Realizing arithmetic invariants of hyperbolic 3–manifolds

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These are notes based on the course of lectures on arithmetic invariants of hyperbolic manifolds given at the workshop associated with the last of three "Volume Conferences," held at Columbia University, LSU Baton Rouge, and Columbia University respectively in March 2006, May/June 2007, June 2009.

The first part of the lecture series was expository, and since most of the material is readily available elsewhere, we move rapidly over it here (the very first lecture was a rapid introduction to algebraic number theory, here compressed to less than 2 pages, but hopefully sufficient for the topologist who has never had a course in algebraic number theory). Section 2 on arithmetic invariants has some new material, while Section 3 describes a question that Alan Reid and the author first asked about 20 years ago, and describes a very tentative approach. It is here promoted to a conjecture, in part because the author believes he is safe from contradiction in his lifetime.

In its simplest form the conjecture says:

Conjecture 1. *Every non-real concrete number field k and every quaternion algebra over it arise as the invariant trace field and invariant quaternion algebra of some hyperbolic manifold.*

With an excess of optimism, one might add to the conjecture that, moreover, every set of primes of $\mathcal{O}_k$ arises as the set of primes in denominators in the invariant trace ring of one of these hyperbolic manifolds.

Section 3 describes the already mentioned tentative first step for a program of proof, which the author has revisited over many years without significant advance.

We discuss also the question whether the Bloch invariants of manifolds with a given invariant trace field k generate the Bloch group $\mathcal{B}(k)$ for that number field, or even whether their extended Bloch invariants generate $K_3^{\mathrm{ind}}(k)$.

1. Notation and terminology for algebraic number theory

1.1. Number fields. A *number field* K is a finite extension of $\mathbb{Q}$. That is, K is a field containing $\mathbb{Q}$, and finite-dimensional as a vector space over $\mathbb{Q}$. This

dimension d, denoted $d = [K : \mathbb{Q}]$, is the *degree* of the number field. K has exactly d embeddings into the complex numbers,

$$\theta_i \colon K \to \mathbb{C}, \qquad i = 1, \ldots, d = r_1 + 2r_2 \,,$$

where r_1 is the number of them with real image, and the remaining embeddings come in r_2 complex conjugate pairs. Indeed, the "Theorem of the Primitive Element" implies that K is generated over $\mathbb{Q}$ by a single element, from which if follows that $K \cong \mathbb{Q}[x]/(f(x))$ with $f(x)$ an irreducible polynomial of degree d; the embeddings $K \to \mathbb{C}$ arise by mapping the generator x of K to each of the d zeros in $\mathbb{C}$ of $f(x)$.

A *concrete number field* is a number field K with a chosen embedding into $\mathbb{C}$, i.e., K given as a subfield of $\mathbb{C}$. The union of all concrete number fields is the field of algebraic numbers in $\mathbb{C}$, which is the concrete algebraic closure $\overline{\mathbb{Q}} \subset \mathbb{C}$ of $\mathbb{Q}$.

An *algebraic integer* is a zero of a monic polynomial with rational integer coefficients. The algebraic integers in K form a subring $\mathcal{O}_K \subset K$, the *ring of integers of K*. It is a Dedekind domain, which is to say that any ideal in $\mathcal{O}_k$ factors uniquely as a product of prime ideals. Each prime ideal $\mathfrak{p}$ (or "prime" for short) of $\mathcal{O}_K$ is a divisor of a unique ideal (p) with $p \in \mathbb{Z}$ a rational prime (determined by $|\mathcal{O}_K/\mathfrak{p}| = p^e$ for some $e > 0$). The factorization of (p) as a product $(p) = \mathfrak{p}_1^{f_1} \ldots \mathfrak{p}_k^{f_k}$ of primes of $\mathcal{O}_K$ follows patterns which can be found in any text on algebraic number theory. In particular, the exponents f_i are 1 for all but a finite number of primes $\mathfrak{p}$ of $\mathcal{O}_K$, which are called *ramified*.

For the ring of integers $\mathcal{O}_{\mathbb{Q}} = \mathbb{Z}$ of $\mathbb{Q}$, every ideal is principal, and the factorization of the ideal (n) into a product of ideals (p_i) expresses the familiar unique prime factorization of rational integers. In general $\mathcal{O}_K$ is a unique factorization domain (UFD) if and only if it is a PID (every ideal is principal), which is somewhat rare. It is presumed to happen infinitely often, but this is not proven.

Given a prime $\mathfrak{p}$ of $\mathcal{O}_k$, there is a multiplicative norm $||.||_{\mathfrak{p}}$ defined for $a \in \mathcal{O}_K$ by $||a||_{\mathfrak{p}} := c^{-r}$, where $\mathfrak{p}^r$ is the largest power of $\mathfrak{p}$ which "divides" a (i.e., contains a) and $c > 1$ is some constant[1]; the norm is then determined for arbitrary elements of K by the multiplicative property $||ab||_{\mathfrak{p}} = ||a||_{\mathfrak{p}}||b||_{\mathfrak{p}}$. This norm determines a translation invariant topology on K and the completion of K in this topology is a field denoted $K_{\mathfrak{p}}$. The unit ball around 0 in $K_{\mathfrak{p}}$ is its *ring of integers $\mathcal{O}_{K_{\mathfrak{p}}}$*, and the open unit ball is the unique maximal ideal in this ring. The norm $||.||_{\mathfrak{p}}$ is *non-Archimedean*, i.e., it satisfies the strong triangle inequality $||a + b||_{\mathfrak{p}} \leq \max(||a||_{\mathfrak{p}}, ||b||_{\mathfrak{p}})$. Up to equivalence (norms are *equivalent* if one is a positive power of the other), the only non-Archimedean multiplicative norms are the ones just described, and the only other multiplicative norms on K are the norms $||a||_{\theta} := |\theta(a)|$ given by absolute value in $\mathbb{C}$ for an embedding $\theta \colon K \to \mathbb{C}$. The completion of K in the topology induced by one of these is $\mathbb{R}$ or $\mathbb{C}$ according as the image of θ lies in $\mathbb{R}$ or not.

The fields $\mathbb{R}$, $\mathbb{C}$, $K_{\mathfrak{p}}$ arising from completions are *local fields*[2]. The name is geometrically motivated: one thinks of $\mathcal{O}_K$ as a ring of functions on a "space" with a "finite point" for each prime ideal, plus $r_1 + r_2$ "infinite points" corresponding to the embeddings in $\mathbb{R}$ and $\mathbb{C}$; "local" means focusing on an individual point. One therefore refers to an embedding of K into K_p as a "finite place" and an embedding

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[1]The value of c is unimportant for topological considerations but is standardly taken as $c = N(\mathfrak{p}) := |\mathcal{O}_K/\mathfrak{p}|$

[2]The definition of local field is: non-discrete locally compact topological field. The ones mentioned here are all that exist in characteristic 0.

into $\mathbb{R}$ or $\mathbb{C}$ as an "infinite place," and if an object A associated with K (e.g., an algebra A over K) has corresponding objects associated to each place (e.g., $A \otimes K_{\mathfrak{p}}$, $A \otimes \mathbb{R}$, $A \otimes \mathbb{C}$) then a "property of A at the (finite or infinite) place" means that property for the associated object. We stress that an "infinite place" refers to the embedding of K in $\mathbb{C}$ up to conjugation (even though conjugate embeddings may have different images), so there are r_1 real places and just r_2 complex places.

1.2. Quaternion algebras. References for this section are [**25**] and [**5**]. A *quaternion algebra* over a field K is a simple algebra over K of dimension 4 and with center K. The simplest example is the algebra $M_2(K)$ of 2×2 matrices over K. This is the only quaternion algebra up to isomorphism for $K = \mathbb{C}$. For $K = \mathbb{R}$ there are exactly two, namely $M_2(\mathbb{R})$ and the Hamiltonian quaternions. The situation for the non-Archimedean local fields $K_{\mathfrak{p}}$ is similar: there are exactly two quaternion algebras over each of them, one being the trivial one $M_2(K_{\mathfrak{p}})$ and the other being a division algebra. In each case the trivial quaternion algebra M_2 is called *unramified* and the division algebra is called *ramified*. For a number field K the classification of quaternion algebras over K is as follows:

Theorem 1.1 (Classification)**.** *A quaternion algebra E over K is ramified at only finitely many places (i.e., only finitely many of the $E \otimes K_{\mathfrak{p}}$ and $E \otimes \mathbb{R}$'s are division algebras) and is determined up to isomorphism by the set of these "ramified places." The number of ramified places is always even, and every set of places of K of even size arises as the set of ramified places of a quaternion algebra over K.*

A quaternion algebra E over K can always be given in terms of generators and relations in the form

$$E = K\langle i, j : i^2 = \alpha, \ j^2 = \beta, \ ij = -ji\rangle ,$$

with $\alpha, \beta \in K^*$. The *Hilbert symbol* notation $\left\{\frac{\alpha,\beta}{K}\right\}$ refers to this quaternion algebra. For example, $\left\{\frac{-1,-1}{\mathbb{R}}\right\}$ is Hamilton's quaternions, and $\left\{\frac{1,\beta}{K}\right\} = M_2(K)$ for any K. The Hilbert symbol for a given quaternion algebra is far from unique, but computing the ramification—and hence the isomorphism class—of a quaternion algebra from the Hilbert symbol is not hard, and is described in [**25**], see also [**5**] for a description tailored to 3-manifold invariants.

In terms of the above presentation, the map $i \mapsto -i, j \mapsto -j, ij \mapsto -ij$ of a quaternion algebra E to itself is an anti-automorphism called conjugation, and the *norm* of $x = a + ib + jc + ijd \in E$ is defined as $N(x) := x\bar{x} = a^2 + \alpha b^2 + \beta c^2 + \alpha\beta d^2$.

1.3. Arithmetic subgroups of $\mathrm{SL}(2, \mathbb{C})$ and $\mathrm{PSL}(2, \mathbb{C})$. For a quaternion algebra E over K the set $\mathcal{O}_E$ of integers of E (elements which are zeros of monic polynomials with coefficients in $\mathcal{O}_K$) does not form a subring. One considers instead an *order* in E: any subring $\mathcal{O}$ of E, contained in $\mathcal{O}_E$ and containing $\mathcal{O}_K$ and of rank 4 over $\mathcal{O}_K$. E has infinitely many orders; we just pick one of them.

The subset $\mathcal{O}^1 \subset \mathcal{O}$ of elements of norm 1 is a subgroup. At any complex place, E becomes $E \otimes \mathbb{C} = M_2(\mathbb{C})$ and $\mathcal{O}^1$ becomes a subgroup of $\mathrm{SL}(2, \mathbb{C})$, while at an unramified real place E becomes $E \otimes \mathbb{R} = M_2(\mathbb{R})$ and $\mathcal{O}^1$ becomes a subgroup of $\mathrm{SL}(2, \mathbb{R})$. We thus get an embedding of $\Gamma := \mathcal{O}^1/\{\pm 1\}$

$$\Gamma \subset \prod_{i=1}^{r_2} \mathrm{PSL}(2, \mathbb{C}) \times \prod_{j=1}^{r_1^u} \mathrm{PSL}(2, \mathbb{R}) ,$$

where r_1^u is the number of unramified real places of K. This subgroup is a lattice (discrete and of finite covolume).

If $r_1^u = 0$ and $r_2 = 1$ this gives an arithmetic subgroup of $\mathrm{PSL}(2,\mathbb{C})$, and similarly for $r_1^u = 1, r_2 = 0$ and $\mathrm{PSL}(2,\mathbb{R})$. Up to commensurability this group only depends on E and not on the choice of order $\mathcal{O}$. Any subgroup commensurable with an arithmetic subgroup—i.e., sharing a finite index subgroup with it up to conjugation—is, by definition, also arithmetic.

The general definition of an arithmetic group is in terms of the set of $\mathbb{Z}$-points of an algebraic group which is defined over $\mathbb{Q}$. But Borel shows in [1] that all arithmetic subgroups of $\mathrm{PSL}(2,\mathbb{C})$ (and $\mathrm{PSL}(2,\mathbb{R})$) can be obtained as above.

Arithmetic orbifolds (orbifolds $\mathbb{H}^3/\Gamma$ with Γ arithmetic) are very rare—there are only finitely many of bounded volume—but surprisingly common among the manifolds and orbifolds of smallest volume.

2. Arithmetic invariants of hyperbolic manifolds

2.1. Invariant trace field and quaternion algebra. A *Kleinian group* Γ is a discrete subgroup of $\mathrm{PSL}(2,\mathbb{C}) = \mathrm{Isom}^+(\mathbb{H}^3)$ for which $M = \mathbb{H}^3/\Gamma$ is finite volume (M may be an orbifold). Let $\overline{\Gamma} \subset \mathrm{SL}(2,\mathbb{C})$ be the inverse image of Γ under the projection $\mathrm{SL}(2,\mathbb{C}) \to \mathrm{PSL}(2,\mathbb{C})$.

Definition 2.1. The *trace field* of Γ (or of $M = \mathbb{H}^3/\Gamma$) is the field $\mathrm{tr}(\Gamma)$ generated by all traces of elements of $\overline{\Gamma}$. We also write $\mathrm{tr}(M)$.

The *invariant trace field* is the field $k(\Gamma) := \mathrm{tr}(\Gamma^{(2)})$ where $\Gamma^{(2)}$ is the group generated by squares of elements of Γ. It can also be computed as $k(\Gamma) = \mathbb{Q}(\{(\mathrm{tr}(\gamma))^2 \mid \gamma \in \overline{\Gamma}\})$ ([23], see also [19]). We also write $k(M)$.

The *invariant quaternion algebra* is the $k(\Gamma)$-subalgebra of $M_2(\mathbb{C})$ (2×2 matrices over $\mathbb{C}$) generated over $k(\Gamma)$ by the elements of $\overline{\Gamma}^{(2)}$. It is denoted $A(\Gamma)$ or $A(M)$.

Theorem 2.2. $k(\Gamma)$ *and* $A(\Gamma)$ *are commensurability invariants of* Γ.

If Γ is arithmetic, then $k(\Gamma)$ and $A(\Gamma)$ equal the defining field and defining quaternion algebra of Γ, so they form a complete commensurability invariant. They are not a complete commensurability invariant in the non-arithmetic case.

An obvious necessary condition for arithmeticity is that $k(\Gamma)$ have only one non-real complex embedding (it always has at least one). Necessary and sufficient is that in addition all traces should be algebraic integers and $A(\Gamma)$ should be ramified at all real places of k. See [23]. Equivalently, each $\gamma \in \overline{\Gamma}$ trace(γ^2) should be an algebraic integer whose absolute value at all real embeddings of k is bounded by 2.

These invariants are already quite powerful invariants of a hyperbolic manifold. For example, if a hyperbolic manifold M is commensurable with an amphichiral manifold N (i.e., N has an orientation reversing self-homeomorphism) then $k(M) = \overline{k(M)}$ and $A(M) = \overline{A(M)}$ (complex conjugation).

If M has cusps then the invariant quaternion algebra is always unramified, so it gives no more information than the invariant trace field, but for closed M unramified invariant quaternion algebras are uncommon; for example among the almost 40 manifolds in the Snappea closed census [26] which have invariant trace field $\mathbb{Q}(\sqrt{-1})$, only two have unramified quaternion algebra.

2.2. The PSL-fundamental class. For details on what we discuss here see [**18, 21, 22**] or the expository article [**17**].

2.2.1. *PSL-fundamental class of a hyperbolic manifold.* The *PSL-fundamental class* of M is a homology class

$$[M]_{PSL} \in H_3(\mathrm{PSL}(2,\mathbb{C})^\delta; \mathbb{Z}) \,,$$

where the superscript δ means "with discrete topology".

This class is easily described if M is compact. Write $M = \mathbb{H}^3/\Gamma$ with $\Gamma \subset \mathrm{PSL}(2,\mathbb{C})$. The PSL-fundamental class is the image of the fundamental class of M under the map $H_3(M;\mathbb{Z}) = H_3(\Gamma; Z) \to H_3(\mathrm{PSL}(2,\mathbb{C})^\delta; \mathbb{Z})$, where the first equality is because M is a $K(\Gamma, 1)$-space. If M has cusps one obtains first a class in $H_3(\mathrm{PSL}(2,\mathbb{C})^\delta, P; \mathbb{Z})$, where P is a maximal parabolic subgroup of $\mathrm{PSL}(2,\mathbb{C})^\delta$. One then uses a natural splitting of the map

$$H_3(\mathrm{PSL}(2,\mathbb{C})^\delta; \mathbb{Z}) \to H_3(\mathrm{PSL}(2,\mathbb{C})^\delta, P; \mathbb{Z})$$

to get $[M]_{\mathrm{PSL}}$. This was described in [**18**] and proved carefully by Zickert in [**8**], who shows that the class in $H_3(\mathrm{PSL}(2,\mathbb{C})^\delta, P; \mathbb{Z})$ depends on choices of horoballs at the cusps, but the image $[M]_{\mathrm{PSL}} \in H_3(\mathrm{PSL}(2,\mathbb{C})^\delta; \mathbb{Z})$ does not.

The group $\Gamma \subset \mathrm{PSL}(2,\mathbb{C})$ can be conjugated to lie in $\mathrm{PSL}(2,K)$ for a number field K (which can always be chosen to be a quadratic extension of the trace field, but there is generally no canonical choice), so the PSL-fundamental class is then defined in $H_3(\mathrm{PSL}(2,K); \mathbb{Z})$.

The following theorem, which holds also with PSL replaced by SL, summarizes results of various people, see [**22**] and [**27**] for more details.

Theorem 2.3. *$H_3(\mathrm{PSL}(2,\mathbb{C}); \mathbb{Z})$ is the direct sum of its torsion subgroup, isomorphic to $\mathbb{Q}/\mathbb{Z}$, and an infinite dimensional $\mathbb{Q}$ vector space.*

If $k \subset \mathbb{C}$ is a number field then $H_3(\mathrm{PSL}(2,k); \mathbb{Z})$ is the direct sum of its torsion subgroup and $\mathbb{Z}^{r_2}$, where r_2 is the number of conjugate pairs of complex embeddings of k. Moreover, the map $H_3(\mathrm{PSL}(2,k); \mathbb{Z}) \to H_3(\mathrm{PSL}(2,\mathbb{C}); \mathbb{Z})$ has torsion kernel.

The *Rigidity Conjecture*, which is about 30 years old (see [**17**] for a discussion), posits that each of the following equivalent statements is true:

Conjecture 2. (1) $H_3(\mathrm{PSL}(2,\mathbb{C})^\delta; \mathbb{Z})$ *is countable.*
(2) $H_3(\mathrm{PSL}(2,\overline{\mathbb{Q}})^\delta; \mathbb{Z}) = H_3(\mathrm{PSL}(2,\mathbb{C})^\delta; \mathbb{Z})$
(3) $H_3(\mathrm{PSL}(2,\mathbb{C})^\delta; \mathbb{Z})$ *is the union of the images of the maps $H_3(\mathrm{PSL}(2,K); \mathbb{Z}) \to H_3(\mathrm{PSL}(2,\mathbb{C})^\delta; \mathbb{Z})$, as K runs through all concrete number fields.*

2.3. Invariants of the PSL-fundamental class. There is a homomorphism

$$\hat{c} \colon H_3(\mathrm{PSL}(2,\mathbb{C}); \mathbb{Z}) \to \mathbb{C}/\pi^2\mathbb{Z}$$

called the "Cheeger-Simons class" ([**3**]) whose real and imaginary parts give Chern-Simons invariant and volume:

$$\hat{c}([M]_{PSL}) = cs(M) + i\,\mathrm{vol}(M) \,.$$

The Chern-Simons invariant here is the Chern-Simons invariant of the flat connection, which is defined for any complete hyperbolic manifold M of finite volume. If M is closed the Riemannian Chern-Simons invariant $\mathrm{CS}(M) \in \mathbb{R}/2\pi^2$ is also defined; it reduces to $cs(M) \bmod \pi^2$. See [**18**] for details.

We denote the homomorphisms given in the obvious way by the real and imaginary parts of $\hat{c}$ by:

$$\mathrm{cs} \colon H_3(\mathrm{PSL}(2,\mathbb{C});\mathbb{Z}) \to \mathbb{R}/\pi^2\mathbb{Z}, \qquad \mathrm{vol} \colon H_3(\mathrm{PSL}(2,\mathbb{C});\mathbb{Z}) \to \mathbb{R}\,.$$

The homomorphism cs is injective on the torsion subgroup of $H_3(\mathrm{PSL}(2,\mathbb{C});\mathbb{Z})$. A standard conjecture that appears in many guises in the literature (see [17] for a discussion) is:

Conjecture 3. *The Cheeger-Simons class is injective. That is, volume and Chern-Simons invariant determine elements of $H_3(\mathrm{PSL}(2,\mathbb{C});\mathbb{Z})$ completely.*

If k is an algebraic number field and $\sigma_1,\ldots,\sigma_{r_2}\colon k \to \mathbb{C}$ are its different complex embeddings up to conjugation then denote by vol_j the composition

$$\mathrm{vol}_j = \mathrm{vol}\circ(\sigma_j)_* \colon H_3(\mathrm{PSL}(2,k);\mathbb{Z}) \to \mathbb{R}.$$

The map

$$\mathrm{Borel} := (\mathrm{vol}_1,\ldots,\mathrm{vol}_{r_2})\colon H_3(\mathrm{PSL}(2,k);\mathbb{Z}) \to \mathbb{R}^{r_2}$$

is called the *Borel regulator*.

Theorem 2.4. *The Borel regulator maps $H_3(\mathrm{PSL}(2,k);\mathbb{Z})/\mathrm{Torsion}$ injectively onto a full sublattice of $\mathbb{R}^{r_2}$.*

By Theorem 2.3 and the discussion above, $\mathrm{cs}(M) \in \mathbb{R}/\mathbb{Z}$ and $\mathrm{Borel}([M]_{PSL}) \in \mathbb{R}^{r_2(k)}$ determine the PSL-fundamental class $[M]_{PSL} \in H_3(\mathrm{PSL}(2,\mathbb{C});\mathbb{Z})$ completely.

These invariants are computed by the program Snap (see [5]). Snap does this via a more easily computed invariant which we describe next.

2.4. Bloch group and Bloch invariant. The *Bloch group* $\mathcal{B}(\mathbb{C})$ is the quotient of $H_3(\mathrm{PSL}(2,\mathbb{C})^\delta;\mathbb{Z})$ by its torsion subgroup. It has the advantage that it has a simple symbolic description and the image of $[M]_{\mathrm{PSL}}$ in $\mathcal{B}(\mathbb{C})$ is readily computed from an ideal triangulation.

The Bloch group is defined for any field. There are different definitions of it in the literature; they differ at most by torsion and agree with each other for algebraically closed fields (see, e.g., [7]). We use the following.

Definition 2.5. Let K be a field. The *pre-Bloch group* $\mathcal{P}(K)$ is the quotient of the free $\mathbb{Z}$-module $\mathbb{Z}(K - \{0,1\})$ by all instances of the following relation:

$$[x] - [y] + [\frac{y}{x}] - [\frac{1-x^{-1}}{1-y^{-1}}] + [\frac{1-x}{1-y}] = 0,$$

called the *five term relation*. The *Bloch group* $\mathcal{B}(K)$ is the kernel of the map

(1) $$\mathcal{P}(k) \to K^* \wedge_\mathbb{Z} K^*, \quad [z] \mapsto 2(z \wedge (1-z))\,.$$

Suppose we have an ideal triangulation of a hyperbolic 3-manifold M using ideal hyperbolic simplices with cross ratio parameters $z_1,\ldots,z_n$. This ideal triangulation can be a genuine ideal triangulation of a cusped 3-manifold, or a deformation of such a one as used by Snap and SnapPea to study Dehn filled manifolds, but it may be more generally any "degree one triangulation"; see [22].

Definition 2.6. The *Bloch invariant* $\beta(M)$ is the element $\sum_1^n \pm[z_j] \in \mathcal{P}(\mathbb{C})$ with signs as explained below. It lies in $\mathcal{B}(\mathbb{C})$ by [22].

The cross-ratio parameter of an ideal simplex depends on a chosen ordering of the vertices, and the sign in the above sum reflects whether or not this ordering orients the simplex as it is oriented as part of the degree one triangulation.

If the z_j's all belong to a subfield $K \subset \mathbb{C}$, we may consider $\beta(M)$ as an element of $\mathcal{B}(K)$. But it is then necessary to assume that the vertex orderings of the simplices match on common faces. If not, then $\sum_1^n \pm[z_j]$ may differ from $\beta(M)$ by a torsion element (of order dividing 12; this torsion issue does not arise in $\mathcal{B}(\mathbb{C})$, which is torsion-free). Not every triangulation has compatible vertex-orderings for the simplices, although a triangulation can always be refined to one which does.

Theorem 2.7. *If M has cusps then $\beta(M)$ is actually defined in $\mathcal{B}(k)$, for the invariant trace field k, while if M is closed this holds for $2\beta(M)$.*

This was known in the cusped case (the simplex parameters of an ideal triangulation then lie in k, see [**19**]) but it was only known up to a higher power of 2 in the closed case ([**17, 22**]). The proof, joint with Zickert (but mostly Zickert), is at the end of this subsection.

Since $\beta(M)$ only loses torsion information over $[M]_{\mathrm{PSL}}$, the Borel regulator $\mathrm{Borel}(M)$ can be computed from $\beta(M)$. It is computed from the simplex parameters z_i as follows. The z_i generate a field K which contains the invariant trace field k of M. The j-th component $\mathrm{vol}_j([M]_{PSL})$ of $\mathrm{Borel}(M)$ is

$$\mathrm{Borel}(M)_j = \sum_{i=1}^{n} \pm D_2(\tau_j(z_i)),$$

where $\tau_j \colon K \to \mathbb{C}$ is any complex embedding which extends $\sigma_j \colon k \to \mathbb{C}$. Here the signs are as above, and D_2 is the "Wigner dilogarithm function"

$$D_2(z) = \mathrm{Im}\ln_2(z) + \log|z|\arg(1-z), \quad z \in \mathbb{C} - \{0,1\},$$

where $\ln_2(z)$ is the classical dilogarithm function. $D_2(z)$ can also be defined as the volume of the ideal simplex with parameter z.

Recall that k is a concrete number field, i.e., it comes as a subfield of $\mathbb{C}$. The component of $\mathrm{Borel}(M)$ corresponding to this embedding in $\mathbb{C}$ is $\pm \mathrm{vol}(M)$, and it has maximal absolute value among the components of $\mathrm{Borel}(M)$ (see [**22**]). This restricts which elements of $\mathcal{B}(k)$ can be the Bloch invariant of a hyperbolic 3-manifold. A related (and conjecturally equivalent) restriction is in terms of the Gromov norm, which is defined on $\mathcal{B}(k)$ (see [**22**]); the Bloch invariants of hyperbolic manifolds are constrained to lie in the cone over a single face of the norm ball.

Nevertheless, it is plausible that the Bloch group can be generated by Bloch invariants of 3-manifolds. No obstructions to this are known, and there is (very mild) experimental evidence for it for low degree fields which appear as invariant trace fields of manifolds in the cusped and Snappea closed censuses [**2, 26**]; some computations related to this are in [**5**]. So we ask:

Question 2.8. *Is $\mathcal{B}(k)$ generated by Bloch invariants of hyperbolic manifolds with invariant trace field in k; how about $\mathcal{B}(k) \otimes \mathbb{Q}$ over $\mathbb{Q}$?*

PROOF OF THEOREM 2.7. (See also [**28**].) Since the theorem is known in the cusped case we assume M is closed. We will need Suslin's version of the Bloch group [**24**], defined by omitting the factor 2 in the map (1) in the definition above. We will denote it $\mathcal{B}_S(K)$. Clearly, $\mathcal{B}_S(K) \subset \mathcal{B}(K)$, and the quotient $\mathcal{B}(K)/\mathcal{B}_S(K)$

is of exponent 2 (one can show it is infinitely generated if K is a number field). We will actually show that $\beta(M) \in \mathcal{B}_S(k)$.

We will use Suslin's theorem that $\mathcal{B}_S(K)$ is a quotient of $K_3^{\mathrm{ind}}(K)$ by a finite cyclic group ([**24**], see also [**28**]).

The geometric PSL–representation of $\pi_1(M)$ lifts to a representation $\pi_1(M) \to \mathrm{SL}(2, \mathbb{C})$. The set of such lifts is in one-one correspondence with spin structures on M; we just pick one for now. The image Γ of such a lifted representation lies in the quaternion algebra $\mathbb{Q}\Gamma$, which can be unramified by extending its scalars to a quadratic extension field K' of the trace field K (such a K' can be taken as $K(\lambda)$ for any eigenvalue λ of a nontrivial element of $\mathbb{Q}\Gamma$; see e.g., [**14**]). We get $\pi_1(M) \hookrightarrow \mathrm{GL}(2, K')$, leading to a "GL–fundamental class" $[M]_{\mathrm{GL}} \in H_3(GL(2, K'); \mathbb{Z})$. There is a natural map $H_3(\mathrm{GL}(2, K'); \mathbb{Z}) \to K_3^{\mathrm{ind}}(K')$ (see, e.g., [**28**]), and we denote the image of $[M]_{\mathrm{GL}}$ by $[M]_K \in K_3^{\mathrm{ind}}(K')$. Now the non-trivial element of $\mathrm{Gal}(K'/K)$ preserves traces of $\pi_1(M) \to \mathrm{GL}(2, \mathbb{C})$, so it takes this representation to a representation which is equivalent over $\mathbb{C}$. It therefore fixes the Borel invariant and Chern-Simons invariant of $[M]_K$. The Chern-Simons invariant on K_3^{ind} takes values in $\mathbb{C}/4\pi^2\mathbb{Z}$, and this Chern-Simons invariant and Borel invariant together determine any element of $K_3^{\mathrm{ind}}(K')$ (see [**28**]). Thus the class $[M]_K$ is invariant under $\mathrm{Gal}(K'/K)$, and since K_3^{ind} satisfies Galois descent (Merkuriev and Suslin [**15**]), this class lies in $K_3^{\mathrm{ind}}(K)$. We note, however, that a priori $[M]_K$ may depend on which lift $\pi_1(M) \hookrightarrow \mathrm{SL}(2, \mathbb{C})$ we started with.

By [**19**, Theorem 2.2] the trace field K is a multi-quadratic extension of the invariant trace field k, with Galois group $\mathrm{Gal}(K/k) \cong (\mathbb{Z}/2)^r$ for some r. This group permutes the lifts $\pi_1(M) \hookrightarrow \mathrm{SL}(2, \mathbb{C})$ and hence acts on the elements $[M]_K \in K_3^{\mathrm{ind}}(K)$ defined by these lifts. In [**12**] it is shown that $[M]_K$ is changed by at most the unique element of order 2 (and such a change can occur). Thus $2[M]_K$ is invariant under this Galois group, and is hence an invariant of M in $K_3^{\mathrm{ind}}(k)$ which is independent of the lift to $SL(2, \mathbb{C})$.

The Bloch invariant $2\beta(M)$ is the image of $2[M]_K$ under a natural transformation from K_3^{ind} to $\mathcal{B}_S$, so the theorem is proved. $\qquad\square$

Problem 2.9. Find a general explicit way to obtain a representative for $\beta(M)$ in $\mathcal{B}_S(k)$ if M is closed.

Remark 2.10. In [**28**] Zickert points out that $\beta(M) \in \mathcal{B}(k)$ may not be in its subgroup $\mathcal{B}_S(k)$ if M has cusps, in contrast to the closed case. An example is the manifold $M009$ in the census [**2**].

2.5. Extended Bloch group. The *extended Bloch group* $\widehat{\mathcal{B}}(\mathbb{C})$ of [**18**] is defined by replacing $\mathbb{C} - \{0, 1\}$ by a $\mathbb{Z} \times \mathbb{Z}$–cover in the definition of Bloch group, and appropriately lifting the 5-term relation and the map λ. There are two different versions of this defined in [**18**]; we will write $\widehat{\mathcal{B}}_{\mathrm{PSL}}(\mathbb{C})$ for the first and $\widehat{\mathcal{B}}(\mathbb{C})$ for the second (they were denoted $\widehat{\mathcal{B}}(\mathbb{C})$ and $\mathcal{E}\mathcal{B}(\mathbb{C})$ respectively in [**18**]).

Theorem 2.11. *There are natural isomorphisms* $H_3(\mathrm{PSL}(2, \mathbb{C})^\delta; \mathbb{Z}) \cong \widehat{\mathcal{B}}_{\mathrm{PSL}}(\mathbb{C})$ *and* $H_3(\mathrm{SL}(2, \mathbb{C})^\delta; \mathbb{Z}) \cong \widehat{\mathcal{B}}(\mathbb{C}) \cong K_3^{\mathrm{ind}}(\mathbb{C})$. *(See* [**18**] *and* [**9**] *respectively.)*

The program Snap actually computes the element in $\widehat{\mathcal{B}}_{\mathrm{PSL}}(\mathbb{C})$, and then prints the Borel regulator and Chern-Simons invariant, which, as already mentioned, determine this element. In [**27**] Zickert gives a much simpler way of computing the

element of $\widehat{\mathcal{B}}_{\mathrm{PSL}}(\mathbb{C})$ than the one currently used by Snap. This has now been implemented in SnapPy [**6**] by Matthias Goerner.

In [**28**] Zickert extends the definition of extended Bloch groups to number fields and shows

Theorem 2.12. *There is a natural isomorphism $\widehat{\mathcal{B}}(K) \cong K_3^{\mathrm{ind}}(K)$ for any number field K.*

Moreover, as mentioned in the proof of Theorem 2.7, for a closed hyperbolic manifold with spin structure Zickert shows that there is a natural invariant $[M]_K \in \widehat{\mathcal{B}}(k) = K_3^{\mathrm{ind}}(k)$ which lifts the Bloch invariant. If M has cusps he shows that $[M]_K$ is defined in $\widehat{\mathcal{B}}(K) \otimes \mathbb{Z}[\frac{1}{2}]$, where K is the trace field, and his arguments show that $8[M]_K$ is well defined in $\widehat{\mathcal{B}}(k) = K_3^{\mathrm{ind}}(k)$ (it is easily seen that $4[M]_K$ is well defined in $K_3^{\mathrm{ind}}(\mathbb{C})$).

2.6. Mutation. If a hyperbolic manifold M contains an essential 4–punctured sphere then one can cut along this embedded surface and re-glue by an involution, and it is well known that this process, called *Conway mutation* or simply *mutation*, yields a hyperbolic manifold which shares many properties with M, for example it has the same volume and Chern-Simons invariant [**11**], and if M was a knot complement, many knot theoretic invariants are preserved too. It is folk knowledge that the Bloch invariant is preserved (and hence also $[M]_{\mathrm{PSL}}$) but there is no proof in the literature, so we give a direct proof here. There are other types of mutation, sometimes called "generalized mutation." For example one can mutate along any essential embedded 3–punctured sphere.

Theorem 2.13. *If M and M' are hyperbolic manifolds related by Conway mutation, then $[M]_{\mathrm{PSL}} = [M']_{\mathrm{PSL}}$.*

If they are related by mutation on a 3–punctured sphere then $[M]_{\mathrm{PSL}}$ and $[M']_{\mathrm{PSL}}$ differ by the element of order 2 in $H_3(\mathrm{PSL}(2,\mathbb{C})^\delta; \mathbb{Z})$.

For any generalized mutation $[M]_{\mathrm{PSL}}$ and $[M']_{\mathrm{PSL}}$ differ by an element of finite order. Every torsion element of $H_3(\mathrm{PSL}(2,\mathbb{C})^\delta; \mathbb{Z})$ arises this way.

PROOF. Suppose we have an essential embedded two-sided surface $\Sigma \subset M$ which has a tubular neighborhood N which admits a (necessarily finite order) isometry Φ which preserves orientation and sides of Σ. Denoting by $\phi \colon \Sigma \to \Sigma$ the restriction of Φ, let M' be the manifold obtained from M by cutting along Σ and re-gluing by ϕ. This is "generalized mutation." If Σ is a 3– or 4–punctured sphere or a closed surface of genus 2 then it can always be positioned to have a $\mathbb{Z}/2$–symmetry.

Let K be a $K(\mathrm{PSL}(2,\mathbb{C})^\delta, 1)$–space, so $H_3(K; \mathbb{Z}) = H_3(\mathrm{PSL}(2,\mathbb{C})^\delta; \mathbb{Z})$. The holonomy map $\alpha \colon \pi_1(M) \to \mathrm{PSL}(2,\mathbb{C})$ induces a map $\alpha_\sharp \colon M \to K$ (well defined up to homotopy), and similarly we have $\alpha'_\sharp \colon M' \to K$. These maps can be chosen to agree outside the tubular neighborhood of $N \cong \Sigma \times [0,1]$ of Σ. They then give a map $N \cup -N \to K$. Thinking of $N \cup -N$ as $\Sigma \times [0,1] \cup -\Sigma \times [0,1]$, it is glued on one end by the identity and on the other end by ϕ, so it is simply the mapping torus $T_\phi \Sigma$ of $\phi \colon \Sigma \to \Sigma$. Its fundamental group is the semidirect product $\pi_1(\Sigma) \rtimes \mathbb{Z}$ and it is represented into $\mathrm{PSL}(2,\mathbb{C})$ by the homomorphism which on $\pi_1(\Sigma)$ is the restriction of α and on a generator of $\mathbb{Z}$ is an isometry of $\mathbb{H}^3$ which restricts to a lift to $\tilde{N}$ of the isometry $\Phi \colon N \to N$.

The images in $H_3(K; \mathbb{Z}) = H_3(\mathrm{PSL}(2, \mathbb{C})^\delta; \mathbb{Z})$ of the fundamental classes of M, M' and $T_\phi\Sigma$ clearly satisfy $[M]_{\mathrm{PSL}} - [M']_{\mathrm{PSL}} = [T_\phi\Sigma]_{\mathrm{PSL}}$. Since $T_\phi\Sigma$ is n–fold covered by $\Sigma \times S^1$, where n is the order of ϕ, the element $[T_\phi\Sigma]_{\mathrm{PSL}}$ is n–torsion, and is hence determined by the Chern-Simons invariant. It describes the change of $[M]_{\mathrm{PSL}}$ under the corresponding mutation. Moreover, it depends only on the finite order map $\Phi\colon N \to N$ and not on the geometry in a neighborhood of N, since if one deforms the geometry in an equivariant fashion then $cs([T_\Phi\Sigma])$ is a continuously varying n–torsion element in $\mathbb{R}/\pi^2\mathbb{Z}$, hence constant. So to compute it one just needs to compute the change in Chern-Simons invariant for a single example. The following two examples thus complete the proof of the first two sentences of the theorem.

The Conway and Kinoshita-Teresaka knots, which are related by Conway mutation, both have Chern-Simons invariant $7.192579607752896703724 0463\dots$ (mod π^2), while the two orientations of the Whitehead link are related by a three-punctured mutation and have Chern-Simons invariants $\pm\pi^2/4$.

The final sentence of the theorem is by section 3 of [13], in which Meyerhoff and Ruberman give examples to show any change of Chern-Simons invariant by a rational multiple of π^2 arises by generalized mutation. $\square$

The Riemannian Chern-Simons invariant, defined for a closed hyperbolic manifold M, is a lift of $\mathrm{cs}(M)$ to an invariant defined modulo $2\pi^2$. It is not uncommon to see claims in the literature that the Chern-Simons invariant of a cusped manifold can be defined modulo $2\pi^2$, but we have the following consequence of the above theorem:

Theorem 2.14. *There is no consistent definition of* $\mathrm{cs}(M)$ *which is well defined modulo* $2\pi^2$ *for cusped manifolds.*

PROOF. Mutation along a thrice-punctured sphere is an involution which changes cs by $\pi^2/2$. Such a change cannot lift to a an order two change modulo $2\pi^2$. $\square$

2.7. Scissors Congruence. Two hyperbolic manifolds M_1 and M_2 are *scissors congruent* if M_1 can be cut into finitely many (possibly partially ideal) polyhedra which can be reassembled to form M_2. They are *stably scissors congruent* if there is some polyhedron Q such that $M_1 + Q$ is scissors congruent to $M_2 + Q$ (disjoint union). If M_1 and M_2 are either both compact or both non-compact then stable scissors congruence implies scissors congruence. The following follows easily from [20] (see Theorem 7.2 of [5]):

Theorem 2.15. *Let K be a field that contains the invariant trace fields of M_1 and M_2. Then M_1 and M_2 are stably scissors congruent if and only if* $\mathrm{Borel}(M_1) - \mathrm{Borel}(M_2)$ *is the Borel regulator of an element of* $\mathcal{B}(K \cap \mathbb{R})$

In particular, if $K \cap \mathbb{R}$ is totally real (as is "usually" the case) then scissors congruence class of M is not only determined by $\mathrm{Borel}(M)$ but also determines it.

As discussed in [17], the following conjecture would be a consequence of Conjecture 3.

Conjecture 4. *The stable scissors congruence class of M is determined by* $\mathrm{vol}(M)$.

In view of the above theorem this conjecture is amenable to experimentation with Snap; all the evidence from this is positive.

3. Realizing invariants

We know of no way to generate examples of manifolds with given Bloch invariant. The program Snap enables extensive experimentation—basically casting a fishing line in an ocean of examples—but even for quadratic number fields, only a few fields allow manifolds of small enough volume that they can easily be caught this way.

Moreover, to try to realize such fine arithmetic invariants, one must first realize any non-real number field as an invariant trace field. Here we address this question, and that of realizing a quaternion algebra. The only general result known in this direction is a result first observed by Reid and the author, described in [14], that any non-real multi-quadratic extension of $\mathbb{Q}$ can be realized.

Let k be number field, A a quaternion algebra over k, $\mathcal{O}$ an order in A, and Γ a torsion free subgroup of finite index in $\mathcal{O}^1$. Then, as described in subsection 1.3, each complex embedding of k induces a map $\Gamma \to \mathrm{PSL}(2,\mathbb{C})$, each real embedding at which A is unramified induces a map $\Gamma \to \mathrm{PSL}(2,\mathbb{R})$ and, via these maps, Γ acts discretely with finite co-volume on the product

$$\mathbb{X} := \prod_{i=1}^{r_2} \mathbb{H}^3 \times \prod_{j=1}^{r_1^u} \mathbb{H}^2$$

of copies of $\mathbb{H}^3$ and $\mathbb{H}^2$ (here r_1^u is the number of real places of k at which A is unramified). Denote $Y = \mathbb{X}/\Gamma$. Each projection of $\mathbb{X}$ to one of the $\mathbb{H}^3$ factors gives a codimension 3 foliation on $\mathbb{X}$ which is preserved by the Γ–action, so Y inherits a codimension 3 foliation from each of these projections. This is a transversally hyperbolic foliation: there is a metric on the normal bundle of the foliation which induces a hyperbolic metric on any local transverse section. Similarly, each projection to $\mathbb{H}^2$ gives a codimension 2 transversally hyperbolic foliation.

Now assume that k is a concrete non-real number field, i.e., it comes with a particular complex embedding singled out (which we call the *concrete embedding*). Pick the corresponding codimension 3 foliation $\mathcal{F}$. Let $M^3 \to Y$ be an immersion of a 3-manifold to Y which is everywhere transverse to $\mathcal{F}$. So M^3 has an induced hyperbolic metric. If M^3 is compact this metric is, of course, complete of finite volume. We are interested also in the case that M^3 is not compact, but we require then that the metric be complete of finite volume (as we will see, this can only happen if A is unramified over k).

Theorem 3.1. *The invariant trace field and quaternion algebra for M^3 embed in k resp. A (as concrete field and quaternion algebra). Moreover, M has integral traces.*

Conversely, up to commensurability, every finite volume hyperbolic 3–manifold with invariant quaternion algebra in A (and hence invariant trace field in k) and with integral traces occurs this way.

PROOF. Suppose first that $M \hookrightarrow Y$ is as described in the theorem. Then M inherits a hyperbolic metric locally from the metric transverse to the foliation $\mathcal{F}$. Consider one component $\tilde{M} \to \mathbb{X}$ of the pullback to the universal cover $\mathbb{X}$ of Y. By assumption the projection $\mathbb{X} \to \mathbb{H}^3$ to the first factor restricts to a proper local isometry, hence an isometry, of $\tilde{M}$ to $\mathbb{H}^3$. It follows that $M = \tilde{M}/\Gamma_0 = \mathbb{H}^3/\Gamma_0$, where Γ_0 is a subgroup of the group Γ. The invariant trace field and invariant quaternion algebra of Γ_0 therefore embed in the invariant trace field k and invariant quaternion algebra A of Γ.

Conversely, suppose M has invariant trace field in k and invariant quaternion algebra in A and has integral traces. By going to a finite cover if necessary, we can assume the trace field of M is in k. Write $M = \mathbb{H}^3/\Delta$ with $\Delta \subset \mathrm{PSL}(2,\mathbb{C})$ and let $\overline{\Delta}$ be the inverse image of Δ in $\mathrm{SL}(2,\mathbb{C})$. Then A is the k–subalgebra of $M_2(\mathbb{C})$ generated by $\overline{\Delta}$. The subring $\mathcal{O} \subset A$ consisting of $\mathcal{O}_k$–linear combinations of elements of $\overline{\Delta}$ is an order in A, and $\overline{\Delta} \subset \mathcal{O}^1$ (see the proof of Theorem 8.3.2 in [**14**]). Now any two orders in A are commensurable, so by going to a finite cover of M if necessary we can assume that the order $\mathcal{O}$ we have here is contained in the order we used to construct Y, and therefore that $\Delta \subset \Gamma$.

The developing map $\tilde{M} \to \mathbb{H}^3$ is Δ–equivariant for the action of $\Delta = \pi_1(M)$ by covering transformations on $\tilde{M}$ and the given action of Δ on $\mathbb{H}^3$. The latter is induced from the inclusion of $\bar{\Delta}$ in $\mathcal{O}^1 \subset SL(2,\mathbb{C})$ coming from the concrete embedding $k \to \mathbb{C}$.

Now the non-concrete complex embeddings of k give actions of $\Delta = \pi_1(M)$ on $\mathbb{H}^3$ which are not discrete. But, by Lemma 3.2 below, for each of these we can construct a smooth Δ–equivariant map $\tilde{M} \to \mathbb{H}^3$. Similarly, we construct smooth equivariant maps $\tilde{M} \to \mathbb{H}^2$ for each unramified real embedding. Together these maps give a Δ-equivariant map of $\tilde{M}$ to $\mathbb{X} = \prod_{i=1}^{r_2} \mathbb{H}^3 \times \prod_{j=1}^{r_1^u} \mathbb{H}^2$, where Δ acts on $\mathbb{X}$ as a subgroup of Γ. We thus get an induced map of $\tilde{M}/\Delta = M$ to $\mathbb{X}/\Gamma = Y$, which clearly does what is required. $\qquad\square$

We used the following well known lemma:

Lemma 3.2. *If X is a simplicial or CW-complex then for any action of $\pi_1(X)$ on a contractible space Y there is a $\pi_1(X)$–equivariant map of $\tilde{X}$ to Y, and it is unique up to equivariant homotopy. Moreover, if X and Y are smooth manifolds and the action of $\pi_1(X)$ on Y is by diffeomorphisms then this map can be chosen to be smooth.*

PROOF. Indeed, one constructs the map inductively over skeleta of $\tilde{X}$. If X is smooth one can triangulate X and construct the smooth map inductively over thin neighborhoods of the skeleta. At the k-th step one chooses a lift of each k–simplex and first extends the smooth map already defined on a neighborhood of the boundary of this lifted k–simplex smoothly to a neighborhood of the whole k–simplex and then defines the map on $\pi_1(X)$–images of this neighborhood by equivariance. $\qquad\square$

One can extend the theorem to remove the restriction that the hyperbolic manifold have integral traces. For each prime $\mathfrak{p} \subset \mathcal{O}_k$ which one wishes to allow in denominators of traces one should add the corresponding Bass-Serre tree for $\mathfrak{p}$ (see, e.g., [**16**, Chapter VI]) as a factor on the right side of the product of factors defining $\mathbb{X}$. Then Y is no longer a manifold, but the foliation is still defined and any transversal to it will be a manifold.

The theorem applies also to hyperbolic surfaces. Suppose k has a chosen real place (and is no longer required to have at least one complex place) and A is now unramified at this chosen real place. We consider the foliation of Y given by projecting $\mathbb{X}$ to the corresponding $\mathbb{H}^2$ factor. An immersion $M^2 \to Y$ transversal to this foliation will induce a hyperbolic structure on M^2 with invariant quaternion algebra in the concrete quaternion algebra A, and again, any finite volume hyperbolic surface with integral traces and invariant quaternion algebra in A occurs this way

up to commensurability. As before, the integral trace restriction can be avoided by allowing also some Bass-Serre tree factors in $\mathbb{X}$.

In the 2-dimensional case the existence of surfaces with given invariant trace field can often be shown. For example, explicit computation for the character varieties of "small" surfaces point to the lack of restriction on what fields occur. And in at least one case the existence of the transversals in Y has been shown directly: if $k = \mathbb{Q}(\sqrt{d})$ is a real quadratic field and $\Gamma = \mathrm{PSL}(2, \mathcal{O}(\sqrt{d}))$ (so A is totally unramified) then $Y = \mathbb{H}^2 \times \mathbb{H}^2/\Gamma$ is a Hilbert modular surface and Hirzebruch and Zagier [10] constructed many Riemann surfaces in Y which are transverse to both foliations. In this case these surfaces are all arithmetic, so they give nothing new (they have invariant trace field $\mathbb{Q}$ and quaternion algebra ramified at a possibly empty set of finite places, but becoming unramified on extending scalars to $\mathbb{Q}(\sqrt{d})$).

In the 3-dimensional case the best evidence that such transversals might always exist may be the richness of the collection of fields and quaternion algebras provided by the 3-manifold census and snap, plus the fact that their existence seems very likely in the 2-dimensional case.

There is a 3–dimensional foliation of Y transverse to $\mathcal{F}$, provided by the projection of $\mathbb{X}$ to the factors other than the one used to construct $\mathcal{F}$. W. Thurston (private communication) has suggested that one might seek immersed 3-manifolds which are everywhere almost tangent to this foliation (and hence transverse to $\mathcal{F}$). This would be very interesting from the point of view of realizing Bloch invariants, since it would realize Bloch invariants for which the components other than the "concrete component" (giving $\mathrm{vol}(M)$) of the Borel regulator are small with respect to volume.

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