Volume and topology III
(Applications)

Marc Culler

June 9, 2009

Theorem (ACCS+A/CG+Density). *Suppose that Γ is a non-abelian free Kleinian group with basis $\{\gamma_1, \ldots, \gamma_n\}$. Fix $p \in \mathbb{H}^3$ and set $d_i = \mathrm{dist}(p, \gamma_i(p))$ for $i = 1, 2, \ldots n$. Then*

$$\sum_{i=1}^{n} \frac{1}{1+e^{d_i}} \leq \frac{1}{2}.$$

Corollary. *If M is a closed hyperbolic 3-manifold and $\pi_1(M)$ is k-free then, for any $p \in \mathbb{H}^3$, we have $d_i \leq \log(2k-1)$ for at least one index i.*

Definition. A group Γ is k-free if every k-generator subgroup of Γ is a free group (possibly with rank $< k$).

(Note that Γ k-free $\implies$ Γ j-free for $0 \leq j \leq k$.)

Theorem (Jaco-Shalen). *If M is a closed hyperbolic 3-manifold then either*

- $\pi_1(M)$ *is 2-free; or*
- *M has a finite cover with 2-generator fundamental group.*

(More generally, if $\pi_1(M)$ is contains no surface subgroups of genus $< k$ and if every k-generator subgroup of $\pi_1(M)$ has infinite index, then $\pi_1(M)$ is k-free.)

It is clear that if $H_1(M;\mathbb{Q})$ has dimension $> k$ then every k-generator subgroup has infinite index. But this can also be detected with mod p homology.

Theorem (Shalen-Wagreich). *Suppose that $H_1(M;\mathbb{Z}_p)$ has rank $> k+1$ for some prime p. Then every k-generator subgroup of $\pi_1(M)$ has infinite index.*

Corollary. *If M is a closed hyperbolic 3-manifold and $\pi_1(M)$ is 2-free then the maximal injectivity radius of M is at least $\frac{1}{2}\log 3$.*

Proof: Given a maximal cyclic subgroup $C < \pi_1(M)$, define

$$Z_\lambda(C) = \{x \in \mathbb{H}^3 |\, \mathrm{dist}(x, \gamma(x)) < \lambda) \text{ for some } \gamma \in C\ \}.$$

(If non-empty, this is an open cylinder around the axis of C.)

Take $\lambda = \log 3$. If $C_1 \neq C_2$ and $p \in Z_{\log 3}(C_1) \cap Z_{\log 3}(C_2)$ then there exist $\gamma_1 \in C_1$, $\gamma_2 \in C_2$, generating a free group of rank 2, with $\mathrm{dist}(p, \gamma_i(p)) < \log 3$. This contradicts the $\log 3$-Theorem, so $Z_{\log 3}(C_1) \cap Z_{\log 3}(C_2) = \emptyset$.

We cannot cover $\mathbb{H}^3$ with disjoint open cylinders. So there is $p \in \mathbb{H}^3$ not contained in any $Z_{\log 3}(C)$. Thus $\mathrm{dist}(p, \gamma(p)) > \log 3$ for all $\gamma \in \pi_1(M)$.

If M contains an embedded hyperbolic ball $B = B(p, R)$ then the lifts of B to $\mathbb{H}^3$ are disjoint, and form a "ball-packing". Each lift $B(\tilde{p}, R)$ has a *Dirichlet domain*:

$$D(\tilde{p}) = \{x \in \mathbb{H}^3 : \ \mathrm{dist}(x, \tilde{p}) \leq \mathrm{dist}(x, \tilde{p}') \text{ for any lift } \tilde{p}' \text{ of } p\}$$

which is a fundamental domain for M.

Böröczky gave an estimate of the "density" of an arbitrary ball-packing.

Theorem (Böröczky). *Suppose $\{B(p_n, R)\}$ is a radius R ball-packing in $\mathbb{H}^3$. Then for each n,*
$\mathrm{vol}\, D(p_n) \geq \mathrm{vol}\, B(p_n, R)/d(R)$.

Corollary. *If M is a closed orientable hyperbolic 3-manifold and $\pi_1(M)$ is 2-free then*
$\mathrm{vol}\, M > \mathrm{vol}\, B(x, \frac{1}{2}\log 3)/d(\frac{1}{2}\log 3) = 0.929\ldots$

Böröczky's result also applies to horoball packings, using the same definition of Dirichlet domain (which only compares distances to points on S^2_∞). The local density of a horoball packing is at most $d(\infty) = \lim_{R\to\infty} d(R) = 0.8532\ldots$

Corollary. *If M is a cusped hyperbolic 3-manifold and $\mathcal{H}$ is a cusp neighborhood in M then* $\operatorname{vol} M > \operatorname{vol} H/d(\infty)$

One can also define a Dirichlet domain for a cylinder (banana) in a cylinder packing, replacing center points by the central axes of the cylinder. Andrew Przeworski has given estimates for the density.

Theorem (Przeworski). *If M is a hyperbolic 3-manifold containing an embedded tube T of radius R then* $\operatorname{vol} M > \operatorname{vol} T/min(0.91, p(R))$.

Theorem (Kerckhoff). *If M is an orientable hyperbolic manifold with finite volume and C is an embedded geodesic in M then $M - C$ admits a finite-volume hyperbolic metric.*

The following theorem used Perelman's estimates for Ricci flow to give explicit estimates for the amount volume decreases under Dehn surgery:

Theorem (Agol, Storm, W. Thurston + Dunfield). *Let M be a closed orientable hyperbolic 3-manifold, let C be a geodesic in M and let N be the hyperbolic manifold homeomorphic to $M - C$. If the maximal embedded tube around C has radius R then*

$$\operatorname{vol} N \leq \coth^3(2R)\left(1 + \frac{1}{\cosh(2R)}\frac{\operatorname{vol} T}{\operatorname{vol} M}\right).$$

Corollary (using Przeworski). *if $R > \frac{1}{2}\log 3$ then* $\operatorname{vol} N < 3.018 \operatorname{vol} M$.

Gabai, Meyerhoff and N. Thurston proved a stronger version of Mostow rigidity: If M is homotopy equivalent to a hyperbolic 3-manifold then M is hyperbolic.

Their proof implies the following result, which involves rigorous computation in the space of 2-generator Kleinian groups:

Theorem. *Let M be a closed orientable hyperbolic 3-manifold and let C be a shortest geodesic in M. Then either*

- *the maximal embedded tube about C has radius $> \frac{1}{2}\log 3$; or*
- *M has a finite cover $\tilde{M}$ with 2-generator fundamental group, and $\pi_1(\tilde{M})$ lies in one of 7 explicit boxes in the space $Hom(F_2, PSL_2(\mathbb{C}))/\sim$.*

On the other hand, the strong form of the $\log 3$ theorem implies

Theorem (Anderson-Canary-C-Shalen). *Suppose that M is a closed orientable hyperbolic 3-manifold with 2-free fundamental group. Let C be a closed geodesic in M of length L. Then the maximal tube about C has volume $> V(L)$, and $V(L) \to \pi$ as*

Theorem (Agol-C-Shalen). *Suppose that M is a closed, orientable hyperbolic 3-manifold with such that $H_1(M;\mathbb{Z}_p)$ has rank > 3 for some prime p. Then $\operatorname{vol} M > 1.22$.*

(In fact, we prove this for $H_1(M;\mathbb{Z}_p)$ of rank > 2 ,$p \neq 2, 7$.)

Proof. Let C be the shortest geodesic in M. Since $\pi_1(M)$ is $2-$*free*, the maximal tube about C has radius $> \frac{1}{2}\log 3$. Drill out C to get a cusped hyperbolic manifold N and let $\mathcal{H}$ be the cusp neighborhood in N.

Consider a framing (μ, λ) where μ is the meridian of N in M. Consider Dehn fillings $M_n = N(1/np)$. Then $H_1(M_n;\mathbb{Z}_p)$ has rank > 3, so $\pi_1(M_n)$ is 2-free.

By the hyperbolic Dehn-filling theorem, M_n is hyperbolic for large n. Let T_n be the maximal tube in M_n about the filling geodesic. The lengths of T_n converge to 0, so $\operatorname{vol} T_n \to \pi$. But $T_n \to H$ geometrically, so $\operatorname{vol} H > \pi$.

Thus $\pi/d(\infty) < \operatorname{vol} N < 3.018 \operatorname{vol} M \implies \operatorname{vol} M > 1.22$.

Theorem (CS, ACCS, Agol-CS+tameness). *Suppose that M is a closed hyperbolic 3-manifold and $\pi_1(M)$ is 3-free. Then the maximal injectivity radius of M is at least $\frac{1}{2}\log 5$. (This implies $\mathrm{vol}\, M > 3.0879$ by sphere-packing.)*

Let $\mathcal{C}_\lambda$ be the set of maximal cyclic subgroups with $Z_\lambda(C) \neq \emptyset$. Take $\lambda = \log 5$.

It suffices to show that the cylinders $Z_\lambda(C)$ cannot cover $\mathbb{H}^3$ with $\lambda = \log 5$. To show this we work with a simplicial "nerve" of the covering. We show that the nerve cannot be contractible, a contradiction.

Given an open covering of $\mathbb{H}^3$ by cylinders $Z_\lambda(C)$, $C \in \mathcal{C}_\lambda$, define a complex K_λ by

- the vertex set is $\mathcal{C}_\lambda$.
- $(C_0, \dots, C_m)$ is an m-simplex if $\cap_{i=0}^m Z_\lambda(C_i) \neq \emptyset$.

For an (open or closed) m-simplex Δ with vertices $C_0, \dots, C_m$ set $\Theta(\Delta) = \langle C_0 \cup \cdots \cup C_m \rangle < \pi_1(M)$.

If $\pi_1(M)$ is k-free and $\Delta = (C_0, \dots, C_{k-1})$ is a $(k-1)$-simplex then $\Theta(\Delta)$ is free, but it has rank *less than* k. If $C_i = \langle \gamma_i \rangle$ then non-trivial relations hold among the γ_i.

If X is a subcomplex of K_λ, or a union of open simplices, define $\Theta(X)$ to be the group generated by the $\Theta(\Delta)$ as Δ ranges over the simplices in X.

Definition. A group has *local rank* $\leq r$ if every finitely generated subgroup is contained in a subgroup of rank $\leq r$.

Lemma. *Suppose* $\pi_1(M)$ *is* k*-free,* $k > 2$*. Set* $\lambda = \log(2k-1)$ *and fix* $r < k$*. Suppose* $X \subset K_\lambda$ *is a connected union of open* $(k-1)$*- and* $(k-2)$*-simplices, where* $\Theta(\Delta)$ *has rank* r *for each simplex* Δ *in* X*. Then* $\Theta(X)$ *has local rank* r*. (And hence is locally free.)*

Induction step: Suppose $\Theta(Y)$ has local rank r and $Y' = Y \cup \Delta$ where Δ is a $(k-1)$-simplex whose $(k-2)$-face Φ is contained in Y. Then $\Theta(Y') = \langle \Theta(Y), C \rangle$ where C is the vertex (maximal cyclic subgroup) opposite the face Φ.

Let $A < \Theta(Y')$ be finitely generated. Then $A < A' = \langle B, C \rangle$ where B is (free) of rank $\leq r$. It suffices to show that A' has rank $\leq r$. If not, then $A' = B \star C$, and B has rank r. But then $\Theta(\Delta) = \Theta(\Phi) * C$ which has rank $r+1$, a contradiction.

We can now sketch the 3-free theorem.

We have $k = 3$, $\lambda = \log 3$. Take X to be the union of the open 1- and 2-simplices in K_λ. The $\log 3$-Theorem implies that $\Theta(\Delta)$ is free of rank 2 for any 2-simplex Δ. Clearly $\Theta(\Delta)$ is free of rank 2 if Δ is a 1-simplex.

Next use geometry to show that the link of each vertex of K_λ is connected, so X is connected. The lemma shows that $\Theta(X)$ has local rank 2. Note that $\Theta(X)$ is a normal subgroup of $\pi_1(M)$.

Lemma. *If Γ is a k-free group with a normal subgroup of local rank $r < k$ then Γ has local rank $\leq r$.*

Since $\Theta(X)$ is normal, $\pi_1(M)$ is free of rank 2, a contradiction.

What if $\pi_1(M)$ is 4-free? Now we take $k = 4$ and $\lambda = \log 7$.

If we could show that the $Z_\lambda(C)$ cannot cover $\mathbb{H}^3$ we would conclude that M has maximal injectivity radius at least $\frac{1}{2}\log 7$, which implies $\mathrm{vol}\, M > 5.7389$. But that would be asking too much (I think).

We can show that there exists a point of $\mathbb{H}^3$ which lies in *at most one* cylinder $Z_\lambda(C)$. Geometrically, this means that there exists $x \in M$ such that any two geodesic loops at x with length $< \log 7$ represent commuting elements of $\pi_1(M)$. Call this a λ-semi-thick point.

Theorem (C-Shalen). *If $\pi_1(M)$ is 4-free, then M has a $\log 7$-semi-thick point.*

Theorem (C-Shalen). *If a closed hyperbolic 3-manifold has a $\log 7$-semi-thick point then $\mathrm{vol}\, M > 3.44$.*

Take $k = 4$, $\lambda = \log 7$, and consider the complex $K = K_\lambda$. Since $\pi_1(M)$ is 4-free, $\Theta(\Delta)$ is free of rank at most 3 (and at least 2) when Δ is a simplex of dimension 1, 2, or 3.

Let X_2 (X_3) be the union of all open simplices Δ in $K^{(3)} - K^{(0)}$ such that $\Theta(\Delta)$ is free of rank 2 (3).

As before, since X_3 contains only 2- and 3-simplices, $\Theta(X_3)$ has local rank at most 3.

We claim that $\Theta(X_2)$ has local rank at most 2. To prove this by induction we need an important special case of the Hanna Neumann conjecture:

Theorem (Kent, Louder-McReynolds 2009). *If A and B are rank-2 subgroups of a free group, and $A \cap B$ has rank 2, then $\langle A \cup B \rangle$ has rank 2.*

To prove the 4-free result, we construct a bipartite graph $\mathcal{G}$ with a $\pi_1(M)$-action as follows:

Vertices are components of X_2 or X_3. Join V and W by an edge if some simplex of V (W) is a face of some simplex of W (V).

Lemma. *If every point of $\mathbb{H}^3$ lies in two cylinders $Z_\lambda(C)$ then vertices of K have contractible links. In particular $K^{(3)} - K^{(0)}$ is simply-connected.*

Lemma. *The graph $\mathcal{G}$ is a homotopy retract of $K^{(3)} - K^{(0)}$. (Hence it is a tree.)*

Thus $\pi_1(M)$ acts on a tree with locally free vertex stabilizers. That is absurd since edge groups must contain surface groups.