FORMULAS FOR ALGEBRAIC CURVES

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Here is a compilation of many of the formulas we have come across so far. We assume throughout that we are over $\mathbb{C}$. We see some of the purely algebraic results reappearing topologically. The last section contains some exercises about rational curves.

1. Families of Plane Curves

Counting the monomials of total degree d in 3 variables, we have:

Formula 1.1. *The projective space $V(d)$ of all plane curves of degree d has dimension $d(d+3)/2$.*

Let $V(d; r_1P_1, \dots, r_nP_n)$ denote the linear subspace of $V(d)$ of curves with at least multiplicity r_i at n given points P_i in the plane.

Formula 1.2. *$V(d; r_1P_1, \dots, r_nP_n)$ has codimension in $V(d)$ at most*

$$c = \sum_{i=1}^{n} \frac{r_i(r_i-1)}{2}.$$

It is therefore non-empty when $d(d+3)/2 \geq c$.

Formula 1.3. *If $d \geq (\sum_i r_i) - 1$, then c is precisely the codimension.*

See Fulton [2], Theorem 1, p. 110. On the other hand:

Formula 1.4. *Given two curves C and D of degrees d and e with no common components and multiplicities r_i and s_i at their points of intersection P_i, $1 \leq i \leq n$, then*

$$\sum_{i=1}^{n} r_i s_i \leq de.$$

See Walker [5], Theorem 3.3 of Chapter III, p. 61. This result is used to prove:

Formula 1.5. *A plane curve of degree d without multiple components with singular points P_i of multiplicity r_i, $1 \leq i \leq n$, satisfies*

$$d(d-1) \geq \sum_{i=1}^{n} r_i(r_i-1).$$

Note that this represents a strengthening of Formula 1.2: just divide by 2 and compare. Furthermore, if the hypothesis of Formula 1.3 is satisfied, then the inequality in Formula 1.5 holds. Indeed the curve made up of d concurrent lines does the trick: it has one singular point of multiplicity d, so Formula 1.5 is sharp for all d.

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Formula 1.6. *An irreducible plane curve C of degree d with multiplicity r_i at P_i, $1 \leq i \leq n$, satisfies*

$$(d-1)(d-2) \geq \sum_{i=1}^{n} r_i(r_i - 1).$$

See Walker [5], Theorems 4.3 and 4.4 of Chapter III, p. 65. The proof of Formula 1.6 in Walker uses Formula 1.2 to show that there is a curve C' of degree $d-1$ passing through the points P_i with multiplicity $r_i - 1$, and also passing through another m non-singular points of C. How big can we make m? As big as the codimension given by Formula 1.2. Then just apply Formula 1.4 to C and C', which is permissible because we know C is irreducible, so it has no common components with a curve of lower degree.

Example 1.7. Walker gives an example showing that Formula 1.6 is sharp for all d: the irreducible curve C with equation $x^d - y^{d-1}z = 0$. It has one singular point of multiplicity $d-1$. The singularity is not an ordinary multiple point, but a higher order cusp. C is a rational curve with parametrization $x = t^{d-1}s$, $y = t^d$, $z = s^d$: a projection of the rational normal curve of degree d discussed below.

2. Coverings of Projective Curves

Suppose you have a projective non-singular algebraic curve C and a finite surjective algebraic map ϕ to another projective non-singular algebraic curve D. Let d be the degree of the covering, and $\nu_\phi(P)$ be the ramification index at a point $P \in C$. For all but a finite number of points Q of D, ϕ is a covering map, so that the ramification index at all P such that $\phi(P) = Q$ is 1. For all Q, $\sum_{\phi(P)=Q} \nu_\phi(P) = d$.

Let $\chi(C)$ be the Euler characteristic of C, and $\chi(D)$ be the Euler characteristic of D. Let g be the genus, so $g = \frac{2-\chi}{2}$.

Formula 2.1.

$$\chi(C) = d\chi(D) - \sum(\nu_\phi(P) - 1),$$

so

$$g(C) = dg(D) - d + 1 + \sum \frac{(\nu_\phi(P) - 1)}{2}$$

where the sums are over all the ramification points.

The proof follows, as in Kirwan [3], by taking a triangulation of D with a vertex at each point $Q \in D$ where there is ramification, and perhaps other vertices. Then lift the triangulation to C. The proof goes through even though D is not $\mathbb{P}^1$ and C is not in $\mathbb{P}^2$.

3. Ramification of Non-Singular Plane Curves

If we now assume that C is a non-singular plane curve, and D is $\mathbb{P}^1$, so that its Euler characteristic is 2. We take for ϕ a suitable projection of C from a point of $\mathbb{P}^2$ not on C, and use the discriminant of the defining equation of C to compute the ramification:

Formula 3.1. *Choose a center of projection that does not lie on C or any of the inflectionary tangents of D. Then there are $d(d-1)$ ramification points, and the ramification index is 2 at each ramification point.*

This is Kirwan [3] Lemma 4.7.

Formula 3.2. *For any non-singular plane curve C of degree d,*

$$\chi(C) = 2d - d(d-1) = 3d - d^2$$

and

$$g(C) = \frac{(d-1)(d-2)}{2}.$$

This follows immediately from 2.1 and 3.1.

4. Ramification of Plane Curves with Ordinary Singularities

Now assume that the plane curve C is irreducible with only ordinary singularities P_i of multiplicities r_1. We make a projection from a point that is not on C, not on any of the inflectionary tangent lines, and not on any of the tangents to to the singular points. Since we have excluded a finite number of lines, such a point can be found. We need to know that C has only a finite number of inflection points: this follows from the proof of Lemma 3.22 of Kirwan.

Compute the discriminant of C in the direction of projection.

Formula 4.1. *An ordinary r-fold point contributes exactly $r(r-1)$ to the discriminant.*

Proof. This is a new computation. Because of the choice of direction, each one of the r non-singular branches intersects the derivative term (of degree r-1) transversely, so with multiplicity $r-1$. □

Now let $\tilde{C}$ be the resolution of all these ordinary singularities, so that $\tilde{C}$ is non-singular. $\tilde{C}$ is just C with the ordinary multiple points pulled apart. So if P is an r-fold point, it is replaced on $\tilde{C}$ by r points. We can apply Formula 2.1, since we can compute the total ramification by simply subtracting the contribution that would have come from the ordinary multiple points.

Formula 4.2. *The Euler characteristic of the resolution $\tilde{C}$ of a curve C in $\mathbb{P}^2$ with n ordinary multiple points R_i of multiplicity r_i is*

$$\chi(\tilde{C}) = 3d - d^2 + \sum r_i(r_i - 1).$$

so the genus is

$$g(\tilde{C}) = \frac{(d-1)(d-2)}{2} - \sum_{i=1}^{n} \frac{r_i(r_i-1)}{2}.$$

Happily, Formula 1.6 tells us that $g(\tilde{C})$ is non-negative, as it must be.

5. Rational Curves

We can map $\mathbb{P}^1$ into $\mathbb{P}^d$, for any $d \geq 2$ as a non-singular curve of degree d by using all $d+1$ monomials of degree d in the homogeneous coordinates x and y of $\mathbb{P}^1$, so that the mapping is given by

$$z_i = x^i y^{d-i}$$

to the homogeneous coordinates z_i, $0 \leq i \leq d$ of $\mathbb{P}^d$. The image of $\mathbb{P}^1$ is called the rational normal curve of degree d. It does not lie in a hyperplane in $\mathbb{P}^d$: indeed it intersects each hyperplane in d points counted with multiplicity.

Exercise 5.1. *Consider the degree 4 plane curve* $x^4 - 2x^2yz + y^2z^2 - y^3z = 0$. *Show it is irreducible and rational, and find its singular points.*

By taking a suitable projection of $\mathbb{P}^2$ as explained in class, we get an irreducible curve C with only ordinary singular points.

By Formula 4.2 we get:

Formula 5.1. *A rational curve of degree d in the plane with only ordinary singular points* P_i *of multiplicity* r_i, $1 \le i \le n$, *satisfies*

$$\frac{(d-1)(d-2)}{2} = \sum_{i=1}^{n} \frac{r_i(r_i-1)}{2}.$$

In particular, if it only has ordinary double points, there are exactly $\frac{(d-1)(d-2)}{2}$ of them.

Example 5.2. In the very first lecture of this class, we consider the nodal cubic, and we parametrized it. In our current notation, we get

$$\begin{aligned} z_0 &= y(x^2 - y^2), \\ z_1 &= x(x^2 - y^2), \\ z_2 &= y^3. \end{aligned}$$

This shows that the node occurs at the parameter values $x/y = \pm 1$, so when $z_0 = z_1 = 0$. The equation of the curve is $z_0^3 + z_0^2 z_2 - z_1^2 z_2 = 0$.

Exercise 5.2. *Using the same method as in the above example, write the parametrization of a rational quartic with three ordinary double points. Also write its equation.*

Do you see how this could be extended (in principle) to all degrees?

Formula 5.3. *If* C *has genus 0, then* D *also has genus 0.*

Exercise 5.3. *Prove Formula 5.3 using the notation and hypotheses of Formula 2.1.*

This proves a special case of a theorem in field theory:

Theorem 5.4 (Lüroth's Theorem)**.** *Let* k *be any field, and let* $k(x)$ *be a purely transcendental extension of* k. *Then any subfield of* $k(x)$ *containing* k *and strictly bigger than* k *is purely transcendental, so can be written* $k(t)$. *Since* $t \in k(x)$ *is a rational function in* x, *it can be written as* $f(x)/g(x)$, *where* $f(x)$ *and* $g(x)$ *are polynomials with no common factors. Then the degree of the field extension* $k(x)$ *over* $k(t)$ *is the max of the degrees of* f *and* g *in* x. *In particular all the generators of* $k(x)$ *are of the form*

$$\frac{ax+b}{cx+d}, \quad ad - bc \neq 0.$$

This theorem is proved algebraically in van der Waerden, [4], Volume 1, §63. It is also proved in Walker [5], Chapter V, Theorem 7.2. It is stated in Dummit and Foote, [1] in §14.9 and partly proved in Exercise 18 of §13.2.

The transformation $x \to \frac{ax+b}{cx+d}$ is a *fractional linear transformation.*

Exercise 5.4. *The rational curve C is a finite cover of the rational curve D, as above. The field of rational functions of C is the purely transcendental extension $k(x)$, and that of D is the subfield $k(t)$. Assume that this field extension is Galois, so that there is a finite group G acting on $k(x)$, with the subfield of invariants being $k(t)$.*

- *Show that every element of G sends x to another generator of $k(x)$, so that by Lüroth's Theorem, G is a finite subgroup of the group of fractional linear transformations. Indeed this exercise requires the description of all these finite subgroups.*
- *Show that G permutes the points of $\mathbb{P}^1$.*
- *Because G acts trivially on $k(t)$, show that this implies that G permutes the points on C above a given point of D. Show that the ramification index of all the points above a given point of D must be the same.*
- *Find all possible combinations of ramification indices using the genus formula for the covering. Hint: first show that you cannot get more than three branch points.*
- *Find a group of the right order that realizes the ramification and acts on $\mathbb{P}^1$ in all cases. You can normalize the branch points so that they are - for example -* 0, 1 *and* ∞.
- *Write t explicitly in terms of x in all cases. Verify that the expression you find is a rational function $f(x)/g(x)$ in reduced form, with the max of the degrees of f and g equal to the degree of the Galois group G. Hint: Read van der Waerden* [4], §*63, and note the exercise p.200.*

References

[1] David S. Dummit, Richard M. Foote, *Abstract Algebra*, Second Edition, Prentice Hall, Upper Saddle River, 1999.
[2] William Fulton, *Algebraic Curves*, Benjamin, New York, 1969.
[3] Frances Kirwan, *Complex Algebraic Curves*, Cambridge U. P., Cambridge, 1992.
[4] B. L. van der Waerden, *Modern Algebra*, Frederick Ungar, New York, 1949.
[5] Robert J. Walker, *Algebraic Curves*, Dover, New York, 1962.