HONORS LINEAR ALGEBRA (MATH V 2020) SPRING 2013

PROFESSOR HENRY C. PINKHAM

1. Prerequisites

The only prerequisite is Calculus III (Math 1201) or the equivalent: the first semester of multivariable calculus. The material from Calculus III we will use is limited: it is contained in Chapter 12 and the Appendix on complex numbers in the text used at Columbia: James Stewart's Calculus, Early Transcendentals.

- Functions of several variables; vectors in two and three space; property of vectors in space; the dot product.
- Equations of lines and planes;
- Complex numbers; polar form; de Moivre's theorem; roots of a complex number.

Most of it will be reviewed in class, and all of it is contained in our course textbook.

2. About the Course

This is a first course in Linear Algebra: a more extensive treatment of the material of Math V2010, with an increased emphasis on proof. Besides getting you to learn the beautiful material of linear algebra, I plan to teach you how to read and write mathematical proofs. This is an ideal course for this, since there are many theorems, especially at the beginning of the course, with short and transparent proofs. There will be less emphasis on computation. I also plan to leave time for a few of the important applications of linear algebra: these are not covered in our textbook.

3. The Course Textbook

Serge Lang, *Linear Algebra*, Third Edition, Springer, 2010, ISBN 978-1-4419-3081-1.

I will refer to this below as Lang. I will assign reading and homework exercises from it, so you need to have a copy available. It will be available at Book Culture on 112th Street, and on reserve in the library. Note the following important resource:

Rami Shakarchi, *Solutions Manual for Lang's Linear Algebra*, Springer, 1996, ISBN 0-387-94760-4.

Date: January 17, 2012.

This book contains solutions to all the exercises in Lang. You do not need to buy it. I will arrange for at least one copy to be put on reserve in the Math Library.

Here are some other linear algebra books that you might enjoy consulting:

Peter D. Lax, *Linear Algebra*, 2007.

Denis Serre, *Matrices, Theory and Applications*, Second Edition, 2010.

Gilbert Strang, *Linear Algebra and its Applications*, Fourth Edition, 2005.

I mention them because all three contain applications of linear algebra. Strang is at about the same level as Lang. Lax and Serre are harder.

4. Homework

There will be weekly problem sets, due at the beginning of class on every Tuesday except on the week there is a midterm. The first set will be due Tuesday Jan. 29th. On some weeks, instead on collecting the homework, I will give an in-class closed-book quiz on one of the homework problems. The midterm and final exams will be closely related to the homework problems, so you should understand how to do all the problems. Collaboration and discussion with your classmates is encouraged, but you must write up your assignment on your own.

Graded homework will be returned at the beginning of the next class. Late homework will not be accepted.

5. Exams

There will be two 75-minute in-class midterm exams, and a final exam. The dates for the exams are:

- First midterm exam, Thursday, February 21
- Second midterm exam, Tuesday, April 2
- Final exam, Thursday, May 16, 4:10-7.

6. Grading

The final course grade will be determined by:

- Homework: 10%
- Each midterm exam: 20%
- Final exam: 50%

7. Help

You can send me email, or come to my office hours:

Monday, 4:30-5:30;

Thursday, 1-2.

8. Detailed Syllabus

Two general remarks. Linear algebra can be done over any *field*, but in this course we will restrict to the real numbers $\mathbb{R}$ and the complex numbers $\mathbb{C}$. Secondly, in this course we will study only *finite-dimensional* vector spaces, and the linear maps between them.

Tuesday, January 22: Lang, Chapter I, §1-2. Vector spaces and bases.

This is where *fields* are defined. $\mathbb{R}$ and $\mathbb{C}$ are fields. Lang p. 2. When we do not need to specify, we will refer to our field by the letter F. We also define *vector spaces.*

Thursday, January 24: Lang, Chapter I, §3-4. Dimension and direct sum.

There is a sequence of important theorems in Chapter I leading up to the notion of dimension: Theorems 2.1, 2.2, 3.1, 3.2, and 3.3. You need to understand their proofs.

Tuesday, January 29: Lang, Chapter II, §1-2. Matrices and their role in linear equations.

Thursday, January 31: Matrix multiplication and invertible matrices. Lang, Chapter II, §3.

The definition of an invertible matrix is given page 35: A square matrix A is invertible if there exists another square matrix B of the same size such that $AB = BA = I$, where I is the identity matrix.

Tuesday, February 5: Matrices and the solution of equations via elimination.

The material for this class and the next class is not in Lang, other that the short §II.2. It is contained in his more elementary textbook, *Introduction to Linear Algebra*, Springer (1986), pages 64-84, the sections:

- Homogeneous Linear Equations and Elimination;
- Row Operations and Gauss Elimination;
- Row Operations and Elementary Matrices.

There will be a handout. In this first class we will cover the first two sections.

Thursday, February 7: Elementary matrices and invertible matrices.

We conclude the discussion of the material from Lang's elementary book, showing that elimination can be realized by left multiplication by elementary matrices, and then showing that every invertible matrix is a product of elementary matrices.

Tuesday, February 12: Lang, Chapter III, §1-3. Linear maps.

§III.1, which discusses mappings, should be mainly review, since you have studied multivariable calculus. However, important definitions are given, so you should read it carefully: in particular the notion of an *injective* and *surjective* mapping. Next the definition of an *inverse mapping* given on page 48, with the Examples 11, 12, and 13. Finally the results on page 50 establish that a mapping f has an inverse mapping if and only if f is both injective and surjective. These definitions and this result will be used constantly in the rest of the book, so learn them now.

The first new section is §III.2: it contains the important definition of a *linear map*. In §III.3, the two key subspaces associated to a linear map are defined: the *kernel* (often called the *nullspace*) and the *image* and one of the key results of linear algebra is proved: Theorem 3.2 p.61, often called the rank-nullity theorem. Theorem 3.3 p.63 is also important, since it says that any two vector spaces of the same dimension are *isomorphic*, meaning that there is a bijection between them.

Thursday, February 14: Lang, Chapter III, §4-5. Composition of linear maps. The inverse of an invertible linear map.

The main result (Theorem 4.1) is that the composite of two linear maps is a linear map. Similarly, if a linear map has an inverse mapping, it is linear: Theorem 4.3.

A linear map $F : V \to V$ is called an *operator*. Geometric examples for Chapter III of Lang are given in §III.5.

When you work Exercise III.4.9 page 71, be sure to compare it to Exercise 18 of §II.3 page 39. Then Exercises 10-13 on the same page all circle around the same theme: the map that is called Q in Exercise 11 is $I - P$ in Exercise 10.

Tuesday, February 19: Lang, Chapter IV, §1-2.

Given a $m \times n$ matrix A, one can associate a linear map $L_A : K^n \to K^m$, by matrix multiplication. Conversely, given a linear map from K^n to K^m we get a unique matrix A. In the case where $n = m$, we prove that A is invertible if and only if the columns of A are linearly independent, in Theorem 2.2 p.86. The careful reader will be bothered that Lang only shows that A has a left-inverse: I will ask you to show as an exercise that this implies it has a right inverse.

Thursday, February 21: First Midterm. On the first three chapters of Lang.

Tuesday, February 26: Lang, Chapter IV, §3. This section is considered difficult by students, probably because it is very succinct in Lang. You should read Example 1, p.89 first.

Remark that the notation $X_{\mathcal{B}}(v)$ applies in any vector space V, given a basis $\mathcal{B}$ and an arbitrary element $v \in V$. The symbol X just means to extract the coordinates of v relative to the basis $\mathcal{B}$. Unfortunately. in the italicized statement in the middle of page 88, Lang uses X also to denote a column coordinate vector in V. A different, although related use.

Next, on the top of page 89, Lang uses A for convenience of notation to denote the matrix $M^{\mathcal{B}}_{\mathcal{B}'}$. Since A_i is the ith row of A, by matrix multiplication when it is dotted with the coefficient vector X of v, you get the coefficients of the image. This explains the first displayed formula on page 89.

In the next set of displayed formulas, marked $(*)$, confusingly the a_{ij} have nothing to do with the matrix A. In fact, the goal of the proof is to show that they are the coefficients of A, and that is what is shown in the rest of the proof.

Finally we have the important result that multiplication of matrices corresponds to composition of linear maps: Theorem 3.4.

At the end of the section, the important special case of operators is dealt with. Theorem 3.6 explains what happens to the matrix of an operator when the basis is changed. There is a typo in the proof: read Theorem 3.4 instead of the nonexistent Theorem 3.2. Then an important definition is made: that of square matrices being *similar*. In class, we will show similarity is an *equivalence relation*.

Thursday, February 28: Lang, Chapter V, §1.

Scalar products over $\mathbb{R}$: the Pythagorian Theorem, the Schwarz inequality, the triangle inequality.

Tuesday, March 5: Lang, Chapter V, §2.

Orthogonal bases. Gram-Schmidt orthogonalization process. Scalar products over $\mathbb{C}$. The Hermitian case.

Thursday, March 7: Lang Chapter V, §3 and §6.

We define the rank of a matrix, and prove two important theorems: Theorem 3.1 p.113, and Theorem 3.2 p.114. The proof uses Theorem 6.4 p.131, which is why we cover §6 on the dual space. The four subspaces associated to a matrix. Note that we will skip §4, 5, 7 and 8 for the time being.

Tuesday, March 12: Lang, Chapter IX.

Polynomials and matrices. If needed, we will review complex numbers at this point.

Thursday, March 14: Lang, Chapter XI, §1-3.

The Euclidean Algorithm (for polynomials), greatest common divisor and unique factorization. This material has very little to do with Linear Algebra, but will be useful later on.

Tuesday, March 26: Lang, Chapter VI, §2, 3, and 5.

Determinants, part 1. The goal of this first lecture is to cover the essential parts of the indicated sections, in order to establish the existence of the determinant with the desired properties: given in Theorem 2.1 p. 144.

Thursday, March 28: Lang, Chapter VI, §6, 7.

Determinants, part 2. Permutations and the expansion formula for the determinant, in order to establish the uniqueness of the determinant. Whatever else time allows. One of the key results is Theorem 7.3 p.172, and its corollaries.

Tuesday, April 2: Second Midterm. On Chapters IV, V, VI, IX and XI, §1-3.

Thursday, April 4: Lang, Chapter VII, §1, 3. Symmetric operators. Unitary operators over $\mathbb{R}$.

We assume the base field is $\mathbb{R}$, so our vector space can be identified with $\mathbb{R}^n$ on which we put the ordinary dot product. Then a *symmetric operator* is a linear map: $\mathbb{R}^n \to \mathbb{R}^n$ that is given by a symmetric matrix. To say this in the proper generality, and without introducing bases, requires Theorem 6.2 of Chapter V, §6, studied when we studied the dual space.

What Lang calls the transpose is often called the adjoint, and a symmetric operator is often called *self-adjoint*, despite the potential confusion with the complex case.

Then, we look at another special kind of (real) operator. Lang calls them *unitary*, they are ofter called orthogonal.

Tuesday, April 9: Lang, Chapter VII, §2, 3. Hermitian operators. Unitary operators over $\mathbb{C}$

We continue our study of special operators. We take the base field to be $\mathbb{C}$, and repeat the discussion of the previous lecture for the new field.

Thursday, April 11: Lang, Chapter VIII, §1-2.

We meet some of the most important concepts in linear algebra: eigenvalues and eigenvectors; the characteristic polynomial. §2 contains some important theorems, especially

Theorem 2.4 p.205: Similar matrices have the same characteristic polynomial.

It implies (see the discussion p.206) that one can define the characteristic polynomial of an operator.

Tuesday, April 16: Lang, Chapter V, §4-5, 7-8. Bilinear Maps, Quadratic Forms and Sylvester's Theorem.

We go back to Chapter V to prove a key theorem (Theorem 8.2, p.137), due to the nineteenth century mathematician Sylvester, concerning vector spaces over $\mathbb{R}$ with a scalar product, not necessarily positive definite..

Thursday, April 18: Lang, Chapter VIII, §3-4.

Diagonalization of symmetric operators. An important step in one of the proofs given by Lang uses the techniques of multivariable calculus: see the bottom of p.214. The key result here is the Spectral Theorem 4.3 p.219.

Tuesday, April 23: Lang, Chapter VIII, §5-6.

The case of Hermitian operators, and then unitary (in the real case usually called orthogonal) operators.

Thursday, April 25: Lang, Chapter X, §1.

Putting a matrix in triangular form.

Tuesday, April 30: Lang, Chapter X, §2-3.

The Theorem of Cayley-Hamilton.

Thursday, May 2: Lang, Chapter XI, §4-6.

Decomposition of a vector space, Schur's lemma and Jordan Canonical Form.

9. Homework Assignments

(partial and unfinished)

In the assignments below, the only exercises I expect you to turn in (and the only ones for which solutions will be distributed) are marked separately. It is very important that you do the others, however. Remember that there are solutions to all the problems in Lang in Rami Shakarchi's Solution Book referred to above.

(1) Assignment 1, due Tuesday Jan. 29: Abstract vector spaces, bases and the notion of dimension.
    Read: Lang, Chapter I;
    Do exercises: I §1: 1-6, 8-11; I §2: 1, 3-5, 7-9; I §4: 2, 4.
    Turn in exercises: I §1: 6, 10; I §2: 4, 8, 9 ; I §4: 4.
(2) Assignment 2, due Tuesday Feb. 5: Matrices, matrix multiplication and linear equations.
    Read: Lang, Chapter II;
    Do exercises: II §1 (Exercises on Matrices): 1, 3, 5, 7, 10, 11; (Exercises on Dimension): 1-6; II, §2: 1-4; II, §3: 1-8, 12-14, 18, 21, 23, 25-28, 32, 33, 36.
    Turn in exercises: II §1 (Exercises on Matrices): 10; (Exercises on Dimension): 2, 4, 6; II, §2: 1-2; II, §3: 5, 8, 18, 21, 23, 27, 32, 33.
(3) Assignment 3, due Tuesday Feb. 12:
    Read: Lang's *Introduction to Linear Algebra*, pp. 64-85.
    Turn in exercises: From Lang's *Introduction to Linear Algebra*, II, §3 p.70: 4, 5 (a), (b), (c), (d), 6 (a), (d),(f); II §4 p.76: 1, 2, 4; II §5 p.85: 1 (a), (b), 3.
(4) Assignment 4, due Tuesday Feb. 19:
    Read: Lang, Chapter III;
    Do exercises: III §2: 1-7, 9-11, 14 -16, 17, 18, 19; III §3: 1-5, 10-18, III §4: 2-6, 8, 9, 10.
    Turn in exercises: III §2, 9, 15; III §3, 10-11, 13, 17-18
(5) Assignment 5, due Tuesday March 5:
    Read: Lang, Chapter IV, and Chapter V, §1;
(6) Assignment 6, due Tuesday March 12:
    Read: Lang, Chapter V, §2, 3, 6;

(7) Assignment 7, due Tuesday March 26:
Read: Lang, Chapter IX and Chapter XI, §1-3.
(8) Assignment 8, due Tuesday April 9:
Read: Lang, Chapter VI.
(9) Assignment 9, due Tuesday April 16:
Read: Lang, Chapter VII, Chapter VIII, §1-2.
(10) Assignment 10, due Tuesday April 23:
Read: Lang, Chapter V, §7-8 and Chapter VIII, §3-4.
(11) Assignment 11, due Tuesday April 30:
Read: Lang, Chapter VIII, §5-6, Chapter X, §1.