Volume and topology

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Slogan: Volume is a topological invariant of hyperbolic 3-manifolds.

References:

W. Thurston, *The Geometry and Topology of* 3-*Manifolds,* `http://www.msri.org/publications/books/gt3m`

H. Munkholm, Springer Lecture Notes #788.

N. Dunfield, "Cyclic surgery, degrees of maps of character curves, and volume rigidity for hyperbolic manifolds", *Invent. Math* **136**(1999)

C. Löh, "Measure homology and singular homology are isometrically isomorphic", *Math. Z.* **253** (2006).

Textbooks: Benedetti and Petronio, Ratcliffe

Definition (Gromov). If $x \in H_k(X;\mathbb{R})$ is a singular homology class, define

$$||x|| = \inf \left\{ \sum |a_i| : \ \sum a_i \sigma_i \in x \right\}.$$

This is a seminorm. If M is an orientable n-manifold, let $[M] \in H_n(M;\mathbb{R})$ denote its fundamental class, and define $||M|| = ||[M]||$.

Pessimists might expect $||M||$ to always be 0, but clearly it is a *topological* invariant, and has these important properties:

- If $f : X \to Y$ and $x \in H_k(X)$ then $||x|| \geq ||f_*(x)||$.
- If M and N are orientable n-manifolds and $f : M \to N$ has degree n then $||M|| \geq n||N||$.

Theorem (Gromov). *If M is an orientable hyperbolic n-manifold then $||M|| = \mathrm{vol}\, M/v_n$ where v_n is the volume of a regular ideal hyperbolic n-simplex.*

W. Thurston improved on Gromov's definition by inventing a variant which makes the proof of Gromov's Theorem clean and elegant. It required a new homology theory.

Definition. A *measure k-chain* on a CW-complex X is a compactly supported, signed Borel measure of bounded total variation on the space $C^1(\Delta_k, X)$.

So an ordinary smooth k-chain is a special case of a measure k-chain, where the measure is a weighted sum of Dirac masses supported on a finite set of singular simplices.

If we let $\mathcal{C}_k(X)$ denote the abelian group of measure k-chains, this gives an inclusion $\iota_k : C_k(X) \to \mathcal{C}_k(X)$.

Let $f_i : \Delta_{k-1} \to \Delta_k$ denote the face inclusions, which induce $f_i^* : C^1(\Delta_k, X) \to C^1(\Delta_{k-1}, X)$. For a measure k-chain μ, define

$$\partial_n(\mu) = \sum_i (-1)^i (f_i^*)_*(\mu).$$

This makes $\mathcal{C}_*$ into a chain complex and ι into a chain map. Write the homology groups as $\mathcal{H}_k$.

Define the *total variation* of a measure-chain as

$$||\mu|| = \sup\left\{ \int f\mu :\ |f(\sigma)| \leq 1 \text{ for all } \sigma \in C^1(\Delta_k, X) \right\}.$$

Define a seminorm on $\mathcal{H}_k$ by $||x|| = \inf\{||\mu|| :\ \mu \in x\}$.

We still have:

- If $f : X \to Y$ and $x \in \mathcal{H}_k(X)$ then $||f_*(x)|| \leq ||x||$.

Theorem (C. Löh, 2006). *For any connected CW-complex X, the inclusion $\iota : C_*(X) \to \mathcal{C}_*(X)$ induces an isomorphism $H_k(X) \to \mathcal{H}_k(X)$ which is an isometry with respect to their seminorms.*

(Isomorphism was proven in 1998, independently by Hanson and Zastrow.)

We may identify $[M]$ with $\iota([M])$ and define $||M|| = ||[M]||$, where now $[M] \in \mathcal{H}_n(M)$. This is Thurston's definition of Gromov's norm, and the two definitions are equivalent by Löh's Theorem.

In particular,

- If M and N are orientable n-manifolds and $f : M \to N$ has degree n then $||M|| \geq n||N||$.

To work with measure chains we need to know that the usual pairing (i.e. integration) between k-chains and k-forms extends to measure chains. Here is the definition. If μ is a measure k-chain and ω is a k-form,

$$\langle \mu, \omega \rangle = \int \left(\int_\sigma \omega \right) \mu.$$

That is, we integrate the function $\sigma \to \int_\sigma \omega$ using the measure μ.

- If μ and μ' are measure k-cycles then $[\mu] = [\mu']$ if and only if $\langle \mu, \omega \rangle = \langle \mu', \omega \rangle$ for all closed k-forms ω.
- If M is a Riemannian n-manifold and μ is a measure n-cycle then $[\mu] = (\langle \mu, dV \rangle / \operatorname{vol} M)[M]$.

From now on $M = \mathbb{H}^n/\Gamma$ is a hyperbolic manifold, and $p : \mathbb{H}^n \to M$ the universal covering projection.

Suppose $\sigma : \Delta_k \to \mathbb{H}^n$ is a singular simplex. The points $\sigma(v_0), \ldots, \sigma(v_k)$ span a hyperbolic k-simplex Δ_σ. There is a canonical map $S(\sigma) : \Delta_k \to \Delta_\sigma$ which preserves barycentric coordinates. There is a homotopy from σ to $S(\sigma)$, constant on the vertices. ($S(\sigma)$ is the straightening of σ.)

Suppose $\sigma : \Delta_k \to M$ is a singular simplex. Lift it to $\tilde{\sigma} : \Delta_k \to \mathbb{H}^n$. Define:

$$S(\sigma) = p \circ S(\tilde{\sigma}).$$

Let $\mathcal{S}_k(M) \subset \mathcal{C}_k(M)$ denote the measures supported on the image of S. Since S commutes with the boundary map, $\mathcal{S}_*(M)$ is a chain sub-complex. The map $\mu \to S_*(\mu)$ is a chain-homotopy inverse to the inclusion.

Straightening gives the easy direction of Gromov's Theorem.

We need the key geometrical fact that the regular ideal tetrahedron is the unique hyperbolic tetrahedron of maximal volume. (In $\mathbb{H}^3$ we have a volume formula. In $\mathbb{H}^n$ this is a theorem of Haagerup and Munkholm, proved after Gromov's proof for $n = 3$. So now we can do this for n-manifolds)

Choose $\mu \in \mathcal{S}_n(M)$ with $[\mu] = [M]$. Since $[\mu] = [M]$ we know $\langle \mu, dV \rangle = \mathrm{vol}\, M$, so

$$\mathrm{vol}\, M = \langle \mu, dV \rangle = \int \left(\int_\sigma dV \right) \mu.$$

But μ is supported on straight simplices, for which $\int_\sigma dV = \mathrm{vol}\, \sigma(\Delta_n) < v_n$. Since $\mathrm{vol}\, M < v_n ||\mu||$ for any $\mu \in \mathcal{S}_n(M)$ with $[\mu] = [M]$; it follows that $\mathrm{vol}\, M \leq v_n ||[M]||$.

There is a natural action of $\mathrm{Isom}_+\mathbb{H}^n$ on $C^1(\Delta_n, M)$:

If $\sigma : \Delta_n \to M$ and $\gamma \in \mathrm{Isom}_+\mathbb{H}^n$, lift σ to $\tilde{\sigma} : \Delta_n \to \mathbb{H}^n$ and define

$$\gamma \cdot \sigma = p \circ \gamma \circ \sigma.$$

The stabilizer of a singular simplex is $\Gamma = \pi_1(M)$. So an orbit is identified with $\mathrm{Isom}_+\mathbb{H}^n/\Gamma$, which is an $SO(n)$-bundle over M.

If $\sigma : \Delta_k \to M$ is a singular simplex, define $\mathrm{Smear}(\sigma)$ to be the measure chain supported on the orbit of σ, with the measure which is locally the product of $\pm dV$ on an open set in M with the unit mass Haar measure on the $SO(n)$-fibers. (Use $+$ if σ is positively oriented, $-$ if it is negatively oriented.)

In particular, $||\mathrm{Smear}(\sigma)|| = \mathrm{vol}\, M$ for *any* singular simplex σ.

Let $\sigma : \Delta_n \to M$ be a **straight** positive singular simplex and let v_σ denote the volume of its image hyperbolic simplex. If $\mu = \text{Smear}(\sigma)$ then

$$\langle \mu, dV \rangle = \int \left(\int_\sigma dV \right) \mu = v_\sigma \int \mu = v_\sigma \operatorname{vol} M.$$

Of course Smear(σ) is not a cycle. However, let $\bar{\sigma}$ be the simplex obtained by composing σ with reflection in one of its faces. (Orient $\bar{\sigma}$ negatively). Notice that each oriented face of σ is mapped to an oriented face of $\bar{\sigma}$ by a hyperbolic rotation.

This implies $\Sigma = \text{Smear}(\sigma) - \text{Smear}(\bar{\sigma})$ is a cycle!

Moreover, since Σ is supported on two disjoint orbits, $||\Sigma|| = 2 \operatorname{vol} M$. Since $\langle \Sigma, dV \rangle = (v_\sigma - (-v_\sigma)) \operatorname{vol} M$, we have $[\Sigma] = 2v_\sigma[M]$. Therefore $||M|| \leq \frac{||\Sigma||}{2v_\sigma} = \frac{\operatorname{vol} M}{v_\sigma}$.

But we may take v_σ arbitrarily close to v_n, so $\operatorname{vol} M \geq ||M|| v_n$.

Gromov used his theorem to give a simple proof of Mostow's Rigidity Theorem.

Theorem (Mostow). *Suppose $M_1 = \mathbb{H}^n/\Gamma_1$ and $M_2 = \mathbb{H}^n/\Gamma_2$ are closed orientable hyperbolic n-manifolds with $n > 2$. If M_1 is homotopy equivalent to M_2 then Γ_1 is conjugate to Γ_2 in $Isom_+\mathbb{H}^n$, and hence M_1 is isometric to M_2.*

Start with a homotopy equivalence $f : M_1 \to M_2$. Lift f to $\tilde{f} : \mathbb{H}^n \to \mathbb{H}^n$. Say an $n+1$-tuple of points on S^n_∞ is *regular* if they span a regular ideal n-simplex.

- Show $\tilde{f}$ is a quasi-isometry, and deduce that $\tilde{f}$ extends continuously, giving $\tilde{f}_\infty : S^2_\infty \to S^2_\infty$. (See Munkholm.)
- Show that $\tilde{f}_\infty$ sends regular 4-tuples to regular 4-tuples.
- Show that this condition on $\tilde{f}_\infty$ implies that f is an isometry.

Suppose $\tilde{f}_\infty$ maps the vertices of a regular ideal simplex Δ to the vertices of an irregular ideal simplex Δ', with $\operatorname{vol}\Delta' < v_n - 2\epsilon$. Take a sequence σ_k of straight (non-ideal) simplices with vertices tending to the ideal vertices of Δ.

Let $\mu_k = \frac{1}{2}\text{Smear}(\sigma_k)$ and $\bar{\mu}_k = \frac{1}{2}\text{Smear}(\bar{\sigma}_k)$. We know that $[\mu_k + \bar{\mu}_k] = (\int_{\sigma_k} dV)/v_n)[M_1]$, and hence that $[S_* f_*(\mu_k + \bar{\mu}_k)] \to [M_2]$ as $k \to \infty$.

There is an open set $U \subset \text{Isom}_+\mathbb{H}^n$ so that, for all $g \in U$, and all sufficiently large n, $\operatorname{vol} S \circ f(g \cdot \sigma_k) < v_n - \epsilon$.Thus

$$\langle S_* f_* \mu_k, dV \rangle < \mu(U)(v_n - \epsilon) + \left(\frac{1}{2} \operatorname{vol} M_1 - \mu(U) \right) v_n,$$

Since $\operatorname{vol} M_1 = \operatorname{vol} M_2$ by Gromov's theorem, this gives $\langle S_* f_*(\mu_k + \bar{\mu}_k), dV \rangle < v_n \operatorname{vol} M_2 - \mu(U)\epsilon$, contradicting that $[S_* f_*(\mu_k + \bar{\mu}_k)] \to [M_2]$.

We know that $\tilde{f}_\infty$ sends regular $(n+1)$-tuples to regular $(n+1)$-tuples. We will show that $\tilde{f}_\infty$ is a Möbius transformation. Since the action of Γ_i is determined by its action on S^2_∞, this implies that Γ_1 is conjugate to Γ_2 in $\mathrm{Isom}_+\mathbb{H}^n$.

Take a regular ideal simplex Δ in $\mathbb{H}^n$. Consider the group generated by reflections in the sides of Δ. The orbit of Δ is a tesselation of $\mathbb{H}^n$ by regular ideal tetrahedra. The vertices are dense in S^2_∞.

Let g be the Möbius transformation that agrees with $\tilde{f}_\infty$ on the vertices of Δ. Since $\tilde{f}_\infty$ takes regular $(n+1)$-tuples to regular $(n+1)$-tuples, f agrees with g on the vertices of each simplex in the the tesselation.

Thus $\tilde{f}_\infty$ agrees with g on a dense set of S^∞. Since $\tilde{f}_\infty$ is continuous, it is equal to g.

Quiz: Where did we use $n > 2$?

Answer

In dimension $n > 2$ there are exactly 2 regular ideal n-simplexes having a given regular ideal $(n-1)$-simplex Φ as a face. The reflection through Φ takes one of these n-simplexes to the other. So there is a unique way to extend the regular ideal simplex Δ to a tesselation of $\mathbb{H}^n$ by regular ideal simplexes.

In dimension 2, every ideal 2-simplex is regular. There are uncountably many ways to extend the regular ideal simplex Δ to a tesellation of $\mathbb{H}^2$ by regular ideal 2-simplexes.

So, if $n = 2$, we could not conclude that $\tilde{f}_\infty$ agrees with g on the vertices of each simplex in the tesselation.