). $c(2k\beta+\alpha,\beta) = c(\alpha,\beta)$ if $2k\beta+\alpha > 0$,

(vi). $c(2k\beta-\alpha,\beta) = -c(\alpha,\beta)$ if $2k\beta-\alpha > 0$.

Proof.

(i) holds by lemma 13.1.

(ii): By (8.8) $X((g),(\alpha,-\beta)) \cong -X((g),(\alpha,\beta))$, so applying (13.3) gives $c(\alpha,-\beta) + \operatorname{sign}(-\beta) - \operatorname{sign}\frac{-\beta}{\alpha} = -(c(\alpha,\beta) + \operatorname{sign}\beta - \operatorname{sign}\frac{\beta}{\alpha})$,

whence $c(\alpha,-\beta) = -c(\alpha,\beta)$.

(iii) and (iv): Assume first $l \geq 0$. By lemma 7.2 $X((g),(\alpha,\beta+l\alpha)) \cong X((g),l(1,1),(\alpha,\beta))$, so applying (13.3) gives

$$c(\alpha,\beta+l\alpha) + \operatorname{sign}(\beta+l\alpha) - \operatorname{sign}\frac{\beta+l\alpha}{\alpha} = l(c(1,1) + \operatorname{sign}1) + c(\alpha,\beta) + \operatorname{sign}\beta - \operatorname{sign}(1+\frac{\beta}{\alpha}) \; .$$

Since $c(1,1) = 0$, this simplifies to

(13.4) $$c(\alpha,\beta+l\alpha) = c(\alpha,\beta) + l + \operatorname{sign}\beta - \operatorname{sign}(\beta+l\alpha) \; .$$

For $l \leq 0$, applying (13.3) to $X((g),(\alpha,\beta+l\alpha)) \cong X((g),(-l)(1,-1),(\alpha,\beta))$ also yields (13.4). Properties (iii) and (iv) are special cases of (13.4).

(v) holds by lemma 13.1.

(vi): This is trivial for $|\beta| \leq 1$, so assume $|\beta| > 1$.

We prove (vi) first for $k > 0$. Then since $2k\beta-\alpha$ and α are positive, we must have $\beta > 0$. Assume first that $k = 1$. Then

$$\begin{aligned}
c(2\beta-\alpha,\beta) &= c(2\beta-\alpha,(\alpha-\beta)+(2\beta-\alpha)) \\
&= c(2\beta-\alpha,\alpha-\beta) + 1 + \operatorname{sign}(\alpha-\beta) - \operatorname{sign}\beta && \text{(by (13.4)} \\
&= c(2\beta-\alpha,\alpha-\beta) + \operatorname{sign}(\alpha-\beta) && \text{(as } \beta > 0 \\
&= c(-2(\alpha-\beta)+\alpha,\alpha-\beta) + \operatorname{sign}(\alpha-\beta) \\
&= c(\alpha,\alpha-\beta) + \operatorname{sign}(\alpha-\beta) && \text{(by (v))} \\
&= c(\alpha,-\beta) + 1 + \operatorname{sign}(-\beta) - \operatorname{sign}(\alpha-\beta) + \operatorname{sign}(\alpha-\beta) && \text{(by (13.4)} \\
&= c(\alpha,-\beta) && \text{(as } -\beta < 0 \\
&= -c(\alpha,\beta) && \text{(by (ii))}
\end{aligned}$$

Now if $k > 1$, then since $-\alpha < 0 < 2k\beta-\alpha$, there exists an integer m with $0 \leq m < k$ such that $2m\beta-\alpha < 0 < 2(m+1)\beta-\alpha$. Then

$$\begin{aligned}
c(2k\beta-\alpha,\beta) &= c(2(k-m-1)\beta+2(m+1)\beta-\alpha,\beta) \\
&= c(2(m+1)\beta-\alpha,\beta) && \text{(by (v))} \\
&= c(2\beta-(\alpha-2m\beta),\beta)
\end{aligned}$$

$$\begin{aligned}
&= -c(\alpha-2m\beta,\beta) && \text{(by (vi) with } k=1\text{)}\\
&= c(\alpha-2m\beta,-\beta) && \text{(by (ii))}\\
&= c(\alpha,-\beta) && \text{(by (v))}\\
&= -c(\alpha,\beta) && \text{(by (ii))}\ .
\end{aligned}$$

This proves (vi) for $k>0$. One can now deduce it for $k<0$ by replacing α by $\alpha'=2k\beta-\alpha$, k by $k'=-k$, and β by $\beta'=-\beta$ in (vi). ||

We now show that $c(\alpha,\beta)$ is as defined in §12. For $\beta=0$ there is nothing to prove. Also, since the function $c(\alpha,\beta)$ defined in §12 clearly satisfies $c(\alpha,-\beta)=-c(\alpha,\beta)$, it suffices to consider only $\beta>0$.

Let (α,β) be a coprime pair with $\beta>0$, $\alpha>0$, α odd. Then setting $\alpha_o=\alpha$, $\beta_o=\beta$, a modification of the euclidean algorithm gives

$$\begin{aligned}
\alpha_o &= 2k_o'\beta_o+(-1)^{\varepsilon_1}\alpha_1 , && 0<\alpha_1<\beta_o , && k_o'\geq 0 ;\\
\beta_o &= l_1\alpha_1+\beta_1 , && 0<\beta_1<\alpha_1 , && l_1>0 ;\\
\alpha_1 &= 2k_1'\beta_1+(-1)^{\varepsilon_2}\alpha_2 , && 0<\alpha_2<\beta_1 , && k_1'>0 ;\\
&\ldots && \ldots && \ldots\\
\alpha_{s-1} &= 2k_{s-1}'\beta_{s-1}+(-1)^{\varepsilon_s}\alpha_s , && 0<\alpha_s<\beta_{s-1} , && k_{s-1}'>0 ;\\
\beta_{s-1} &= l_s\alpha_s+\beta_s , && 0<\beta_s\leq\alpha_s , && l_s>0 ;\\
\alpha_s &= k_s'\beta_s , && && k_s'>0 .
\end{aligned}$$

If one defines $k_i=2k_i'$ for $i=0,\dots,s-1$, and $k_s=k_s'$, then $\frac{\alpha}{\beta}$ is given by the continued fraction of (12.1).

Observe that as in the usual euclidean algorithm, β_s is the greatest common divisor of α_o and β_o, and hence is equal to 1.

Thus $c(\alpha_s,\beta_s)=0$. Furthermore for $i=1,\dots,s$, properties (iii), (v), and (vi) of lemma 13.2 give that

$$\begin{aligned}
c(\alpha_{i-1},\beta_{i-1}) &= c(2k_{i-1}'\beta_{i-1}+(-1)^{\varepsilon_i}\alpha_i,\beta_{i-1})\\
&= (-1)^{\varepsilon_i}c(\alpha_i,\beta_{i-1})\\
&= (-1)^{\varepsilon_i}c(\alpha_i,l_i\alpha_i+\beta_i)\\
&= (-1)^{\varepsilon_i}(c(\alpha_i,\beta_i)+l_i)\ .
\end{aligned}$$

A trivial induction now gives

$$c(\alpha,\beta)=c(\alpha_o,\beta_o)=(-1)^{\varepsilon_1}l_1+(-1)^{\varepsilon_1+\varepsilon_2}l_2+\dots+(-1)^{\varepsilon_1+\dots+\varepsilon_s}l_s ,$$

as was to be proved. ||

The above proof shows that the properties (i), (ii), (iii), (v), and (vi) already suffice to define $c(\alpha,\beta)$ uniquely. In general it is quicker to calculate $c(\alpha,\beta)$ by means of the properties given in lemma 13.2, rather than the definition in §12. A table of values for $\alpha\leq 27$ and $\beta\leq 26$ is given in Appendix I.

§14. A characteristic submanifold for $(X(\mathfrak{X}),J)$

Throughout the following let

$$X = X((g),(\alpha_1,\beta_1),\ \dots\ ,(\alpha_n,\beta_n))$$

with α_j odd and $\beta_j\neq 0$ for each j. X is constructed as in §6 as

$$X = X_o\cup T_1\cup\ \dots\ \cup T_n\ .$$

Here $X_o=X_o^*\times S^1$, where X_o^* is a surface of genus g with n holes cut in it. The T_i are solid tori with S^1-actions as described in

(6.3), and they are sewn into X_o as described in §6.

If S^1 is parametrised by $e^{i\psi}$ define

$$S^+ = \{e^{i\psi} \mid 0 \le \psi \le \pi\} \subset S^1$$
$$S^- = \{e^{i\psi} \mid \pi \le \psi \le 2\pi\} \subset S^1 .$$

Let

$$A^+ = X_o^* \times S^+ \subset X_o$$
$$A^- = X_o^* \times S^- \subset X_o .$$

For each j we have, as in §6, that T_j is a copy of $D^2 \times S^1$. Let

$$T_j^+ = D^2 \times S^+ \subset T_j$$
$$T_j^- = D^2 \times S^- \subset T_j .$$

By (6.3) the involution J on T_j is given by

(14.1) $$J(re^{i\theta}, e^{i\psi}) = ((-1)^{\nu_j} re^{i\theta}, -e^{i\psi})$$

Thus the involution J on X exchanges T_j^+ and T_j^- for each j. It clearly also exchanges A^+ and A^-.

<u>Define</u>

(14.2) $$A := A^+ \cup T_1^+ \cup \dots \cup T_n^+ .$$

A is then a piecewise smooth compact submanifold of X with the properties $A \cup JA = X$ and $A \cap JA = \partial A$. Hence if we write

(14.3) $$W := \partial A ,$$

then W is a piecewise smooth characteristic submanifold for (X,J). We now describe W in more detail.

Let $V_j = W \cap T_j$ for each j. Then clearly

(14.4) $$W = R \cup -JR \cup V_1 \cup \dots \cup V_n$$

(recall that $R = X_o^* \times \{1\} \subset X_o$ and J is the involution, so $JR = X_o^* \times \{-1\}$. $-JR$ is of course JR with reversed orientation).

Using the map φ_j defined in §6 gives that

$$A^+ \cap T_j = \{(e^{i(\rho_j\theta + \nu_j\psi)}, e^{i(\beta_j\theta + \alpha_j\psi)}) \mid 0 \le \theta < 2\pi , 0 \le \psi \le \pi\}$$

Replacing θ by $\theta - \frac{\alpha_j}{\beta_j}\psi$ and using (6.2), this gives

(14.5) $$A^+ \cap T_j = \{(e^{i(\rho_j\theta + \frac{\psi}{\beta_j})}, e^{i\beta_j\theta}) \mid 0 \le \theta < 2\pi , 0 \le \psi \le \pi\} .$$

That is $A^+ \cap T_j$ is the strip in ∂T_j which "lies between"

$$Q_j = \{(e^{i\rho_j\theta}, e^{i\beta_j\theta}) \mid 0 \le \theta < 2\pi\}$$

and

$$JQ_j = \{(e^{i(\rho_j\theta + \frac{\pi}{\beta_j})}, e^{i\beta_j\theta}) \mid 0 \le \theta < 2\pi\} .$$

Figure 1 (next page) shows $A^+ \cap T_j$ for $\beta_j = 3$ and $\rho_j = 2$.

Now V_j clearly consists of the points of T_j which lie in one of $A^+ \cap T_j$ and ∂T_j^+, but not in both. By the above comments this gives (see figure 1)

$$V_j = \{(e^{i(\rho_j\theta - \frac{\psi}{\beta_j})}, e^{i\beta_j\theta}) \mid 0 \le \beta_j\theta \le \pi \text{ mod } 2\pi , 0 \le \psi \le \pi\}$$

(14.6) $$\cup \{(e^{i(\rho_j\theta + \frac{\psi}{\beta_j})}, e^{i\beta_j\theta}) \mid \pi \le \beta_j\theta \le 2\pi \text{ mod } 2\pi , 0 \le \psi \le \pi\}$$

$$\cup \{(re^{i\theta}, \pm 1) \mid 0 \le r \le 1 , 0 \le \theta \le 2\pi\} .$$

W can be smoothed (to do this equivariantly, smooth $W/\mathbb{Z}_2$ in $X/\mathbb{Z}_2$ and lift back to X), but since this makes no homological difference, we shall work with the unsmoothed W.

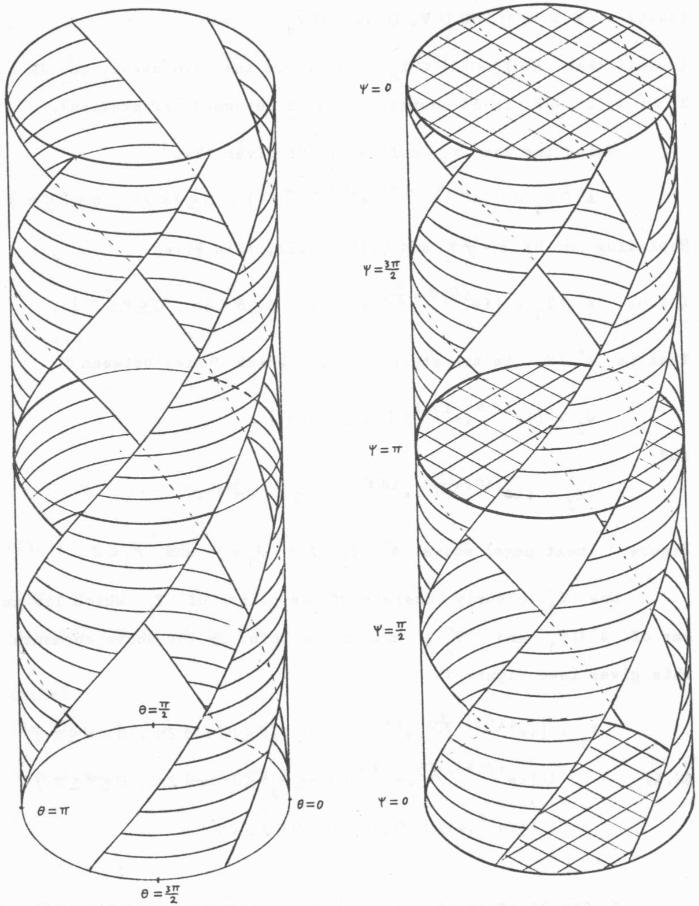

Figure 1 : $A^+\cap T_j$ and V_j lying in T_j for $\beta_j=3$, $\rho_j=2$ (T_j cut open along $D^2\times\{1\}$).

Institut für Angewandte Mathematik
der Universität Bonn

Homology of W and A.

Let $B_1,\dots,B_{2g}$ be closed paths in the interior of X_o^* which represent a basis for the first homology $H_1(X^*,\mathbb{Z})$ of X^*. Since $R=X_o^*\times\{1\}$, we can consider $B_1,\dots,B_{2g}$ to be paths in the interior $\mathring{R}$ of R. $JB_1,\dots,JB_{2g}$ are closed paths in $J\mathring{R}$, and one has the homology relations

(14.7) $\quad B_i\sim JB_i$ in $A \quad (i=1,\dots,2g)$,

since one can deform JB_i into B_i in $A^+\subset A$ by sliding JB_i down the fibres of the projection $A^+=X_o^*\times S^+\to X_o^*$.

Let $1\le j\le n$. If $T_j=D^2\times S^1$ is parametrised by $(re^{i\theta},e^{i\psi})$, then a "path running in the positive direction around T_j" will mean a path running around T_j in the direction of increasing ψ. For $0\le k\le|\beta_j|-1$ let C_j^k be the path in V_j which starts at the point $(0,1)\in T_j$, runs in the disc $\psi=0$ to the point $(e^{i2\pi\frac{k}{\beta_j}},1)\in T_j$, from there along Q_j half way around T_j in the positive direction to the point $(e^{i(2\pi\frac{k}{\beta_j}+\rho_j\frac{\pi}{\beta_j})},-1)$, and thence in the disc $\psi=\pi$ to the point $(0,-1)\in T_j$. That is

$$C_j^k=\{(re^{i2\pi\frac{k}{\beta_j}},1)\mid 0\le r\le 1\}$$
$$\cup\{(e^{i(2\pi\frac{k}{\beta_j}+\rho_j\frac{\theta}{\beta_j})},e^{i\theta})\mid 0\le\theta\le\pi\}$$
$$\cup\{(re^{i(2\pi\frac{k}{\beta_j}+\rho_j\frac{\pi}{\beta_j})},-1)\mid 1\ge r\ge 0\},\quad (0\le k\le|\beta_j|-1).$$

For $1\le k\le|\beta_j|-1$ let E_j^k be the closed path in V_j which runs along C_j^{k-1} from $(0,1)$ to $(0,-1)$, and back along C_j^k to $(0,1)$. That is

(14.8) $\quad E_j^k=C_j^{k-1}\cdot(-C_j^k)\quad,\quad(1\le k\le|\beta_j|-1)$.

Note that E_j^k lies in the boundary of T_j^+, which is topologically a 3-ball, so E_j^k is homologous to zero in T_j^+. Since $T_j^+ \subset A$

(14.9) $\quad E_j^k \sim 0$ in A.

Let

(14.10) $\quad F_j^k = C_j^o \cdot JC_j^{k-1} \quad , \quad (1 \leq k \leq |\beta_j|)$.

One has the homology relations

(14.11) $\quad JE_j^k \sim F_j^k - F_j^{k+1}$ in A, $(1 \leq k \leq |\beta_j|-1)$,

since $JE_j^k = JC_j^{k-1} \cdot (-JC_j^k) \sim C_j^o \cdot JC_j^{k-1} \cdot (-JC_j^k) \cdot (-C_j^o) = F_j^k - F_j^{k+1}$. This holds in V_j, and hence certainly also in A.

Let K_j be the path in V_j which starts at $(1,1) \varepsilon T_j$ and runs $\frac{\alpha_j}{2}$ times in the positive direction around T_j along Q_j to the point $(e^{i\rho_j \frac{\alpha_j \pi}{\beta_j}}, -1)$, and from there along the meridial path $(e^{i(\rho_j \frac{\alpha_j \pi}{\beta_j} + \frac{\psi}{\beta_j})}, -1)$, $0 \leq \psi \leq \pi$, to the point $(e^{i(\rho_j \frac{\alpha_j \pi}{\beta_j} + \frac{\pi}{\beta_j})}, -1) \varepsilon T_j$. This latter point is the point $((-1)^{\nu_j}, -1) = J(1,1)$, since $\frac{\rho_j \alpha_j + 1}{\beta_j} = \nu_j$ by (6.2). Thus

$$K_j = \{(e^{i\rho_j \frac{\alpha_j \theta}{\beta_j}}, e^{i\alpha_j \theta}) \mid 0 \leq \theta \leq \pi\}$$

(14.12)

$$\cup \{(e^{i((\nu_j - \frac{1}{\beta_j})\pi + \frac{\psi}{\beta_j})}, -1) \mid 0 \leq \psi \leq \pi\} .$$

Here the first part of the path is parametrised by $0 \leq \theta \leq \pi$, and may have multiple points.

Let K'_j be the path in T_j which starts at $(1,1)$ and runs half way around the orbit through this point to end at $((-1)^{\nu_j}, -1) = J(1,1)$. That is

$$K'_j = \{(e^{i\nu_j \theta}, e^{i\alpha_j \theta}) \mid 0 \leq \theta \leq \pi\}$$

$$= \{(e^{i(\frac{\rho_j \alpha_j + 1}{\beta_j})\theta}, e^{i\alpha_j \theta}) \mid 0 \leq \theta \leq \pi\}$$

(14.13) $$= \{(e^{i(\rho_j \frac{\alpha_j \theta}{\beta_j} + \frac{\theta}{\beta_j})}, e^{i\alpha_j \theta}) \mid 0 \leq \theta \leq \pi\} .$$

Comparing (14.13), (14.12), and (14.5), one sees that the paths K_j and K'_j are homotopic in $A^+ \cap T_j$ by a homotopy which fixes the end-points. Hence the path $K_j \cdot (-K'_j)$ is homologous to zero in $A^+ \cap T_j$, and hence certainly also in A:

(14.14) $\quad K_j \cdot (-K'_j) \sim 0$ in A.

Now for $1 \leq j \leq n-1$ let G_j be any path in R from the point $(1,1) \varepsilon T_n$ to the point $(1,1) \varepsilon T_j$, and define

(14.15) $\quad L_j^o = G_j \cdot K_j \cdot (-JG_j) \cdot (-K_n) \quad , \quad (1 \leq j \leq n-1)$.

That is L_j^o is the closed path in W which runs from the point $(1,1) \varepsilon T_n$ along G_j to $(1,1) \varepsilon T_j$, thence along K_j to $J(1,1) \varepsilon T_j$, thence backwards along JG_j in JR to $J(1,1) \varepsilon T_n$, and finally backwards along K_n back to $(1,1) \varepsilon T_n$.

By (14.14) L_j^o is homologous in A to the path $L'_j = G_j \cdot K'_j \cdot (-JG_j) \cdot (-K'_n)$. But by sliding along the fibres of the projection $A^+ \to X_o^*$, L'_j can be deformed into the path $G_j \cdot (-G_j)$. Hence

(14.16) $\quad L_j^o \sim 0$ in A.

We shall denote a homology class in $H_1(W, \mathbb{Q})$ by the same symbol as the path by which we represent it. The Mayer-Vietoris sequence for the pair $(R \cup -JR, V_1 \cup V_2 \cup \dots \cup V_n)$ yields easily that $H_1(W,\mathbb{Q})$ is the $\mathbb{Q}$-module, freely generated by the elements :

$$
(14.17)\qquad
\begin{array}{ll}
B_i \; , \; JB_i & (i=1,\dots,2g) \\
E_j^k \; , \; JE_j^k & (j=1,\dots,n \; ; \; k=1,\dots,|\beta_j|-1) \\
F_j^1 & (j=1,\dots,n-1) \\
L_j^o & (j=1,\dots,n-1) \; .
\end{array}
$$

If P is a closed path in A, we shall denote the represented homology class in $H_1(A,\mathbb{Q})$ by $\bar{P}$. The Mayer - Vietoris sequence for the pair $(A^+, (T_1 \cup \dots \cup T_n) \cap A)$ (note that the nicer looking pair $(A^+, T_1^+ \cup \dots \cup T_n^+)$ is not an excisive pair, so it cannot be used) yields that $H_1(A,\mathbb{Q})$ is the $\mathbb{Q}$-module, freely generated by the elements

$$
(14.18)\qquad
\begin{array}{ll}
\bar{B}_i & (i=1,\dots,2g) \\
\bar{F}_j^k & (j=1,\dots,n-1 \; ; \; k=1,\dots,|\beta_j|) \\
\bar{F}_n^k & (k=1,\dots,|\beta_j|-1) \; .
\end{array}
$$

If the homology map induced by the inclusion $W \subset A$ is

$$i_* : H_1(W,\mathbb{Q}) \longrightarrow H_1(A,\mathbb{Q}) \; ,$$

then (14.7), (14.9), (14.11), and (14.16) give that

$$
(14.19)\qquad
\begin{array}{ll}
i_*B_i = \bar{B}_i & (i=1,\dots,2g) \\
i_*JB_i = \bar{B}_i & (i=1,\dots,2g) \\
i_*E_j^k = 0 & (j=1,\dots,n \; ; \; k=1,\dots,|\beta_j|-1)
\end{array}
$$

$$
\begin{array}{ll}
i_*JE_j^k = \bar{F}_j^k - \bar{F}_j^{k+1} & (j=1,\dots,n \; ; \; k=1,\dots,|\beta_j|-1) \\
i_*F_j^1 = \bar{F}_j^1 & (j=1,\dots,n-1) \\
i_*L_j^o = 0 & (j=1,\dots,n-1) \; .
\end{array}
\qquad \text{(14.19) ctd.}
$$

Hence $\mathcal{L} = \mathrm{kern}(i_* : H_1(W,\mathbb{Q}) \longrightarrow H_1(A,\mathbb{Q}))$ is the $\mathbb{Q}$-module freely generated by the elements

$$
(14.20)\qquad
\begin{array}{ll}
B_i - JB_i & (i=1,\dots,2g) \\
E_j^k & (j=1,\dots,n \; ; \; k=1,\dots,|\beta_j|-1) \\
L_j^o & (j=1,\dots,n-1) \; .
\end{array}
$$

§15. Proof of lemma 13.1

We keep the notation of §14, but we now make the following additional assumptions: $\beta_j \geq -1$ for all j, and $(\alpha_n, \beta_n) = (1,1)$. Without loss of generality $\beta_j > 1$ for $j=1,\dots,m$, and $\beta_j = \pm 1$ for $j=m+1,\dots,n$ $(0 \leq m \leq n-1)$.

We first alter the basis (14.20) of $\mathcal{L}$.

For $0 \leq j \leq m$ and $0 \leq i \leq \beta_j - 1$ let K_j^i be the path constructed in the same way as K_j, but starting from the point $(e^{i2\pi\frac{k}{\beta_j}}, 1) \in T_j$ instead of $(1,1) \in T_j$. That is K_j^i starts at $(e^{i2\pi\frac{k}{\beta_j}}, 1)$, runs $\frac{\alpha_j}{2}$ times around T_j along Q_j to the point $(e^{i(2\pi\frac{k}{\beta_j} + \rho_j \frac{\alpha_j\pi}{\beta_j})}, -1) = (e^{i(2\pi\frac{k}{\beta_j} + (\nu_j - \frac{1}{\beta_j})\pi)}, -1) \in T_j$, and from there along the meridial curve $(e^{i(2\pi\frac{k}{\beta_j} + (\nu_j - \frac{1}{\beta_j})\pi + \frac{\psi}{\beta_j})}, -1)$, $0 \leq \psi \leq \pi$, to the point $(e^{i(2\pi\frac{k}{\beta_j}+\nu_j\pi)},-1) = J(e^{i2\pi\frac{k}{\beta_j}},1)$. Let G_j^i be any path in R from the point $(1,1)\in T_n$ to the point $(e^{i2\pi\frac{k}{\beta_j}},1)\in T_j$ (for i=0 choose $G_j^o = G_j$), and define

(15.1) $$L_j^i = G_j^i \cdot K_j^i \cdot (-JG_j^i)\cdot(-K_n) .$$

For i=0 this coincides with the previous definition of L_j^o . Also, just as for L_j^o , we have

(15.2) $$L_j^i \sim 0 \quad \text{in } A ,$$

so L_j^i represents an element of $\mathcal{L}\subset H_1(W,Q)$. Define

(15.3)
$$L_j = \sum_{i=0}^{\beta_j-1} L_j^i \qquad (1\le j\le m)$$
$$L_j = L_j^o \qquad (m+1\le j\le n-1) .$$

We shall calculate the matrix of the quadratic form $f:(x,y)\mapsto x\circ Jy$ on $\mathcal{L}$ with respect to the following basis of $\mathcal{L}$:

(15.4)
$$B_i - JB_i \qquad (i=1,\dots,2g)$$
$$E_j^k \qquad (j=1,\dots,m\ ;\ k=1,\dots,\beta_j-1\ )$$
$$\frac{1}{|\beta_j|}L_j \qquad (j=1,\dots,n-1\ ) .$$

Lemma 15.1. (i). $(B_i - JB_i)\circ Jx = 0$ for all $x\in\mathcal{L}$,

(ii). $E_i^k \circ JE_j^l = 0$ for $i\neq j$,

(iii). $L_i \circ JE_j^k = 0$.

Proof. (i): It suffices to prove (i) as x runs through the basis

elements (14.20).

$$\begin{aligned}(B_i - JB_i)\cdot J(B_j - JB_j) &= (B_i - JB_i)\circ(JB_j - B_j)\\ &= B_i\circ JB_j + JB_i\circ B_j - B_i\circ B_j - JB_i\circ JB_j\\ &= B_i\circ JB_j + J(B_i\circ JB_j) - B_i\circ B_j - J(B_i\circ B_j)\\ &= B_i\circ JB_j - B_i\circ JB_j - B_i\circ B_j - (-B_i\circ B_j)\\ &= 0 .\end{aligned}$$

Since $B_i - JB_i$ and JE_j^k are represented by disjoint paths in W ,

$$(B_i - JB_i)\cdot JE_j^k = 0 .$$

Finally note that JL_j^o coincides with $-L_j^o$ in the interior $\mathring{R}$ of R , and B_i lies completely in $\mathring{R}$, so $B_i\circ JL_j^o = -B_i\circ L_j^o$. Since $JB_i\circ JL_j^o = J(B_i\circ L_j^o) = -B_i\circ L_j^o$, it follows that

$$(B_i - JB_i)\circ JL_j^o = B_i\circ JL_j^o - JB_i\circ JL_j^o = -B_i\circ L_j^o - (-B_i\circ L_j^o) = 0 .$$

This completes the proof of (i).

(ii) is trivial, since for $i\neq j$ E_i^k and JE_j^l are represented by disjoint paths in W .

(iii): If $i\neq j$, then L_i and JE_j^k are represented by disjoint paths in W , so we need only consider the case $i=j$.

$$L_j\circ JE_j^k = \Big(\sum_{i=0}^{\beta_j-1} L_j^i\Big)\circ JE_j^k = \sum_{i=0}^{\beta_j-1}(L_j^i\circ JE_j^k) .$$

L_j^i and JE_j^k are represented by disjoint paths in W for $i\neq k-1,k$. By inspection one sees that L_j^{k-1} and L_j^k each intersect JE_j^k once (in the points $J(e^{i2\pi\frac{k-1}{\beta_j}},1)\in T_j$ and $J(e^{i2\pi\frac{k}{\beta_j}},1)\in T_j$ respectively), but with opposite parities. Hence the above sum is equal to $\pm1\mp1=0$. ||

For $1 \leq j \leq m$ let $\mathcal{E}_j$ be the $(\beta_j - 1)$ - square matrix $(e_{k\,l})$ with $e_{k\,l} = E_j^k \circ J E_j^l$. Let $\mathcal{F}$ be the $(n-1)$ - square matrix $(f_{i\,j})$ with $f_{i\,j} = \frac{1}{|\beta_i|} L_i \circ J(\frac{1}{|\beta_j|} L_j)$. Then lemma 15.1 states that the matrix of the quadratic form $f : (x,y) \mapsto x \circ Jy$ on $\mathcal{L}$, with respect to the basis (15.4) is equal to

$$\begin{pmatrix} 0 & & & & & 0 \\ & \mathcal{E}_1 & & & & \\ & & \mathcal{E}_2 & & & \\ & & & \ddots & & \\ & & & & \mathcal{E}_m & \\ 0 & & & & & \mathcal{F} \end{pmatrix}$$

Hence the signature $\tau(f)$ of f is given by

(15.5) $\quad \tau(f) = \tau(\mathcal{E}_1) + \ldots + \tau(\mathcal{E}_m) + \tau(\mathcal{F})$.

Now let $1 \leq j \leq m$. By (14.1), (14.6), and (14.8), the involution J on T_j, the surface V_j in T_j, and the closed curves E_j^k $(k=1,\ldots,\beta_j-1)$ in V_j depend only on the numbers ρ_j, β_j, and $(-1)^{\nu_j}$. Thus the matrix $\mathcal{E}_j$ depends only on ρ_j, β_j, and $(-1)^{\nu_j}$. Since these numbers are determined by α_j and β_j, we can define

(15.6) $\quad c(\alpha_j, \beta_j) := \tau(\mathcal{E}_j)$.

This defines $c(\alpha,\beta)$ for $\beta > 1$. For $\beta = \pm 1$ define

(15.7) $\quad c(\alpha, \pm 1) := 0$.

Part (ii) of lemma 13.1 then holds automatically. We now prove part (iii). This is trivial for $\beta = \pm 1$, so we may assume $\beta > 0$. We have already remarked that $c(\alpha,\beta)$ depends only on the numbers ρ, β, and $(-1)^{\nu}$ determined by α and β, so it suffices to show that if we replace α by $\alpha + 2k\beta$ with $k \in \mathbb{Z}$ and $\alpha + 2k\beta > 0$, then ρ and $(-1)^{\nu}$ remain unchanged.

Recall ((6.1) and (6.2)) that ν was defined as the unique integer with $0 < \nu \leq \alpha$ and $\nu\beta \equiv 1 \bmod \alpha$, and ρ was defined by

$$\rho = \frac{\nu\beta - 1}{\alpha} .$$

Hence $\rho = \frac{\nu}{\alpha}\beta - \frac{1}{\alpha} < \frac{\nu}{\alpha}\beta \leq \beta$. Also, since $\nu\beta \geq 1$, $\rho \geq 0$, and since $\rho\alpha = \nu\beta - 1$, $\rho\alpha \equiv -1 \bmod \beta$. Thus ρ can be defined as the unique integer with $0 \leq \rho < \beta$ and $\rho\alpha \equiv -1 \bmod \beta$, and then ν is given by

$$\nu = \frac{\rho\alpha + 1}{\beta} .$$

Since $\rho(\alpha + 2k\beta) \equiv \rho\alpha \equiv -1 \bmod \beta$, replacing α by $\alpha + 2k\beta$ does not change ρ. ν is then replaced by $\frac{\rho(\alpha + 2k\beta) + 1}{\beta} = \frac{\rho\alpha + 1}{\beta} + 2k\rho = \nu + 2k\rho$, which is congruent to ν mod 2, so $(-1)^{\nu}$ remains unchanged also. This completes the proof of part (iii) of lemma 13.1. For an alternative proof see §17.

Only part (i) of lemma 13.1 remains to be proved. Using the definitions (15.6) and (15.7) and the fact that $\beta_j = \pm 1$ for $j = m+1, \ldots, n$, (15.5) can be written as

(15.8) $\quad \tau(f) = \sum_{j=1}^{n} c(\alpha_j, \beta_j) + \tau(\mathcal{F})$.

Since $\tau(f)$ is equal to $\alpha(X,J)$ by definition, part (i) of lemma 13.1

now follows from the following lemma :

Lemma 15.2. $\tau(\mathcal{F}) = \sum_{j=1}^{n} \operatorname{sign}\beta_j - \operatorname{sign}\sum_{j=1}^{n}\frac{\beta_j}{\alpha_j}$.

Proof. Note that the paths L_j^i of (15.1) are always of the form $G\cdot K\cdot(-JG)\cdot(-K_n)$, where G is a path in R from some point $p \in V_n$ to a point $q \in V_j$, K is a path in V_j from the point q to the point Jq, and K_n is a path in V_n from the point p to the point Jp. Suppose $L' = G'\cdot K'\cdot(-JG')\cdot(-K_n')$ is any other path of the same form, such that $JL' = JG'\cdot JK'\cdot(-G')\cdot(-JK_n')$ intersects L_j^i "nicely". Then to calculate $L_j^i \circ JL'$ we need only consider intersection points of L_j^i and JL' which lie in V_j or V_n, for any intersection point which lies in the interior $\mathring{R}$ of R is an intersection point of G and $-G'$, and is cancelled by a corresponding intersection point of $-JG$ and JG' in $J\mathring{R}$, and vice versa.

We shall denote by $(L_j^i \circ JL')_j$ and $(L_j^i \circ JL')_n$ the contributions to $L_j^i \circ JL'$ which come from intersection points in V_j and V_n respectively. By the above comments $L_j^i \circ JL' = (L_j^i \circ JL')_j + (L_j^i \circ JL')_n$.

We first calculate $(L_j \circ JL_j^0)_j$ for $1 \le j \le n-1$. Observe that $L_j = \sum_{i=0}^{|\beta_j|-1} L_j^i$ covers each of the portions of Q_j which lie in the boundary of T_j^+ precisely $\frac{\alpha_j+1}{2}$ times, and goes through each of the points $J(e^{i2\pi\frac{k}{\beta_j}},1)$ $(k=0,\dots,|\beta_j|-1)$ precisely once. The latter are the only points in which L_j intersects the curve JQ_j .

We now assume that we have slid the curve L_j^0 slightly, so that it enters T_j in the point $q = (e^{i\rho_j\varepsilon}, e^{i\beta_j\varepsilon}) \in T_j$ ($\beta_j\varepsilon$ small and >0) instead of the point (1,1), and after running $\frac{\alpha_j}{2}$ times around T_j

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along the curve Q_j , runs along the meridial curve $(e^{i(\rho_j\varepsilon+(\nu_j-\frac{1}{\beta_j})\pi+\frac{\psi}{\beta_j})}, -e^{i\beta_j\varepsilon})$, $0 \le \psi \le \pi$, to the point $(e^{i(\rho_j\varepsilon+\nu_j\pi)}, -e^{i\beta_j\varepsilon}) = Jq$, where it leaves T_j again. JL_j^0 then starts at the point Jq , and first runs $\frac{\alpha_j}{2}$ times around T_j along the curve JQ_j . In doing so it crosses the meridial circle $(e^{i\theta},-1)$ in T_j $\frac{\alpha_j-1}{2}$ times, each time in a point of the form $J(e^{i2\pi\frac{k}{\beta_j}},1)$. This gives $\frac{\alpha_j-1}{2}$ points of intersection with L_j .

JL_j^0 has then arrived at the point $(e^{i(\rho_j\varepsilon-\frac{\pi}{\beta_j})}, e^{i\beta_j\varepsilon})$, and runs from this point along a meridial curve to the point q , where it leaves T_j again. At q it has an $\frac{\alpha_j+1}{2}$-fold intersection with L_j . There are no other points of intersection. At each of these intersection points the direction of L_j , followed by the direction of JL_j^0 , followed by a normal into A gives the orientation of A if $\beta_j > 0$, and otherwise the opposite orientation. Thus each of these intersections must be counted positively if $\beta_j > 0$ and negatively if $\beta_j < 0$, so

$$(L_j \circ JL_j^0)_j = \left(\frac{\alpha_j-1}{2} + \frac{\alpha_j+1}{2}\right)\operatorname{sign}\beta_j = \alpha_j \operatorname{sign}\beta_j \ .$$

By symmetry one has more generally

(15.9) $$(L_j \circ JL_j^i)_j = \alpha_j \operatorname{sign}\beta_j \qquad (i=0,\dots,|\beta_j|-1) \ .$$

Hence $(L_j \circ JL_j)_j = (L_j \circ J\sum_{i=0}^{|\beta_j|-1} L_j^i)_j = \sum_{i=0}^{|\beta_j|-1}(L_j \circ JL_j^i)_j = |\beta_j|\alpha_j \operatorname{sign}\beta_j = \beta_j\alpha_j$. Thus

(15.10) $$\left(\frac{1}{|\beta_j|}L_j \circ J\left(\frac{1}{|\beta_j|}L_j\right)\right)_j = \frac{1}{|\beta_j|^2}\beta_j\alpha_j = \frac{\alpha_j}{\beta_j} \ .$$

Similarly to the above, but rather more easily, one verifies the following for $j,k = 1,\dots,n-1$ (recall that $(\alpha_n,\beta_n) = (1,1)$) :

(15.11) $\quad (L_j^r \circ JL_k^s)_n = 1 \quad (r=0,\dots,|\beta_j|-1 \; ; \; s=0,\dots,|\beta_k|-1)$.

Hence $(L_j \circ JL_k)_n = \sum_{r=0}^{|\beta_j|-1} \sum_{s=0}^{|\beta_k|-1} (L_j^r \circ JL_k^s)_n = |\beta_j||\beta_k|$, so

(15.12) $\quad (\frac{1}{|\beta_j|}L_j \circ J(\frac{1}{|\beta_k|}L_k))_n = 1$.

Also trivially

(15.13) $\quad (L_j \circ JL_k)_j = 0$ for $j \neq k$.

Combining (15.10), (15.12), and (15.13) gives altogether

$$\frac{1}{|\beta_j|}L_j \circ J(\frac{1}{|\beta_k|}L_k) = \begin{cases} 1 & \text{if } j \neq k , \\ \frac{\alpha_j}{\beta_j} + 1 & \text{if } j = k . \end{cases} \tag{15.14}$$

Thus

$$\mathcal{F} = \begin{pmatrix} \frac{\alpha_1}{\beta_1}+1 & 1 & \cdots & 1 \\ 1 & \frac{\alpha_2}{\beta_2}+1 & \cdots & 1 \\ \vdots & \vdots & & \vdots \\ 1 & 1 & \cdots & \frac{\alpha_{n-1}}{\beta_{n-1}}+1 \end{pmatrix}$$

For brevity we write γ_i for $\frac{\alpha_i}{\beta_i}$ $(i=1,\dots,n-1)$, and reindex so that $\gamma_1 \leq \gamma_2 \leq \dots \leq \gamma_{n-1}$.

A simple induction argument shows that the characteristic polynomial $\det(\mathcal{F} - tI)$ of $\mathcal{F}$ is

$$g(t) = \det(\mathcal{F} - tI) = \prod_{j=1}^{n-1}(\gamma_j - t) + \sum_{i=1}^{n-1} \prod_{j \neq i}(\gamma_j - t) .$$

If for fixed i , $\gamma_j = \gamma_i$ occurs for m values of j , then $t = \gamma_i$ is an $(m-1)$-fold root of $g(t)$. The roots of $g(t)$ which are not equal to a γ_i are precisely the roots of

$$h(t) = g(t)/\prod_{j=1}^{n-1}(\gamma_j - t) = 1 + \sum_{j=1}^{n-1} \frac{1}{(\gamma_j - t)} .$$

If $\gamma_i < \gamma_{i+1}$, then $h(t)$ is negative for t just above γ_i and positive for t just below γ_{i+1} , so $h(t)$ has a root between γ_i and γ_{i+1} . Also $h(t)$ is negative for t just above γ_{n-1} and is positive for t large, so it has a root above γ_{n-1} . Thus the roots $t_1, t_2, \dots, t_{n-1}$ of $g(t)$ are distributed as follows :

$$\gamma_1 \leq t_1 \leq \gamma_2 \leq t_2 \leq \dots \leq \gamma_{n-1} < t_{n-1} .$$

The t_i , as roots of the characteristic polynomial of $\mathcal{F}$, are just the eigenvalues of $\mathcal{F}$, so

(15.15) $\quad \tau(\mathcal{F}) = \sum_{i=1}^{n-1} \operatorname{sign} t_i$.

By looking at the value of $h(t)$ at $t = 0$ one sees :

$0 < \gamma_1 \Rightarrow \operatorname{sign} t_i = \operatorname{sign} \gamma_i$ for each i , and $\operatorname{sign}(1 + \sum_{i=1}^{n-1} \frac{1}{\gamma_i}) = 1$.

$\gamma_k < 0 < \gamma_{k+1}$ for some k with $1 \leq k \leq n-2$ $\Rightarrow$ $\operatorname{sign} t_i = \operatorname{sign} \gamma_i$ for each $i \neq k$, and $\operatorname{sign} t_k = -\operatorname{sign}(1 + \sum_{i=1}^{n-1} \frac{1}{\gamma_i})$.

$\gamma_{n-1} < 0 \Rightarrow \operatorname{sign} t_i = \operatorname{sign} \gamma_i$ for each $i \neq n-1$, and $\operatorname{sign} t_{n-1} = -\operatorname{sign}(1 + \sum_{i=1}^{n-1} \frac{1}{\gamma_i})$.

In each case one obtains

$$\sum_{i=1}^{n-1} \operatorname{sign} t_i = \sum_{i=1}^{n-1} \operatorname{sign} \gamma_i + 1 - \operatorname{sign}\Big(1 + \sum_{i=1}^{n-1} \frac{1}{\gamma_i}\Big) .$$

Now $\operatorname{sign}\gamma_i = \operatorname{sign}\beta_i$ for $1 \le i \le n-1$, since $\gamma_i = \frac{\alpha_i}{\beta_i}$ and $\alpha_i > 0$.

Also, $\operatorname{sign}\beta_n = 1 = \frac{\beta_n}{\alpha_n}$, since by assumption $(\alpha_n, \beta_n) = (1,1)$. Hence

$$\sum_{i=1}^{n-1} \operatorname{sign} t_i = \sum_{i=1}^{n-1} \operatorname{sign}\beta_i + \operatorname{sign}\beta_n - \operatorname{sign}\Big(\frac{\beta_n}{\alpha_n} + \sum_{i=1}^{n-1} \frac{\beta_i}{\alpha_i}\Big) .$$

With (15.5) this proves lemma 15.2, and hence also completes the proof of lemma 13.1. ||||

<u>Remark.</u> By actually calculating the matrix $\mathcal{E}_j$ which occurs in the above proof, one can use (15.6) to give a further definition of $c(\alpha,\beta)$ for $\beta > 1$. The calculation is canonical but messy, so we will not carry it out. The result is as follows.

If (α,β) is a coprime pair with $\alpha > 0$, $\beta > 1$, α odd, then define ν and ρ as usual; that is ν is defined by

$$0 < \nu \le \alpha \quad \text{and} \quad \nu\beta \equiv 1 \mod \alpha ,$$

and ρ by

$$\rho = \frac{\nu\beta - 1}{\alpha} ,$$

(see also p61). Define

$$\rho' = \begin{cases} \rho & \text{if } \nu \text{ is even,} \\ \rho + \beta & \text{if } \nu \text{ is odd.} \end{cases}$$

One verifies that ρ' is then odd. Let $A[\alpha,\beta]$ be the $(\beta-1)$-square matrix

$$A[\alpha,\beta] = \left(\begin{array}{c|c} 0 & \begin{matrix} -1 & 1 & & & \\ & -1 & 1 & & \\ & & \ddots & & \\ & & & -1 & 1 \\ & & & & -1 \end{matrix} \\ \hline \begin{matrix} 1 & & & & \\ -1 & 1 & & & \\ & -1 & 1 & & \\ & & \ddots & & \\ & & & 1 & \\ & & & -1 & 1 \end{matrix} & 0 \end{array}\right) \begin{array}{l} \Big\} \beta - \frac{\rho'+1}{2} \\ \\ \Big\} \frac{\rho'-1}{2} \end{array} ,$$

and put

$$A(\alpha,\beta) = A[\alpha,\beta] + {}^{t}A[\alpha,\beta] ,$$

where the superscript ^{t} means transpose.

Then the matrix $\mathcal{E}_j$ occuring in (15.6) is equal to $A(\alpha_j,\beta_j)$, so we have in general

$$\underline{c(\alpha,\beta) = \tau(A(\alpha,\beta)) .}$$

This is of course useless for practical calculation.

§16. Examples; other methods of calculation

Example 16.1. $\Sigma(a_o,a_1,a_2) = \Sigma(3,6j-1,18j-1)$.

In this case the a_i are pairwise coprime, so in the notation of §9 :

$(t_o,t_1,t_2) = (a_o,a_1,a_2)$ and $(a'_o,a'_1,a'_2) = (t_1t_2,t_ot_2,t_ot_1) =$ $(108j^2-24j+1,54j-3,18j-3)$. Since $-2a'_o + ja'_1 + (9j-1)a'_2 = 1$ we can choose

$$(\beta_o,\beta_1,\beta_2) = (-2,j,9j-1) .$$

Thus by theorem 9.1

$$\Sigma(a_o,a_1,a_2) \cong X((0),(3,-2),(6j-1,j),(18j-1,9j-1) .$$

We now calculate $c(t_i,\beta_i)$ by (12.2) .

$$\frac{3}{-2} = -2 + \cfrac{1}{1 + \cfrac{1}{1}} ,$$

so $c(3,-2) = 1$.

$$\frac{6j-1}{j} = 2.3 + \cfrac{-1}{j-1 + \cfrac{1}{1}} ,$$

so $c(6j-1,j) = -(j-1)$.

$$\frac{18j-1}{9j-1} = 2.1 + \cfrac{1}{9j-2 + \cfrac{1}{1}} ,$$

so $c(18j-1,9j-1) = 9j-2$.

Hence by corollary 13.2

$$\begin{aligned}\alpha(\Sigma(3,6j-1,18j-1),J) &= 1 + \text{sign}\,{-2} + -(j-1) + \text{sign}\, j + \\ &\quad + (9j-2) + \text{sign}(9j-1) - 1 \\ &= 8j .\end{aligned}$$

This checks with the result of Hirzebruch [6] §6, see also [7]. In [7],

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Hirzebruch and Jänich give an algorithm to calculate $\alpha(\Sigma(a_o,a_1,a_2),J)$ in the case that the a_i are pairwise coprime and odd. A purely number theoretic proof that their algorithm gives the same result as our corollary 13.2 seems very difficult.

The numbers s_o , s_1 , and s_2 of §9 are pairwise coprime, so at most one of them is even. Hence by (12.4) the involution J , if it is fixpoint free, is equal (possibly after permuting indices) to one of the involutions

$$T : (z_o,z_1,z_2) \mapsto (-z_o,-z_1,-z_2) ,$$

which is defined on $\Sigma(a_o,a_1,a_2)$ if $a_o \equiv a_1 \equiv a_2 \mod 2$, or

$$T_o : (z_o,z_1,z_2) \mapsto (z_o,-z_1,-z_2) ,$$

which is defined on $\Sigma(a_o,a_1,a_2)$ if $a_1 \equiv a_2 \equiv 0 \mod 2$.

The Hirzebruch invariant can always be calculated for both these involutions. Namely if $a_o \equiv a_1 \equiv a_2 \equiv 1 \mod 2$, then $T = J$, and corollary 13.2 is applicable. If $a_o \equiv a_1 \equiv a_2 \equiv 0 \mod 2$, then $\alpha(\Sigma(a),T)$ is given by the following theorem :

Theorem 16.1 (Hirzebruch and Jänich [7]).

If $a_o \equiv a_1 \equiv a_2 \equiv 0 \mod 2$, then

$$\alpha(\Sigma(a),T) = \sum_j \varepsilon(j)(-1)^{j_o+j_1+j_2} ,$$

where the sum is over all $j = (j_o,j_1,j_2) \in \mathbb{Z}^3$ with $0 < j_r < a_r$, and $\varepsilon(j)$ is 1, -1, or 0 , depending on whether $\frac{j_o}{a_o} + \frac{j_1}{a_1} + \frac{j_2}{a_2}$ lies strictly between 0 and 1 mod 2, or strictly between 1 and 2 mod 2, or is integral.

The proof of this theorem uses the results of Pham [12], see also [8]. Using essentially the same proof one has also

<u>Theorem 16.2.</u> If $a_1 \equiv a_2 \equiv 0 \mod 2$, then

$$\alpha(\Sigma(a),T_o) = \sum_j \varepsilon(j)(-1)^{j_1+j_2} ,$$

where the range of summation and the function $\varepsilon(j)$ are as in theorem 16.1. ||

<u>Example 16.2.</u> $(a_o,a_1,a_2) = (6,4,4)$.
Then $(a_o',a_1',a_2') = (2,3,3)$, so since $2.2+(-1).3+0.3 = 1$, we can choose $(\beta_o,\beta_1,\beta_2) = (2,-1,0)$. Hence

$$\Sigma(6,4,4) \cong X((2),4(3,2),2(1,-1)) ,$$

and applying corollary 13.2 gives

$$\alpha(\Sigma(6,4,4),J) = -3 .$$

In this case $J = T_o$. Checking with theorem 16.2 also gives
$\alpha(\Sigma(6,4,4),T_o) = -3$.

Theorem 16.1 gives

$$\alpha(\Sigma(6,4,4),T) = 7 .$$

The involution $T_1 : (z_o,z_1,z_2) \mapsto (-z_o,z_1,-z_2)$ on $\Sigma(6,4,4)$ is essentially the same as the involution T_o on $\Sigma(4,6,4)$, and theorem 16.2 gives

$$\alpha(\Sigma(6,4,4),T_1) = -1 .$$

<u>§17. Involutions on lens spaces</u>

We recall the definition of the lens space $L(p,q)$ ($p>0$, p and q coprime). Let S^3 be the unit sphere $|z_1|^2+|z_2|^2 = 1$ in $\mathbb{C}^2$. $\mathbb{Z}_p$ acts on S^3 by (g a generator of $\mathbb{Z}_p$):

$$g(z_1,z_2) = (e^{i2\pi/p}z_1, e^{i2\pi q/p}z_2) .$$

<u>Definition.</u> $L(p,q) := S^3/\mathbb{Z}_p$, with inherited orientation. One generally also defines $L(0,\pm1) = S^2 \times S^1$.

As is well known, $L(p,q)$ can also be obtained by sewing together two copies of the solid torus $D^2 \times S^1$ by means of the diffeomorphism

$$\partial D^2 \times S^1 = S^1 \times S^1 \longrightarrow S^1 \times S^1 = \partial D^2 \times S^1$$

which is defined by the matrix

$$\begin{pmatrix} -q & x \\ p & y \end{pmatrix}$$

where x and y are numbers with $px+qy = 1$.

Lens spaces are classified up to diffeomorphism (and homeomorphism) as follows (see Brody [4]).

$$L(p,q) \cong L(p',q') \Leftrightarrow p = p' \text{ and either } q \equiv q' \bmod p \text{ or } qq' \equiv 1 \bmod p .$$

(17.1)

$$L(p,q) \cong -L(p',q') \Leftrightarrow p = p' \text{ and either } q \equiv -q' \bmod p \text{ or } qq' \equiv -1 \bmod p .$$

An oriented fixpoint free 3-dimensional S^1-manifold X is a lens space if and only if its orbit space X* is the 2-sphere and X has at most 2 exceptional orbits ([11], [15]). The calculation of which lens space occurs for given Seifert invariants has been done by von Randow [13], see also Raymond [11]. In our notation the result is:

<u>Theorem 17.1.</u> Let $X = X((0),(\alpha_1,\beta_1),(\alpha_2,\beta_2))$ and let ρ_2 and ν_2 be integers with $\alpha_2\rho_2 - \beta_2\nu_2 = -1$. Then $X \cong L(p,q)$, where $p = |\alpha_1\beta_2 + \alpha_2\beta_1|$, $q = -(\alpha_1\rho_2 + \beta_1\nu_2)\,\mathrm{sign}\,p$. ||

This can easily be checked directly by comparing the construction of §6 with the representation of a lens space as the union of two solid tori.

Let X be as in the above theorem. The involution $J \in S^1$ is free on X if and only if α_1 and α_2 are odd. Then by theorem 8.1 $X/Z_2 = X((0),(\alpha_1,2\beta_1),(\alpha_2,2\beta_2))$, which is again a lens space, by the above theorem.

The classification of coverings gives that there is at most one free $\mathbb{Z}_2$-manifold X (up to equivariant diffeomorphism) whose orbit space $X/\mathbb{Z}_2$ is a given lens space $L(p',q)$, for the fundamental group $\pi_1 L(p',q) = \mathbb{Z}_{p'}$ contains at most one subgroup of index 2. Such a subgroup, and hence also such a $\mathbb{Z}_2$-manifold X exists if and only if p′ is even.

<u>Theorem 17.2.</u> Let 2p and q be coprime positive integers. Let (X,J) be the unique manifold with free involution, such that $X/\mathbb{Z}_2 = L(2p,q)$. Then $X = L(p,q)$ and

$$\alpha(X,J) = c(q,p).$$

<u>Proof.</u> As $X((0),(q,p)) = X((0),(q,p),(1,0))$, choosing $\rho_2 = -1$ and $\nu_2 = 0$, theorem 17.1 gives that

$$X((0),(q,p)) \cong L(p,q).$$

Further the involution J on $X((0),(q,p))$ is the involution we want, since by theorem 8.1 the orbit space is $X((0),(q,2p))$, and this is $L(2p,q)$. Hence by theorem 12.1

$$\alpha(L(p,q),J) = c(q,p) + \mathrm{sign}\,q - \mathrm{sign}\,\frac{q}{p} = c(q,p) \quad . \quad ||$$

This theorem gives immediately the promised alternative proof that $c(\alpha,\beta) = c(2k\beta+\alpha,\beta)$ if $\alpha, \beta, 2k\beta+\alpha > 0$. Namely this follows by applying the theorem to the equation

$$L(2\beta,\alpha) \cong L(2\beta,2k\beta+\alpha),$$

which holds by (17.1). A further property of $c(\alpha,\beta)$ which follows in the same way is:

Let $\alpha, \alpha', \beta > 0$; α odd; $\alpha\alpha' \equiv 1 \mod 2\beta$. Then $c(\alpha,\beta) = c(\alpha',\beta)$.

For by (17.1) $L(2\beta,\alpha) \cong L(2\beta,\alpha')$.

§18. Remarks on the characteristic submanifold.

In [1] Bredon and Wood consider the problem of which closed non-orientable surfaces can be embedded in a given closed orientable 3-manifold, and solve this for all finite connected sums of manifolds out of the following class: $M \times S^1$, M any orientable closed surface; lens spaces. By Raymond's classification, such finite connected sums include all closed orientable 3-manifolds which admit S^1-actions with fixpoints.

For closed orientable 3-manifolds which admit fixpoint free S^1-actions the problem still seems to be open in general, though some results follow immediately by the methods of Bredon and Wood. For instance their proof that the non-orientable surface U_{2k} of genus 2k, $k \geq 2$, (that is the connected sum of 2k copies of the projective plane) is always embeddable in $M \times S^1$, M an orientable surface, can be used to show that U_{2k}, $k \geq 2$, is embeddable in $X' = X((g), (\alpha'_1,\beta'_1), \ldots, (\alpha'_n,\beta'_n))$ if $g > 0$. This embedding is such that the mod 2 homology classes $[U_{2k}]_2 \in H_2(X',\mathbb{Z}_2)$ and $[H]_2 \in H_1(X',\mathbb{Z}_2)$ have zero intersection number, where H is a principal orbit.

Let X' be an orientable closed 3-dimensional S^1-manifold without fixpoints, and let H be a principal orbit in X'. Let the non-orientable surface U_h be embedded in X' in such a way that $[U_h]_2$ and $[H]_2$ have non-zero intersection number. In this case we can obtain weak results on the minimum possible value of the genus h.

Theorem 18.1. If X' and U_h are as above, then X' is of the form $X' = X(\mathcal{X}')$ with

$$\mathcal{X}' = ((g), (\alpha_1, 2\beta_1), \ldots, (\alpha_n, 2\beta_n))$$

and all α_i odd. Let

$$\mathcal{X} = ((g), (\alpha_1, \beta_1), \ldots, (\alpha_n, \beta_n)).$$

Then

$$h \geq |\alpha(X(\mathcal{X}), J)| + 1.$$

Proof. The 1-codimensional submanifold $U = U_h$ of X' defines in the well known way a 2-fold unbranched covering $\pi: X \to X'$ of X' with the property that if J denotes the non-trivial covering transformation of X, then $W = \pi^{-1}(U)$ is a characteristic submanifold for (X,J).

The lifting theorem for paths gives that X has an S^1-action such that any orbit in X is mapped by π onto an orbit of S^1 in X'. The fact that $[U]_2 \cdot [H]_2 \neq 0$ gives that the map π restricted to a typical principal orbit of X is a 2-fold covering of a principal orbit in X', whence follows that J is the involution contained in the S^1-action on X. In particular X is connected, say

$$(18.1) \qquad X = X((g), (\alpha_1, \beta_1), \ldots, (\alpha_n, \beta_n)),$$

and by §8 the α_i are odd, and

$$(18.2) \qquad X' = X((g), (\alpha_1, 2\beta_1), \ldots, (\alpha_n, 2\beta_n)).$$

Let A be the compact submanifold of X with $A \cup JA = X$ and $A \cap JA = \partial A = W$, and define

$$\mathcal{L} = \mathrm{kern}(i_*: H_1(W,\mathbb{Q}) \to H_1(A,\mathbb{Q})),$$

where $i: W \subset A$ is the inclusion. Since $\alpha(X,J)$ is the signature of a quadratic form on $\mathcal{L}$, $|\alpha(X,J)| \leq \dim \mathcal{L}$, so the theorem is proved once we show that

$$(18.3) \qquad \dim \mathcal{L} = h-1.$$

Consider the following exact sequences, which follow from the exact reduced homology sequences for the pairs (X',U) and (A,W) (coefficients in $\mathbb{Q}$).

$$(18.4) \qquad 0 \to H_2(X') \to H_2(X',U) \to H_1(U) \to H_1(X') \to H_1(X',U) \to 0$$

(18.5) $0 \to H_3(A,W) \to H_2(W) \to H_2(A) \to H_2(A,W) \to \mathcal{L} \to 0$.

These give the following relations for rational Betti numbers :

(18.6) $b_2(X') - b_2(X',U) + b_1(U) - b_1(X') + b_1(X',U) = 0$,

(18.7) $b_3(A,W) - b_2(W) + b_2(A) - b_2(A,W) + \dim \mathcal{L} = 0$.

$b_2(X') = b_1(X')$ by the Poincaré isomorphism and exact coefficient sequence. Also $b_j(X',U) = b_j(A,W)$ for all j , since one can replace U by a thin tubular neighbourhood of U and then excise the interior of this neighbourhood to get a homology equivalence between the pairs (X',U) and (A,W) . Hence (18.6) reduces to

(18.8) $-b_2(A,W) + b_1(U) + b_1(A,W) = 0$.

Also, $b_2(A) = b_1(A,W)$ by the Poincaré isomorphism and exact coefficient sequence, and $b_3(A,W) = b_2(W) = 1$, so (18.7) reduces to

(18.9) $b_1(A,W) - b_2(A,W) + \dim \mathcal{L} = 0$.

(18.8) and (18.9) together give that $\dim \mathcal{L} = b_1(U)$. Since $b_1(U) = \text{genus}\, U - 1 = h-1$, (18.3) is thereby proved. ||

<u>Example.</u> $X' = L(p,q)$ $(p,q>0)$.

If U_h is embedded in $L(p,q)$ then by Bredon and Wood (loc. cit.) p is even, say

$$L(p,q) = L(2k,q) ,$$

and for a suitable S^1-action on $L(2k,q)$, $[U_h]_2 \circ [H]_2 \neq 0$ (in fact this holds for any fixpoint free S^1-action on $L(2k,q)$). Hence by

theorems 18.1 and 17.2

$$h \geq |c(q,k)| + 1 .$$

Bredon and Wood have actually calculated the minimum possible value of h in this case. It is the integer valued function $N(2k,q)$ defined by the properties

$$N(2k,1) = k ,$$
$$N(2k,q) = N(2k,q-2k) \quad (q > 2k) ,$$
$$N(2(k-q),q) = N(2k,q) - 1 \quad (q < k) .$$

We have thus shown that

$$N(2k,q) \geq |c(q,k)| + 1 ,$$

and equality holds if and only if the bound of theorem 18.1 is exact for $X' = L(2k,q)$. Comparison of $N(2k,q)$ and $|c(q,k)| + 1$ (see Appendix II), shows that our bound is not very good except for small values of k .

Let X' be as in theorem 18.1 and denote by $N(X')$ the smallest genus of a non-orientable surface U which can be embedded in X' such that $[U]_2 \circ [H]_2 \neq 0$. Then theorem 18.1 says that

(18.10) $N(X') \geq |\alpha(X,J)| + 1$,

where (X,J) is as in theorem 18.1.

By actually embedding a non-orientable surface in X' one can obtain an upper bound for $N(X')$. We could use the characteristic submanifold for (X,J) constructed in §14 and project it down to $X' = X/\mathbb{Z}_2$. If $X' = L(2k,q)$, then by choosing the S^1-action on $X = L(k,q)$ suitably, one obtains a surface of genus $N(2k,q)$ in this way for $k \leq 14$, but for $L(30,11)$ the best one obtains this way is a surface of genus 5, while $N(30,11) = 3$. One can improve this by using, instead of the method of §14, the method of Bredon and Wood to define the surface within the tubes $T_1/\mathbb{Z}_2, \dots, T_n/\mathbb{Z}_2$ (T_i as in §14). This leads to the bound

$$(18.11) \qquad N(X') \leq 2g + \sum_{j=1}^{n} N(2|\beta_j|, \alpha_j) .$$

By varying the $(\alpha_j, 2\beta_j)$ in accordance with lemma 7.2 one can optimise this bound, and if $g=0$, then (18.10) and (18.11) often do (but in general do not) yield a precise value for $N(X')$.

Appendix I: Table of $c(\alpha,\beta)$

$\alpha \backslash \beta$	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26
1	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25
3	0	-1		1	0		2	1		3	2		4	3		5	4		6	5		7	6		8	7
5	0	1	-2	-1		1	2	-1	0		2	3	0	1		3	4	1	2		4	5	2	3		5
7	0	-1	2	-3	0	-1		1	0	3	-2	1	0		2	1	4	-1	2	1		3	2	5	0	3
9	0	1		3	-4		-2	-1		1	2		4	-3		-1	0		2	3		5	-2		0	1
11	0	-1	-2	1	4	-5	-2	1	0	-1		1	0	-1	2	5	-4	-1	2	1	0		2	1	0	3
13	0	1	2	-1	0	5	-6	-1	0	-3	-2	-1		1	2	3	0	1	6	-5	0	1	-2	-1	0	
15	0	-1		-3			6	-7			2		0	-1		1	0		-2			7	-6			3
17	0	1	-2	3	0	1	2	7	-8	-3	-2	-1	-4	1	-2	-1		1	2	-1	4	1	2	3	8	-7
19	0	-1	2	1	-4	-1	2	1	8	-9	-2	-3	0	3	-2	-3	0	-1		1	0	3	2	-3	0	3
21	0	1		-1	4			-1		9	-10		0			-5	0		-2	-1		1	2		0	5
23	0	-1	-2	-3	0	-5	-2	1	0	3	10	-11	-4	-1	-2	1	4	-1	2	1	0	-1		1	0	-1
25	0	1	2	3		5	-1	-2	0		2	11	-12	-3		-1	0	1	-6		-4	-3	-2	-1		1
27	0	-1		1	0		-6	1		3	2		12	-13		-3	-4		-2	5		-1	-2		0	-1

Appendix II: Table of $N(2\beta,\alpha) - (|c(\alpha,\beta)|+1)$

$\alpha \backslash \beta$	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26
1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
3	0	0		0	2		0	2		0	2		0	2		0	2		0	2		0	2		0	2
5	0	0	0	0		0	0	2	2		0	0	4	2		0	0	4	2		0	0	4	2		0
7	0	0	0	0	2	0		0	2	0	2	2	2		0	2	0	4	2	2		0	2	0	6	2
9	0	0		0	0		0	0		0	0		0	2		2	2		0	0		0	4		4	2
11	0	0	0	0	0	0	0	2	2	0		0	2	2	0	0	2	2	2	2	2		0	2	4	0
13	0	0	0	0	2	0	0	2	2	0	0	0		0	0	0	2	2	0	2	4	2	2	2	2	
15	0	0		0			0	0			2		2	0		0	2		2			0	2			2
17	0	0	0	0	2	0	0	0	0	0	0	2	0	2	0	0		0	0	2	0	2	0	0	0	2
19	0	0	0	0	0	0	0	2	0	0	2	0	2	2	0	0	2	0		0	2	0	0	2	2	0
21	0	0		0	0			2		0	0		4			0	2		0	0		0	0		2	0
23	0	0	0	0	2	0	0	0	2	0	0	0	0	2	0	2	2	2	2	2	2	0		0	2	2
25	0	0	0	0		0	0	0	2		2	0	0	2		2	2	2	0		0	0	0	0		0
27	0	0		0	2		0	2		0	0		0	0		0	0		2	2		2	0		2	0

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Institut für Angewandte Mathematik
der Universität Bonn