An Introduction to the Volume Conjecture, II
Why we expect the conjecture is true.

Hitoshi Murakami

Tokyo Institute of Technology

8th June, 2009

1. Example of calculation
2. Geometric interpretation of the R-matrix
3. Approximation of the colored Jones polynomial
4. Geometric interpretation of the limit

Review of the definition

Review of the definition

$$R^{ij}_{kl} := \sum_{m=0}^{\min(N-1-i,j)} \delta_{l,i+m}\delta_{k,j-m}\frac{\{l\}!\{N-1-k\}!}{\{i\}!\{m\}!\{N-1-j\}!} \times q^{\left(i-(N-1)/2\right)\left(j-(N-1)/2\right)-m(i-j)/2-m(m+1)/4},$$

Review of the definition

$$R^{ij}_{kl} := \sum_{m=0}^{\min(N-1-i,j)} \delta_{l,i+m}\delta_{k,j-m}\frac{\{l\}!\{N-1-k\}!}{\{i\}!\{m\}!\{N-1-j\}!} \times q^{\left(i-(N-1)/2\right)\left(j-(N-1)/2\right)-m(i-j)/2-m(m+1)/4},$$

$$(R^{-1})^{ij}_{kl} := \sum_{m=0}^{\min(N-1-i,j)} \delta_{l,i-m}\delta_{k,j+m}\frac{\{k\}!\{N-1-l\}!}{\{j\}!\{m\}!\{N-1-i\}!} \times q^{-\left(i-(N-1)/2\right)\left(j-(N-1)/2\right)-m(i-j)/2+m(m+1)/4},$$

Review of the definition

$$R^{ij}_{kl} := \sum_{m=0}^{\min(N-1-i,j)} \delta_{l,i+m}\delta_{k,j-m}\frac{\{l\}!\{N-1-k\}!}{\{i\}!\{m\}!\{N-1-j\}!} \times q^{\left(i-(N-1)/2\right)\left(j-(N-1)/2\right)-m(i-j)/2-m(m+1)/4},$$

$$(R^{-1})^{ij}_{kl} := \sum_{m=0}^{\min(N-1-i,j)} \delta_{l,i-m}\delta_{k,j+m}\frac{\{k\}!\{N-1-l\}!}{\{j\}!\{m\}!\{N-1-i\}!} \times q^{-\left(i-(N-1)/2\right)\left(j-(N-1)/2\right)-m(i-j)/2+m(m+1)/4},$$

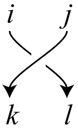
with $\{m\} := q^{m/2} - q^{-m/2}$ and $\{m\}! := \{1\}\{2\}\cdots\{m\}$.

Review of the definition

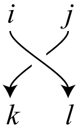
$$R^{ij}_{kl} := \sum_{m=0}^{\min(N-1-i,j)} \delta_{l,i+m}\delta_{k,j-m}\frac{\{l\}!\{N-1-k\}!}{\{i\}!\{m\}!\{N-1-j\}!} \times q^{\left(i-(N-1)/2\right)\left(j-(N-1)/2\right)-m(i-j)/2-m(m+1)/4},$$

$$(R^{-1})^{ij}_{kl} := \sum_{m=0}^{\min(N-1-i,j)} \delta_{l,i-m}\delta_{k,j+m}\frac{\{k\}!\{N-1-l\}!}{\{j\}!\{m\}!\{N-1-i\}!} \times q^{-\left(i-(N-1)/2\right)\left(j-(N-1)/2\right)-m(i-j)/2+m(m+1)/4},$$

with $\{m\} := q^{m/2} - q^{-m/2}$ and $\{m\}! := \{1\}\{2\}\cdots\{m\}$.



$\Rightarrow$ R^{ij}_{kl},



$\Rightarrow$ $(R^{-1})^{ij}_{kl}$

An example of calculation

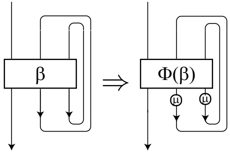
An example of calculation

$$J_N(L;q) := T_{(R,\mu,q^{(N^2-1)/4},1)}(L) \times \frac{\{1\}}{\{N\}}$$

An example of calculation

$$J_N(L;q) := T_{(R,\mu,q^{(N^2-1)/4},1)}(L) \times \frac{\{1\}}{\{N\}}$$

To calculate $J_N(L;q)$ we leave the left-most strand without closing.

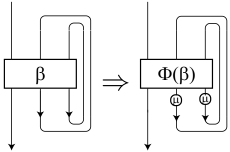


This gives a linear map $\varphi\colon \mathbb{C}^N \to \mathbb{C}^N$, which is a scalar multiple by Schur's lemma.

An example of calculation

$$J_N(L;q) := T_{(R,\mu,q^{(N^2-1)/4},1)}(L) \times \frac{\{1\}}{\{N\}}$$

To calculate $J_N(L;q)$ we leave the left-most strand without closing.



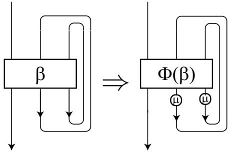
This gives a linear map $\varphi\colon \mathbb{C}^N \to \mathbb{C}^N$, which is a scalar multiple by Schur's lemma.

We fix a basis $\{e_0, e_1, \ldots, e_{N-1}\}$ of C^N.

An example of calculation

$$J_N(L;q) := T_{(R,\mu,q^{(N^2-1)/4},1)}(L) \times \frac{\{1\}}{\{N\}}$$

To calculate $J_N(L;q)$ we leave the left-most strand without closing.



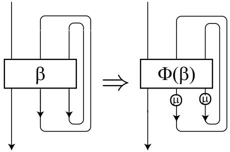
This gives a linear map $\varphi\colon \mathbb{C}^N \to \mathbb{C}^N$, which is a scalar multiple by Schur's lemma.

We fix a basis $\{e_0, e_1, \ldots, e_{N-1}\}$ of C^N. The linear map is a scalar multiple and so e_i is multiplied by S for any i.

An example of calculation

$$J_N(L;q) := T_{(R,\mu,q^{(N^2-1)/4},1)}(L) \times \frac{\{1\}}{\{N\}}$$

To calculate $J_N(L;q)$ we leave the left-most strand without closing.



This gives a linear map $\varphi\colon \mathbb{C}^N \to \mathbb{C}^N$, which is a scalar multiple by Schur's lemma.

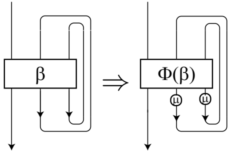
We fix a basis $\{e_0, e_1, \dots, e_{N-1}\}$ of C^N. The linear map is a scalar multiple and so e_i is multiplied by S for any i. Since

$$\begin{aligned} T_{(R,\mu,q^{(N^2-1)/4},1)}(L) &= q^{-w(\beta)(N^2-1)/4}\,\mathrm{Tr}_1(\phi\mu) \\ &= q^{-w(\beta)(N^2-1)/4}\sum_{i=0}^{N-1} S\,q^{(2i-N+1)/2} \\ &= q^{-w(\beta)(N^2-1)/4}\frac{\{N\}}{\{1\}}S, \end{aligned}$$

An example of calculation

$$J_N(L;q) := T_{(R,\mu,q^{(N^2-1)/4},1)}(L) \times \frac{\{1\}}{\{N\}}$$

To calculate $J_N(L;q)$ we leave the left-most strand without closing.



This gives a linear map $\varphi\colon \mathbb{C}^N \to \mathbb{C}^N$, which is a scalar multiple by Schur's lemma.

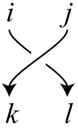
We fix a basis $\{e_0, e_1, \dots, e_{N-1}\}$ of C^N. The linear map is a scalar multiple and so e_i is multiplied by S for any i. Since

$$\begin{aligned} T_{(R,\mu,q^{(N^2-1)/4},1)}(L) &= q^{-w(\beta)(N^2-1)/4}\,\mathrm{Tr}_1(\phi\mu) \\ &= q^{-w(\beta)(N^2-1)/4}\sum_{i=0}^{N-1} S\, q^{(2i-N+1)/2} \\ &= q^{-w(\beta)(N^2-1)/4}\frac{\{N\}}{\{1\}}S, \end{aligned}$$

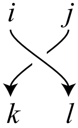
we have $J_N(L;q) = q^{-w(\beta)(N^2-1)/4}S$.

How to label arcs

How to label arcs

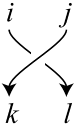


$\Rightarrow i + j = k + l,\ l \geq i,\ k \leq j,$

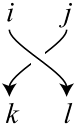


$\Rightarrow i + j = k + l,\ l \leq i,\ k \geq j.$

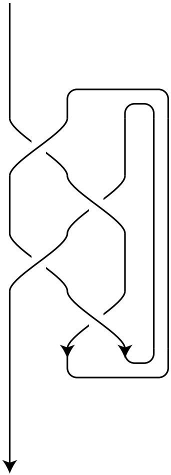
How to label arcs



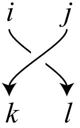
$\Rightarrow i + j = k + l,\ l \geq i,\ k \leq j,$



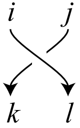
$\Rightarrow i + j = k + l,\ l \leq i,\ k \geq j.$



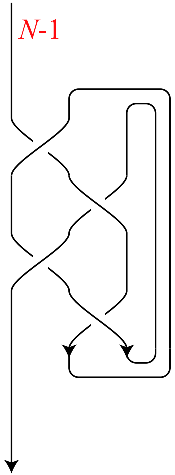
How to label arcs



$\Rightarrow i + j = k + l,\ l \geq i,\ k \leq j,$

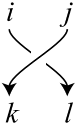


$\Rightarrow i + j = k + l,\ l \leq i,\ k \geq j.$

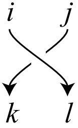


Label the incoming arc with $N - 1$.

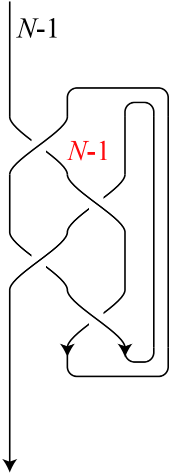
How to label arcs



$\Rightarrow i + j = k + l,\ l \geq i,\ k \leq j,$

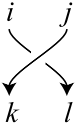


$\Rightarrow i + j = k + l,\ l \leq i,\ k \geq j.$

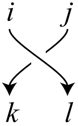


The next one should be $N-1$, since it is $\geq N-1$.

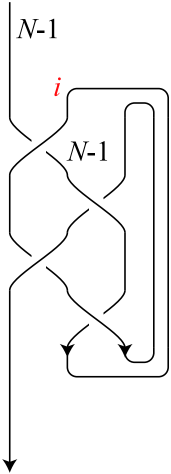
How to label arcs



$\Rightarrow i + j = k + l,\ l \geq i,\ k \leq j,$

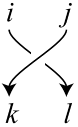


$\Rightarrow i + j = k + l,\ l \leq i,\ k \geq j.$

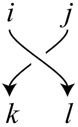


Choose i.

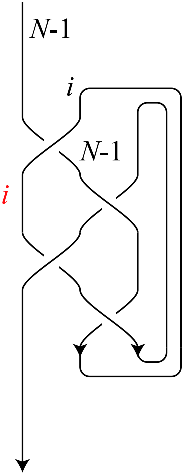
How to label arcs



$\Rightarrow i + j = k + l,\ l \geq i,\ k \leq j,$

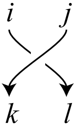


$\Rightarrow i + j = k + l,\ l \leq i,\ k \geq j.$

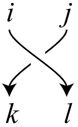


This is also i, since the sum of the labels of the incoming arcs equals the sum of the labels of the outgoing arcs.

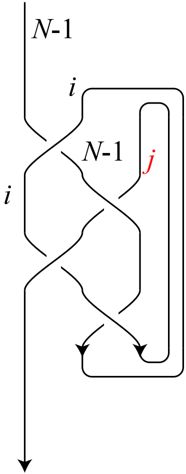
How to label arcs



$\Rightarrow i + j = k + l,\ l \geq i,\ k \leq j,$

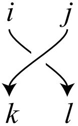


$\Rightarrow i + j = k + l,\ l \leq i,\ k \geq j.$

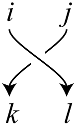


Choose j.

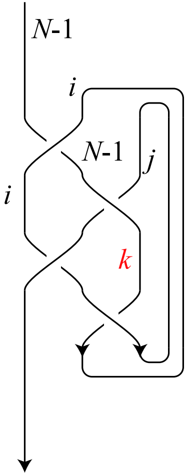
How to label arcs



$\Rightarrow i + j = k + l,\ l \geq i,\ k \leq j,$

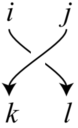


$\Rightarrow i + j = k + l,\ l \leq i,\ k \geq j.$

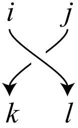


Choose k.

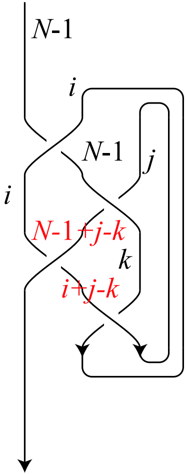
How to label arcs



$\Rightarrow i + j = k + l,\ l \geq i,\ k \leq j,$

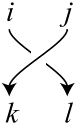


$\Rightarrow i + j = k + l,\ l \leq i,\ k \geq j.$

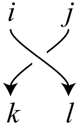


The sum of the labels of the incoming arcs equals the sum of the labels of the outgoing arcs.

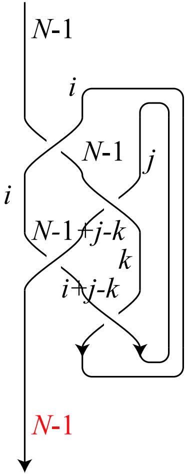
How to label arcs



$\Rightarrow i + j = k + l,\ l \geq i,\ k \leq j,$

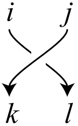


$\Rightarrow i + j = k + l,\ l \leq i,\ k \geq j.$

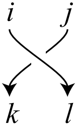


It should be $N - 1$ by the same reason.

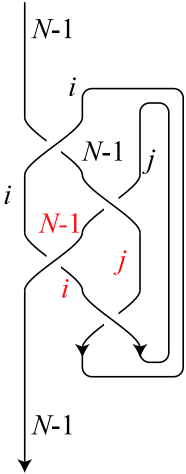
How to label arcs



$\Rightarrow i + j = k + l,\ l \geq i,\ k \leq j,$



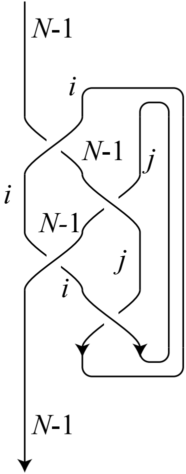
$\Rightarrow i + j = k + l,\ l \leq i,\ k \geq j.$



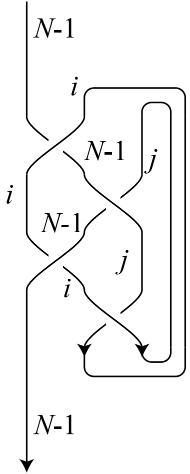
$k = j$, since $N - 1 \leq N - 1 + j + k$

colored Jones polynomial

colored Jones polynomial



colored Jones polynomial



$J_N($$;q)$

$$= \sum_{i\geq j} R^{N-1,i}_{i,N-1}\,(R^{-1})^{N-1,j}_{N-1,j}\,R^{i,N-1}_{N-1,i}\,(R^{-1})^{i,j}_{i,j}\,\mu^j_j\,\mu^i_i$$

$$= \sum_{i\geq j} (-1)^{N-1+i}\frac{\{N-1\}!\{i\}!\{N-1-j\}!}{(\{j\}!)^2\{i-j\}!\{N-1-i\}!}$$

$$\times\, q^{(-i-i^2-2ij-2j^2+3N+6Ni+2Nj-3N^2)/4}$$

Quantum factorial at the N-th root of unity

Quantum factorial at the N-th root of unity

$q = \zeta_N := \exp(2\pi\sqrt{-1}/N)$

Quantum factorial at the N-th root of unity

$q = \zeta_N := \exp(2\pi\sqrt{-1}/N)$
$\Rightarrow$

Quantum factorial at the N-th root of unity

$q = \zeta_N := \exp(2\pi\sqrt{-1}/N)$

$\Rightarrow$
$$\{k\}!\{N-k-1\}!$$

Quantum factorial at the N-th root of unity

$q = \zeta_N := \exp(2\pi\sqrt{-1}/N)$

$$\Rightarrow \quad \begin{aligned}&\{k\}!\{N-k-1\}! \\ &= \pm\,(\text{a power of } \zeta_N) \times (1-\zeta_N)(1-\zeta_N^2)\cdots(1-\zeta_N^k) \\ &\quad \times (1-\zeta_N)(1-\zeta_N^2)\cdots(1-\zeta_N^{N-1-k})\end{aligned}$$

Quantum factorial at the N-th root of unity

$q = \zeta_N := \exp(2\pi\sqrt{-1}/N)$

$$
\Rightarrow \quad \begin{aligned}
&\{k\}!\{N-k-1\}! \\
&= \pm\,(\text{a power of } \zeta_N) \times (1-\zeta_N)(1-\zeta_N^2)\cdots(1-\zeta_N^k) \\
&\quad \times (1-\zeta_N)(1-\zeta_N^2)\cdots(1-\zeta_N^{N-1-k}) \\
&= \pm\,(\text{a power of } \zeta_N) \times (1-\zeta_N)(1-\zeta_N^2)\cdots(1-\zeta_N^k) \\
&\quad \times (1-\zeta_N^{N-1})(1-\zeta_N^{N-2})\cdots(1-\zeta_N^{k+1})
\end{aligned}
$$

Quantum factorial at the N-th root of unity

$q = \zeta_N := \exp(2\pi\sqrt{-1}/N)$

$\Rightarrow$

$$
\begin{aligned}
&\{k\}!\{N-k-1\}! \\
&= \pm\,(\text{a power of } \zeta_N) \times (1-\zeta_N)(1-\zeta_N^2)\cdots(1-\zeta_N^k) \\
&\quad \times (1-\zeta_N)(1-\zeta_N^2)\cdots(1-\zeta_N^{N-1-k}) \\
&= \pm\,(\text{a power of } \zeta_N) \times (1-\zeta_N)(1-\zeta_N^2)\cdots(1-\zeta_N^k) \\
&\quad \times (1-\zeta_N^{N-1})(1-\zeta_N^{N-2})\cdots(1-\zeta_N^{k+1}) \\
&= \pm\,(\text{a power of } \zeta_N) \times 2^{N-1}\sin(\pi/N)\sin(2\pi/N)\cdots\sin((N-1)\pi/N)
\end{aligned}
$$

Quantum factorial at the N-th root of unity

$q = \zeta_N := \exp(2\pi\sqrt{-1}/N)$

$\Rightarrow$

$$
\begin{aligned}
&\{k\}!\{N-k-1\}! \\
&= \pm\,(\text{a power of } \zeta_N) \times (1-\zeta_N)(1-\zeta_N^2)\cdots(1-\zeta_N^k) \\
&\quad \times (1-\zeta_N)(1-\zeta_N^2)\cdots(1-\zeta_N^{N-1-k}) \\
&= \pm\,(\text{a power of } \zeta_N) \times (1-\zeta_N)(1-\zeta_N^2)\cdots(1-\zeta_N^k) \\
&\quad \times (1-\zeta_N^{N-1})(1-\zeta_N^{N-2})\cdots(1-\zeta_N^{k+1}) \\
&= \pm\,(\text{a power of } \zeta_N) \times 2^{N-1}\sin(\pi/N)\sin(2\pi/N)\cdots\sin((N-1)\pi/N) \\
&= \pm\,(\text{a power of } \zeta_N) \times N
\end{aligned}
$$

Quantum factorial at the N-th root of unity

$q = \zeta_N := \exp(2\pi\sqrt{-1}/N)$

$$
\begin{aligned}
\Rightarrow\quad & \{k\}!\{N-k-1\}! \\
&= \pm\,(\text{a power of } \zeta_N) \times (1-\zeta_N)(1-\zeta_N^2)\cdots(1-\zeta_N^k) \\
&\quad \times (1-\zeta_N)(1-\zeta_N^2)\cdots(1-\zeta_N^{N-1-k}) \\
&= \pm\,(\text{a power of } \zeta_N) \times (1-\zeta_N)(1-\zeta_N^2)\cdots(1-\zeta_N^k) \\
&\quad \times (1-\zeta_N^{N-1})(1-\zeta_N^{N-2})\cdots(1-\zeta_N^{k+1}) \\
&= \pm\,(\text{a power of } \zeta_N) \times 2^{N-1}\sin(\pi/N)\sin(2\pi/N)\cdots\sin((N-1)\pi/N) \\
&= \pm\,(\text{a power of } \zeta_N) \times N
\end{aligned}
$$

- $(\zeta_N)_{k^+} := (1-\zeta_N)\cdots(1-\zeta_N^k)$, $(\zeta_N)_{k^-} := (1-\zeta_N)\cdots(1-\zeta_N^{N-1-k})$.

Quantum factorial at the N-th root of unity

$q = \zeta_N := \exp(2\pi\sqrt{-1}/N)$

$$
\begin{aligned}
\Rightarrow\quad & \{k\}!\{N-k-1\}! \\
&= \pm\,(\text{a power of } \zeta_N) \times (1-\zeta_N)(1-\zeta_N^2)\cdots(1-\zeta_N^k) \\
&\quad \times (1-\zeta_N)(1-\zeta_N^2)\cdots(1-\zeta_N^{N-1-k}) \\
&= \pm\,(\text{a power of } \zeta_N) \times (1-\zeta_N)(1-\zeta_N^2)\cdots(1-\zeta_N^k) \\
&\quad \times (1-\zeta_N^{N-1})(1-\zeta_N^{N-2})\cdots(1-\zeta_N^{k+1}) \\
&= \pm\,(\text{a power of } \zeta_N) \times 2^{N-1}\sin(\pi/N)\sin(2\pi/N)\cdots\sin((N-1)\pi/N) \\
&= \pm\,(\text{a power of } \zeta_N) \times N
\end{aligned}
$$

- $(\zeta_N)_{k^+} := (1-\zeta_N)\cdots(1-\zeta_N^k)$, $(\zeta_N)_{k^-} := (1-\zeta_N)\cdots(1-\zeta_N^{N-1-k})$.
- $(\zeta_N)_{k^+}(\zeta_N)_{k^-} = \pm(\text{a power of } \zeta_N) \times N$.

Quantum factorial at the N-th root of unity

$q = \zeta_N := \exp(2\pi\sqrt{-1}/N)$

$$\Rightarrow \quad \begin{aligned} &\{k\}!\{N-k-1\}! \\ &= \pm\,(\text{a power of } \zeta_N) \times (1-\zeta_N)(1-\zeta_N^2)\cdots(1-\zeta_N^k) \\ &\quad \times (1-\zeta_N)(1-\zeta_N^2)\cdots(1-\zeta_N^{N-1-k}) \\ &= \pm\,(\text{a power of } \zeta_N) \times (1-\zeta_N)(1-\zeta_N^2)\cdots(1-\zeta_N^k) \\ &\quad \times (1-\zeta_N^{N-1})(1-\zeta_N^{N-2})\cdots(1-\zeta_N^{k+1}) \\ &= \pm\,(\text{a power of } \zeta_N) \times 2^{N-1}\sin(\pi/N)\sin(2\pi/N)\cdots\sin((N-1)\pi/N) \\ &= \pm\,(\text{a power of } \zeta_N) \times N \end{aligned}$$

- $(\zeta_N)_{k^+} := (1-\zeta_N)\cdots(1-\zeta_N^k)$, $(\zeta_N)_{k^-} := (1-\zeta_N)\cdots(1-\zeta_N^{N-1-k})$.
- $(\zeta_N)_{k^+}(\zeta_N)_{k^-} = \pm(\text{a power of } \zeta_N) \times N$.
- $\{k\}! = \pm(\text{a power of } \zeta_N) \times (\zeta_N)_{k^+}$,
  $\{N-1-k\}! = \pm(\text{a power of } \zeta_N) \times (\zeta_N)_{k^-}$.

R-matrix as a product of quantum factorial

R-matrix as a product of quantum factorial

$$R^{ij}_{kl} = \sum_{m} \pm(\text{a power of } \zeta_N) \times \delta_{l,i+m}\delta_{k,j-m} \frac{\{l\}!\{N-1-k\}!}{\{i\}!\{m\}!\{N-1-j\}!}$$

R-matrix as a product of quantum factorial

$$
\begin{aligned}
R^{ij}_{kl} &= \sum_m \pm(\text{a power of } \zeta_N) \times \delta_{l,i+m}\delta_{k,j-m} \frac{\{l\}!\{N-1-k\}!}{\{i\}!\{m\}!\{N-1-j\}!} \\
&= \sum_m \delta_{l,i+m}\delta_{k,j-m} \frac{\pm(\text{a power of } \zeta_N) \times N^2}{(\zeta_N)_{m^+}(\zeta_N)_{i^+}(\zeta_N)_{k^+}(\zeta_N)_{j^-}(\zeta_N)_{l^-}}
\end{aligned}
$$

R-matrix as a product of quantum factorial

$$R^{ij}_{kl} = \sum_m \pm(\text{a power of } \zeta_N) \times \delta_{l,i+m}\delta_{k,j-m} \frac{\{l\}!\{N-1-k\}!}{\{i\}!\{m\}!\{N-1-j\}!}$$

$$= \sum_m \delta_{l,i+m}\delta_{k,j-m} \frac{\pm(\text{a power of } \zeta_N) \times N^2}{(\zeta_N)_{m^+}(\zeta_N)_{i^+}(\zeta_N)_{k^+}(\zeta_N)_{j^-}(\zeta_N)_{l^-}}$$

$$(R^{-1})^{ij}_{kl} = \sum_m \delta_{l,i-m}\delta_{k,j+m} \frac{\pm(\text{a power of } \zeta_N) \times N^{-2}}{(\zeta_N)_{m^+}(\zeta_N)_{i^-}(\zeta_N)_{k^-}(\zeta_N)_{j^+}(\zeta_N)_{l^+}}$$

R-matrix as a product of quantum factorial

$$R^{ij}_{kl} = \sum_m \pm(\text{a power of } \zeta_N) \times \delta_{l,i+m}\delta_{k,j-m} \frac{\{l\}!\{N-1-k\}!}{\{i\}!\{m\}!\{N-1-j\}!}$$

$$= \sum_m \delta_{l,i+m}\delta_{k,j-m} \frac{\pm(\text{a power of } \zeta_N) \times N^2}{(\zeta_N)_{m^+}(\zeta_N)_{i^+}(\zeta_N)_{k^+}(\zeta_N)_{j^-}(\zeta_N)_{l^-}}$$

$$(R^{-1})^{ij}_{kl} = \sum_m \delta_{l,i-m}\delta_{k,j+m} \frac{\pm(\text{a power of } \zeta_N) \times N^{-2}}{(\zeta_N)_{m^+}(\zeta_N)_{i^-}(\zeta_N)_{k^-}(\zeta_N)_{j^+}(\zeta_N)_{l^+}}$$

$\Rightarrow$

$$J_N(K;\zeta_N) = \sum_{\substack{\text{labellings}\\ i,j,k,l\\ \text{on arcs}}} \left(\prod_{\pm\text{-crossings}} \frac{\pm(\text{a power of } \zeta_N) \times N^{\pm 2}}{(\zeta_N)_{m^+}(\zeta_N)_{i^\pm}(\zeta_N)_{k^\pm}(\zeta_N)_{j^\mp}(\zeta_N)_{l^\mp}} \right)$$

Approximation of the quantum factorial

$$\log(\zeta_N)_{k^+} = \sum_{j=1}^{k} \log(1 - \zeta_N^j)$$

Approximation of the quantum factorial

$$\log(\zeta_N)_{k^+} = \sum_{j=1}^{k} \log(1-\zeta_N^j) = \sum_{j=1}^{k} \log\bigl(1-\exp(2\pi\sqrt{-1}j/N)\bigr)$$

Approximation of the quantum factorial

$$\log(\zeta_N)_{k^+} = \sum_{j=1}^{k} \log(1 - \zeta_N^j)$$

$$= \sum_{j=1}^{k} \log\bigl(1 - \exp(2\pi\sqrt{-1}j/N)\bigr)$$

$$(x := j/N)$$

Approximation of the quantum factorial

$$\begin{aligned}\log(\zeta_N)_{k^+} &= \sum_{j=1}^{k} \log(1-\zeta_N^j) \\ &= \sum_{j=1}^{k} \log\left(1-\exp(2\pi\sqrt{-1}j/N)\right) \\ &(x := j/N) \\ &\underset{N\to\infty}{\approx} N\int_0^{k/N} \log\left(1-\exp(2\pi\sqrt{-1}x)\right)\,dx\end{aligned}$$

Approximation of the quantum factorial

$$
\begin{aligned}
\log(\zeta_N)_{k^+} &= \sum_{j=1}^{k} \log(1-\zeta_N^j) \\
&= \sum_{j=1}^{k} \log\bigl(1-\exp(2\pi\sqrt{-1}j/N)\bigr) \\
&\quad (x := j/N) \\
&\underset{N\to\infty}{\approx} N\int_0^{k/N} \log\bigl(1-\exp(2\pi\sqrt{-1}x)\bigr)\,dx \\
&\quad (y := \exp(2\pi\sqrt{-1}x))
\end{aligned}
$$

Approximation of the quantum factorial

$$\begin{aligned}
\log(\zeta_N)_{k^+} &= \sum_{j=1}^{k} \log(1-\zeta_N^j) \\
&= \sum_{j=1}^{k} \log\left(1-\exp(2\pi\sqrt{-1}j/N)\right) \\
&(x := j/N) \\
&\underset{N\to\infty}{\approx} N\int_0^{k/N} \log\left(1-\exp(2\pi\sqrt{-1}x)\right)dx \\
&(y := \exp(2\pi\sqrt{-1}x)) \\
&= \frac{N}{2\pi\sqrt{-1}}\int_1^{\exp(2\pi\sqrt{-1}k/N)} \frac{\log(1-y)}{y}\,dy
\end{aligned}$$

Approximation of the quantum factorial by dilogarithm

Approximation of the quantum factorial by dilogarithm

- (dilog function)

$$\mathrm{Li}_2(z) := -\int_0^z \frac{\log(1-y)}{y}\,dy = \sum_{n=1}^{\infty} \frac{z^n}{n^2}.$$

Approximation of the quantum factorial by dilogarithm

- (dilog function)

$$\mathrm{Li}_2(z) := -\int_0^z \frac{\log(1-y)}{y}\,dy = \sum_{n=1}^{\infty} \frac{z^n}{n^2}.$$

- $$\log(\zeta_N)_{k^+} \underset{N\to\infty}{\approx} \frac{N}{2\pi\sqrt{-1}} \left[\mathrm{Li}_2(1) - \mathrm{Li}_2(\zeta_N^k)\right].$$

Approximation of the quantum factorial by dilogarithm

- (dilog function)

$$\mathrm{Li}_2(z) := -\int_0^z \frac{\log(1-y)}{y}\,dy = \sum_{n=1}^{\infty} \frac{z^n}{n^2}.$$

-

$$\log(\zeta_N)_{k^+} \underset{N\to\infty}{\approx} \frac{N}{2\pi\sqrt{-1}}\left[\mathrm{Li}_2(1) - \mathrm{Li}_2(\zeta_N^k)\right].$$

-

$$(\zeta_N)_{k^\pm} \underset{N\to\infty}{\approx} \exp\left[-\frac{N}{2\pi\sqrt{-1}}\,\mathrm{Li}_2(\zeta_N^{\pm k})\right].$$

Approximation of the colored Jones polynomial by dilogarithm

Approximation of the colored Jones polynomial by dilogarithm

$$J_N(K;\zeta_N) \underset{N\to\infty}{\approx} \sum_{\text{labellings}} (\text{polynomial of } N) \exp\left[\frac{N}{2\pi\sqrt{-1}} \sum_{\text{crossings}} \left\{\mathrm{Li}_2(\zeta_N^{m}) + \mathrm{Li}_2(\zeta_N^{\pm i}) + \mathrm{Li}_2(\zeta_N^{\mp j}) + \mathrm{Li}_2(\zeta_N^{\pm k}) + \mathrm{Li}_2(\zeta_N^{\mp l}) + \text{log terms}\right\}\right]$$

Approximation of the colored Jones polynomial by dilogarithm

$$J_N(K;\zeta_N) \underset{N\to\infty}{\approx} \sum_{\text{labellings}} (\text{polynomial of } N) \exp\left[\frac{N}{2\pi\sqrt{-1}} \sum_{\text{crossings}} \left\{\mathrm{Li}_2(\zeta_N^{m}) + \mathrm{Li}_2(\zeta_N^{\pm i}) + \mathrm{Li}_2(\zeta_N^{\mp j}) + \mathrm{Li}_2(\zeta_N^{\pm k}) + \mathrm{Li}_2(\zeta_N^{\mp l}) + \text{log terms}\right\}\right]$$

where a log term comes from powers of ζ_N. For example

$$q^{k^2} = \exp\left(\frac{N}{2\pi\sqrt{-1}}\left(\frac{2\pi\sqrt{-1}k}{N}\right)^2\right) = \exp\left[\frac{N}{2\pi\sqrt{-1}}(\log\zeta_N^{k})^2\right].$$

Approximation of the colored Jones polynomial by integral

Approximation of the colored Jones polynomial by integral

Put

$$V(\zeta_N^{i_1},\dots,\zeta_N^{i_c}) :=$$
$$\sum_{\text{crossings}} \left\{ \mathrm{Li}_2(\zeta_N^{m}) + \mathrm{Li}_2(\zeta_N^{\pm i}) + \mathrm{Li}_2(\zeta_N^{\mp j}) + \mathrm{Li}_2(\zeta_N^{\pm k}) + \mathrm{Li}_2(\zeta_N^{\mp l}) + \text{log terms} \right\},$$

where $i_1,\dots,i_c$ are labellings on arcs.

Approximation of the colored Jones polynomial by integral

Put

$$V(\zeta_N^{i_1}, \dots, \zeta_N^{i_c}) :=$$

$$\sum_{\text{crossings}} \left\{ \mathrm{Li}_2(\zeta_N^{m}) + \mathrm{Li}_2(\zeta_N^{\pm i}) + \mathrm{Li}_2(\zeta_N^{\mp j}) + \mathrm{Li}_2(\zeta_N^{\pm k}) + \mathrm{Li}_2(\zeta_N^{\mp l}) + \log \text{ terms} \right\},$$

where $i_1, \dots, i_c$ are labellings on arcs.

$$J_N(K; \zeta_N) \underset{N\to\infty}{\approx} \sum_{i_1,\dots,i_c} (\text{polynomial of } N) \exp\left[\frac{N}{2\pi\sqrt{-1}} V(\zeta_N^{i_1}, \dots, \zeta_N^{i_c})\right]$$

(ignore polynomials since exp grows much bigger)

Approximation of the colored Jones polynomial by integral

Put

$$V(\zeta_N^{i_1}, \dots, \zeta_N^{i_c}) :=$$

$$\sum_{\text{crossings}} \left\{ \mathrm{Li}_2(\zeta_N^{m}) + \mathrm{Li}_2(\zeta_N^{\pm i}) + \mathrm{Li}_2(\zeta_N^{\mp j}) + \mathrm{Li}_2(\zeta_N^{\pm k}) + \mathrm{Li}_2(\zeta_N^{\mp l}) + \log \text{ terms} \right\},$$

where $i_1, \dots, i_c$ are labellings on arcs.

$$J_N(K; \zeta_N) \underset{N\to\infty}{\approx} \sum_{i_1,\dots,i_c} (\text{polynomial of } N) \exp\left[\frac{N}{2\pi\sqrt{-1}} V(\zeta_N^{i_1}, \dots, \zeta_N^{i_c}) \right]$$

(ignore polynomials since exp grows much bigger)

$$\underset{N\to\infty}{\approx} \sum_{i_1,\dots,i_c} \exp\left[\frac{N}{2\pi\sqrt{-1}} V(\zeta_N^{i_1}, \dots, \zeta_N^{i_c}) \right]$$

Approximation of the colored Jones polynomial by integral

Put

$$V(\zeta_N^{i_1},\dots,\zeta_N^{i_c}) :=$$
$$\sum_{\text{crossings}} \left\{ \mathrm{Li}_2(\zeta_N^{m}) + \mathrm{Li}_2(\zeta_N^{\pm i}) + \mathrm{Li}_2(\zeta_N^{\mp j}) + \mathrm{Li}_2(\zeta_N^{\pm k}) + \mathrm{Li}_2(\zeta_N^{\mp l}) + \text{log terms} \right\},$$

where $i_1,\dots,i_c$ are labellings on arcs.

$$J_N(K;\zeta_N) \underset{N\to\infty}{\approx} \sum_{i_1,\dots,i_c} (\text{polynomial of } N) \exp\left[\frac{N}{2\pi\sqrt{-1}} V(\zeta_N^{i_1},\dots,\zeta_N^{i_c})\right]$$

(ignore polynomials since exp grows much bigger)

$$\underset{N\to\infty}{\approx} \sum_{i_1,\dots,i_c} \exp\left[\frac{N}{2\pi\sqrt{-1}} V(\zeta_N^{i_1},\dots,\zeta_N^{i_c})\right]$$

$$\underset{N\to\infty}{\approx} \int_{J_1}\cdots\int_{J_c} \exp\left[\frac{N}{2\pi\sqrt{-1}} V(z_1,\dots,z_c)\, dz_1\cdots dz_c\right],$$

Approximation of the colored Jones polynomial by integral

Put

$$V(\zeta_N^{i_1},\dots,\zeta_N^{i_c}) :=$$
$$\sum_{\text{crossings}} \left\{ \mathrm{Li}_2(\zeta_N^{m}) + \mathrm{Li}_2(\zeta_N^{\pm i}) + \mathrm{Li}_2(\zeta_N^{\mp j}) + \mathrm{Li}_2(\zeta_N^{\pm k}) + \mathrm{Li}_2(\zeta_N^{\mp l}) + \text{log terms} \right\},$$

where $i_1,\dots,i_c$ are labellings on arcs.

$$J_N(K;\zeta_N) \underset{N\to\infty}{\approx} \sum_{i_1,\dots,i_c} (\text{polynomial of } N) \exp\left[\frac{N}{2\pi\sqrt{-1}} V(\zeta_N^{i_1},\dots,\zeta_N^{i_c})\right]$$

(ignore polynomials since exp grows much bigger)

$$\underset{N\to\infty}{\approx} \sum_{i_1,\dots,i_c} \exp\left[\frac{N}{2\pi\sqrt{-1}} V(\zeta_N^{i_1},\dots,\zeta_N^{i_c})\right]$$
$$\underset{N\to\infty}{\approx} \int_{J_1}\cdots\int_{J_c} \exp\left[\frac{N}{2\pi\sqrt{-1}} V(z_1,\dots,z_c)\,dz_1\cdots dz_c\right],$$

where $J_1,\dots,J_c$ are suitable contours.

Saddle point method

Saddle point method

As in the case of  , we want to find the maximum of

$$\left| \exp\left[\frac{N}{2\pi\sqrt{-1}} V(z_1, \dots, z_c)\, dz_1 \cdots dz_c \right] \right|.$$

Saddle point method

As in the case of , we want to find the maximum of

$$\left|\exp\left[\frac{N}{2\pi\sqrt{-1}}V(z_1,\dots,z_c)\,dz_1\cdots dz_c\right]\right|.$$

$V(x_1,\dots,x_c)$: the maximum of $\{\operatorname{Im} V(z_1,\dots,z_c)\}_{(z_1,\dots,z_c)\in J_1\times\cdots\times J_c}$.

Saddle point method

As in the case of , we want to find the maximum of

$$\left|\exp\left[\frac{N}{2\pi\sqrt{-1}}V(z_1,\dots,z_c)\,dz_1\cdots dz_c\right]\right|.$$

$V(x_1,\dots,x_c)$: the maximum of $\{\operatorname{Im} V(z_1,\dots,z_c)\}_{(z_1,\dots,z_c)\in J_1\times\cdots\times J_c}$.

- We may need to change the contours J_k.
- $(x_1,\dots,x_c)$ is called the saddle point.

Saddle point method

As in the case of , we want to find the maximum of
$\left|\exp\left[\frac{N}{2\pi\sqrt{-1}}V(z_1,\dots,z_c)\,dz_1\cdots dz_c\right]\right|$.
$V(x_1,\dots,x_c)$: the maximum of $\{\operatorname{Im} V(z_1,\dots,z_c)\}_{(z_1,\dots,z_c)\in J_1\times\cdots\times J_c}$.

- We may need to change the contours J_k.
- $(x_1,\dots,x_c)$ is called the saddle point.

$\Rightarrow$

$$J_N(K;\zeta_N)\underset{N\to\infty}{\approx}\exp\left[\frac{N}{2\pi\sqrt{-1}}V(x_1,\dots,x_c)\right],$$

Saddle point method

As in the case of , we want to find the maximum of $\left|\exp\left[\frac{N}{2\pi\sqrt{-1}}V(z_1,\dots,z_c)\,dz_1\cdots dz_c\right]\right|$.

$V(x_1,\dots,x_c)$: the maximum of $\{\operatorname{Im} V(z_1,\dots,z_c)\}_{(z_1,\dots,z_c)\in J_1\times\cdots\times J_c}$.

- We may need to change the contours J_k.
- $(x_1,\dots,x_c)$ is called the saddle point.

$\Rightarrow$

$$J_N(K;\zeta_N) \underset{N\to\infty}{\approx} \exp\left[\frac{N}{2\pi\sqrt{-1}}V(x_1,\dots,x_c)\right],$$

$(x_1,\dots,x_c)$ satisfies the following equations.

Saddle point method

As in the case of , we want to find the maximum of
$\left|\exp\left[\frac{N}{2\pi\sqrt{-1}}V(z_1,\dots,z_c)\,dz_1\cdots dz_c\right]\right|$.
$V(x_1,\dots,x_c)$: the maximum of $\{\operatorname{Im} V(z_1,\dots,z_c)\}_{(z_1,\dots,z_c)\in J_1\times\cdots\times J_c}$.

- We may need to change the contours J_k.
- $(x_1,\dots,x_c)$ is called the saddle point.

$\Rightarrow$

$$J_N(K;\zeta_N) \underset{N\to\infty}{\approx} \exp\left[\frac{N}{2\pi\sqrt{-1}}V(x_1,\dots,x_c)\right],$$

$(x_1,\dots,x_c)$ satisfies the following equations.

$$\frac{\partial V}{\partial z_k}(x_1,\dots,x_c)=0 \quad (k=1,\dots,c)$$

Saddle point method

As in the case of , we want to find the maximum of
$\left|\exp\left[\frac{N}{2\pi\sqrt{-1}} V(z_1,\dots,z_c)\,dz_1\cdots dz_c\right]\right|$.
$V(x_1,\dots,x_c)$: the maximum of $\{\operatorname{Im} V(z_1,\dots,z_c)\}_{(z_1,\dots,z_c)\in J_1\times\cdots\times J_c}$.

- We may need to change the contours J_k.
- $(x_1,\dots,x_c)$ is called the saddle point.

$\Rightarrow$

$$J_N(K;\zeta_N) \underset{N\to\infty}{\approx} \exp\left[\frac{N}{2\pi\sqrt{-1}} V(x_1,\dots,x_c)\right],$$

$(x_1,\dots,x_c)$ satisfies the following equations.

$$\frac{\partial V}{\partial z_k}(x_1,\dots,x_c) = 0 \quad (k=1,\dots,c)$$

$\Rightarrow$

$$2\pi\sqrt{-1}\lim_{N\to\infty}\frac{J_N(K;\zeta_N)}{N} = V(x_1,\dots,x_c)$$

Difficulties

Difficulties

Difficulties so far:

Difficulties

Difficulties so far:

- Replacing the summation with an integral

$$\sum_{i_1,\dots,i_c} \exp\left[\frac{N}{2\pi\sqrt{-1}} V(\zeta_N^{i_1},\dots,\zeta_N^{i_c})\right]$$

$$\underset{N\to\infty}{\approx} \int_{J_1}\cdots\int_{J_c} \exp\left[\frac{N}{2\pi\sqrt{-1}} V(z_1,\dots,z_c)\,dz_1\cdots dz_c\right].$$

Difficulties

Difficulties so far:

- Replacing the summation with an integral

$$\sum_{i_1,\dots,i_c} \exp\left[\frac{N}{2\pi\sqrt{-1}} V(\zeta_N^{i_1},\dots,\zeta_N^{i_c})\right] \underset{N\to\infty}{\approx} \int_{J_1}\cdots\int_{J_c} \exp\left[\frac{N}{2\pi\sqrt{-1}} V(z_1,\dots,z_c)\,dz_1\cdots dz_c\right].$$

- How to apply the saddle point method. In particular, which saddle point to choose. In general, we have many solutions to the system of equations.

$$\int_{J_1}\cdots\int_{J_c} \exp\left[\frac{N}{2\pi\sqrt{-1}} V(z_1,\dots,z_c)\,dz_1\cdots dz_c\right] \underset{N\to\infty}{\approx} \exp\left[\frac{N}{2\pi\sqrt{-1}} V(x_1,\dots,x_c)\right].$$

Decomposition into octahedra (by D. Thurston)

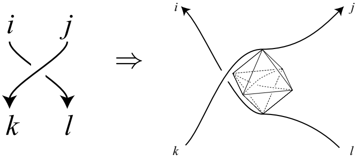
Decomposition into octahedra (by D. Thurston)

Decompose the knot complement into (topological, truncated) tetrahedra.

Decomposition into octahedra (by D. Thurston)

Decompose the knot complement into (topological, truncated) tetrahedra.

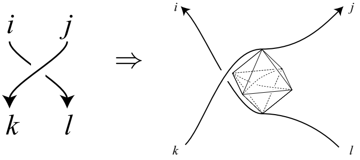
- Around each crossing, put an octahedron:



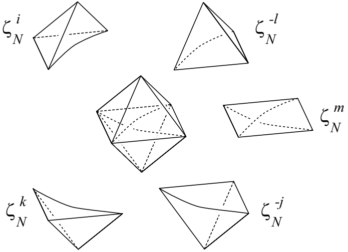
Decomposition into octahedra (by D. Thurston)

Decompose the knot complement into (topological, truncated) tetrahedra.

- Around each crossing, put an octahedron:



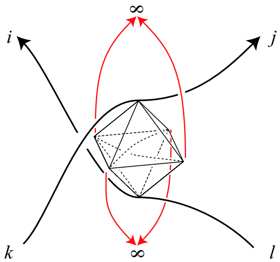
- Decompose the octahedron into five tetrahedra:



Decomposition into topological tetrahedra

Decomposition into topological tetrahedra

- Pull the vertices to the point at infinity:



- $S^3 \setminus K$ is now decomposed into topological, truncated tetrahedra, decorated with complex numbers $\zeta_N^{i_k}$.

Decomposition into hyperbolic tetrahedra

Decomposition into hyperbolic tetrahedra

- Each topological, truncated tetrahedron is decorated with a complex number $\zeta_N^{i_k}$.

Decomposition into hyperbolic tetrahedra

- Each topological, truncated tetrahedron is decorated with a complex number $\zeta_N^{i_k}$.
- We want to regard it as a hyperbolic, ideal tetrahedron.

Decomposition into hyperbolic tetrahedra

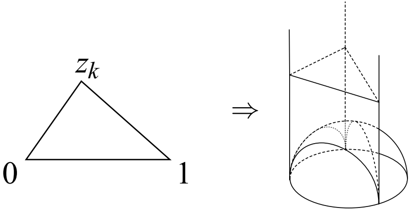
- Each topological, truncated tetrahedron is decorated with a complex number $\zeta_N^{i_k}$.
- We want to regard it as a hyperbolic, ideal tetrahedron.
- Recall that we have replaced a summation over i_k into an integral over z_k.

Decomposition into hyperbolic tetrahedra

- Each topological, truncated tetrahedron is decorated with a complex number $\zeta_N^{i_k}$.
- We want to regard it as a hyperbolic, ideal tetrahedron.
- Recall that we have replaced a summation over i_k into an integral over z_k.
- Replace $\zeta_N^{i_k}$ with a complex variable z_k.

Decomposition into hyperbolic tetrahedra

- Each topological, truncated tetrahedron is decorated with a complex number $\zeta_N^{i_k}$.
- We want to regard it as a hyperbolic, ideal tetrahedron.
- Recall that we have replaced a summation over i_k into an integral over z_k.
- Replace $\zeta_N^{i_k}$ with a complex variable z_k.
- Regard the tetrahedron decorated with z_k as an hyperbolic, ideal tetrahedron parametrized by z_k.



Hyperbolic structure on the knot complement

Hyperbolic structure on the knot complement

- Now the knot complement is decomposed into ideal, hyperbolic tetrahedra parametrized by $z_1, \dots, z_c$.

Hyperbolic structure on the knot complement

- Now the knot complement is decomposed into ideal, hyperbolic tetrahedra parametrized by $z_1, \dots, z_c$.
- Choose $z_1, \dots, z_c$ so that we can glue these tetrahedra well, that is,

Hyperbolic structure on the knot complement

- Now the knot complement is decomposed into ideal, hyperbolic tetrahedra parametrized by $z_1, \dots, z_c$.
- Choose $z_1, \dots, z_c$ so that we can glue these tetrahedra well, that is,
  - around each edge, the sum of angles is 2π,

Hyperbolic structure on the knot complement

- Now the knot complement is decomposed into ideal, hyperbolic tetrahedra parametrized by $z_1, \ldots, z_c$.
- Choose $z_1, \ldots, z_c$ so that we can glue these tetrahedra well, that is,
  - around each edge, the sum of angles is 2π,
  - the triangles that appear in the boundary torus make the torus Euclidean.

Hyperbolic structure on the knot complement

- Now the knot complement is decomposed into ideal, hyperbolic tetrahedra parametrized by $z_1, \dots, z_c$.
- Choose $z_1, \dots, z_c$ so that we can glue these tetrahedra well, that is,
  - around each edge, the sum of angles is 2π,
  - the triangles that appear in the boundary torus make the torus Euclidean.
- These conditions are the same as the system of equations that we used in the saddle point method!

$$\frac{\partial V}{\partial z_k}(x_1, \dots, x_c) = 0 \quad (k = 1, \dots, c)$$

Hyperbolic structure on the knot complement

- Now the knot complement is decomposed into ideal, hyperbolic tetrahedra parametrized by $z_1, \dots, z_c$.
- Choose $z_1, \dots, z_c$ so that we can glue these tetrahedra well, that is,
  - around each edge, the sum of angles is 2π,
  - the triangles that appear in the boundary torus make the torus Euclidean.
- These conditions are the same as the system of equations that we used in the saddle point method!

$$\frac{\partial V}{\partial z_k}(x_1, \dots, x_c) = 0 \quad (k = 1, \dots, c)$$

- $\Rightarrow (x_1, \dots, x_c)$ gives the complete hyperbolic structure.

Hyperbolic structure on the knot complement

- Now the knot complement is decomposed into ideal, hyperbolic tetrahedra parametrized by $z_1, \dots, z_c$.
- Choose $z_1, \dots, z_c$ so that we can glue these tetrahedra well, that is,
  - around each edge, the sum of angles is 2π,
  - the triangles that appear in the boundary torus make the torus Euclidean.
- These conditions are the same as the system of equations that we used in the saddle point method!

$$\frac{\partial V}{\partial z_k}(x_1, \dots, x_c) = 0 \quad (k = 1, \dots, c)$$

- $\Rightarrow (x_1, \dots, x_c)$ gives the complete hyperbolic structure.
- Then, what does $V(x_1, \dots, x_c)(= 2\pi\sqrt{-1} \lim_{N\to\infty} \frac{J_N(K, \zeta_N)}{N})$ mean?

Geometric meaning of the limit

Geometric meaning of the limit

Recall: $V(x_1, \dots, x_c)$ is the sum of $\mathrm{Li}_2(x_k)$ (and log), where x_k defines an ideal hyperbolic tetrahedron.

Geometric meaning of the limit

Recall: $V(x_1, \ldots, x_c)$ is the sum of $\mathrm{Li}_2(x_k)$ (and log), where x_k defines an ideal hyperbolic tetrahedron. We use the following formula:

$$\mathrm{Vol}(\text{tetrahedron parametrized by } z) = \operatorname{Im}\mathrm{Li}_2(z) - \log|z|\arg(1-z).$$

Geometric meaning of the limit

Recall: $V(x_1, \dots, x_c)$ is the sum of $\mathrm{Li}_2(x_k)$ (and log), where x_k defines an ideal hyperbolic tetrahedron. We use the following formula:

$$\mathrm{Vol}(\text{tetrahedron parametrized by } z) = \operatorname{Im}\mathrm{Li}_2(z) - \log|z|\arg(1-z).$$

Therefore we finally have

Geometric meaning of the limit

Recall: $V(x_1, \ldots, x_c)$ is the sum of $\mathrm{Li}_2(x_k)$ (and log), where x_k defines an ideal hyperbolic tetrahedron. We use the following formula:

$$\mathrm{Vol}(\text{tetrahedron parametrized by } z) = \operatorname{Im} \mathrm{Li}_2(z) - \log|z| \arg(1-z).$$

Therefore we finally have

$$\operatorname{Im}\left(2\pi\sqrt{-1} \lim_{N\to\infty} \frac{J_N(K, \zeta_N)}{N}\right) = \mathrm{Vol}(S^3 \setminus K).$$

Geometric meaning of the limit

Recall: $V(x_1,\dots,x_c)$ is the sum of $\mathrm{Li}_2(x_k)$ (and log), where x_k defines an ideal hyperbolic tetrahedron. We use the following formula:

$$\mathrm{Vol}(\text{tetrahedron parametrized by } z) = \operatorname{Im}\mathrm{Li}_2(z) - \log|z|\arg(1-z).$$

Therefore we finally have

$$\operatorname{Im}\left(2\pi\sqrt{-1}\lim_{N\to\infty}\frac{J_N(K,\zeta_N)}{N}\right) = \mathrm{Vol}(S^3\setminus K).$$

$\Rightarrow$

$$2\pi\lim_{N\to\infty}\frac{|J_N(K,\zeta_N)|}{N} = \mathrm{Vol}(S^3\setminus K),$$

which is the Volume Conjecture.