An Introduction to the Volume Conjecture, II
Why we expect the conjecture is true.

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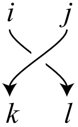
Review of the definition

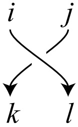
$$R_{kl}^{ij} := \sum_{m=0}^{\min(N-1-i,j)} \delta_{l,i+m}\delta_{k,j-m}\frac{\{l\}!\{N-1-k\}!}{\{i\}!\{m\}!\{N-1-j\}!} \times q^{\left(i-(N-1)/2\right)\left(j-(N-1)/2\right)-m(i-j)/2-m(m+1)/4},$$

$$(R^{-1})_{kl}^{ij} := \sum_{m=0}^{\min(N-1-i,j)} \delta_{l,i-m}\delta_{k,j+m}\frac{\{k\}!\{N-1-l\}!}{\{j\}!\{m\}!\{N-1-i\}!} \times q^{-\left(i-(N-1)/2\right)\left(j-(N-1)/2\right)-m(i-j)/2+m(m+1)/4},$$

with $\{m\} := q^{m/2} - q^{-m/2}$ and $\{m\}! := \{1\}\{2\}\cdots\{m\}$.



$\Rightarrow \quad R_{kl}^{ij}$,

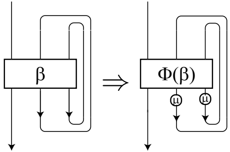


$\Rightarrow \quad (R^{-1})_{kl}^{ij}$

An example of calculation

$$J_N(L;q) := T_{(R,\mu,q^{(N^2-1)/4},1)}(L) \times \frac{\{1\}}{\{N\}}$$

To calculate $J_N(L;q)$ we leave the left-most strand without closing.



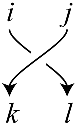
This gives a linear map $\varphi\colon \mathbb{C}^N \to \mathbb{C}^N$, which is a scalar multiple by Schur's lemma.

We fix a basis $\{e_0, e_1, \dots, e_{N-1}\}$ of C^N. The linear map is a scalar multiple and so e_i is multiplied by S for any i. Since

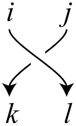
$$\begin{aligned} T_{(R,\mu,q^{(N^2-1)/4},1)}(L) &= q^{-w(\beta)(N^2-1)/4}\,\mathrm{Tr}_1(\phi\mu) \\ &= q^{-w(\beta)(N^2-1)/4}\sum_{i=0}^{N-1} S\,q^{(2i-N+1)/2} \\ &= q^{-w(\beta)(N^2-1)/4}\frac{\{N\}}{\{1\}}S, \end{aligned}$$

we have $J_N(L;q) = S$.

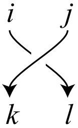
How to label arcs



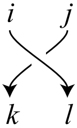
$\Rightarrow i + j = k + l,\ l \geq i,\ k \leq j,$



$i + j = k + l,\ l \leq i,\ k \geq j.$

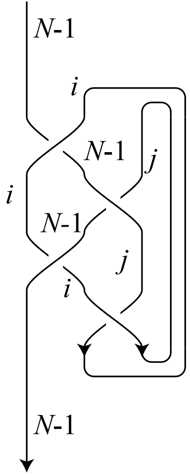


$\Rightarrow i + j = k + l,\ l \geq i,\ k \leq j,$



$i + j = k + l,\ l \leq i,\ k \geq j.$

colored Jones polynomial



$J_N($  $; q)$

$$= \sum_{i\geq j} R^{N-1,i}_{i,N-1}\,(R^{-1})^{N-1,j}_{N-1,j}\,R^{i,N-1}_{N-1,i}\,(R^{-1})^{i,j}_{i,j}\,\mu^{j}_{j}\,\mu^{i}_{i}$$

$$= \sum_{i\geq j}(-1)^{N-1+i}\frac{\{N-1\}!\{i\}!\{N-1-j\}!}{(\{j\}!)^2\{i-j\}!\{N-1-i\}!}$$

$$\times\, q^{(-i-i^2-2ij-2j^2+3N+6Ni+2Nj-3N^2)/4}$$

Quantum factorial at the N-th root of unity

$q = \zeta_N := \exp(2\pi\sqrt{-1}/N)$

$\Rightarrow$

$$
\begin{aligned}
&\{k\}!\{N-k-1\}! \\
&= \pm\,(\text{a power of } \zeta_N) \times (1-\zeta_N)(1-\zeta_N^2)\cdots(1-\zeta_N^k) \\
&\quad \times (1-\zeta_N)(1-\zeta_N^2)\cdots(1-\zeta_N^{N-1-k}) \\
&= \pm\,(\text{a power of } \zeta_N) \times (1-\zeta_N)(1-\zeta_N^2)\cdots(1-\zeta_N^k) \\
&\quad \times (1-\zeta_N^{N-1})(1-\zeta_N^{N-2})\cdots(1-\zeta_N^{k+1}) \\
&= \pm\,(\text{a power of } \zeta_N) \times 2^{N-1}\sin(\pi/N)\sin(2\pi/N)\cdots\sin((N-1)\pi/N) \\
&= \pm\,(\text{a power of } \zeta_N) \times N
\end{aligned}
$$

- $(\zeta_N)_{k^+} := (1-\zeta_N)\cdots(1-\zeta_N^k)$, $(\zeta_N)_{k^-} := (1-\zeta_N)\cdots(1-\zeta_N^{N-1-k})$.
- $(\zeta_N)_{k^+}(\zeta_N)_{k^-} = \pm(\text{a power of } \zeta_N) \times N$.
- $\{k\}! = \pm(\text{a power of } \zeta_N) \times (\zeta_N)_{k^+}$,
  $\{N-1-k\}! = \pm(\text{a power of } \zeta_N) \times (\zeta_N)_{k^-}$.

R-matrix as a product of quantum factorial

$$R^{ij}_{kl} = \sum_m \pm(\text{a power of } \zeta_N) \times \delta_{l,i+m}\delta_{k,j-m} \frac{\{l\}!\{N-1-k\}!}{\{i\}!\{m\}!\{N-1-j\}!}$$

$$= \sum_m \delta_{l,i+m}\delta_{k,j-m} \frac{\pm(\text{a power of } \zeta_N) \times N^2}{(\zeta_N)_{m^+}(\zeta_N)_{i^+}(\zeta_N)_{k^+}(\zeta_N)_{j^-}(\zeta_N)_{l^-}}$$

$$(R^{-1})^{ij}_{kl} = \sum_m \delta_{l,i-m}\delta_{k,j+m} \frac{\pm(\text{a power of } \zeta_N) \times N^{-2}}{(\zeta_N)_{m^+}(\zeta_N)_{i^-}(\zeta_N)_{k^-}(\zeta_N)_{j^+}(\zeta_N)_{l^+}}$$

$\Rightarrow$

$$J_N(K;\zeta_N) = \sum_{\substack{\text{labellings}\\ i,j,k,l\\ \text{on arcs}}} \left(\prod_{\pm\text{-crossings}} \frac{\pm(\text{a power of } \zeta_N) \times N^{\pm 2}}{(\zeta_N)_{m^+}(\zeta_N)_{i^\pm}(\zeta_N)_{k^\pm}(\zeta_N)_{j^\mp}(\zeta_N)_{l^\mp}} \right)$$

Approximation of the quantum factorial

$$\begin{aligned}
\log(\zeta_N)_{k^+} &= \sum_{j=1}^{k} \log(1-\zeta_N^j) \\
&= \sum_{j=1}^{k} \log\bigl(1-\exp(2\pi\sqrt{-1}j/N)\bigr) \\
&(x := j/N) \\
&\underset{N\to\infty}{\approx} N\int_0^{k/N} \log\bigl(1-\exp(2\pi\sqrt{-1}x)\bigr)\,dx \\
&(y := \exp(2\pi\sqrt{-1}x)) \\
&= \frac{N}{2\pi\sqrt{-1}} \int_1^{\exp(2\pi\sqrt{-1}k/N)} \frac{\log(1-y)}{y}\,dy
\end{aligned}$$

Approximation of the quantum factorial by dilogarithm

- (dilog function)

$$\mathrm{Li}_2(z) := -\int_0^z \frac{\log(1-y)}{y}\,dy = \sum_{n=1}^{\infty} \frac{z^n}{n^2}.$$

- $$\log(\zeta_N)_{k^+} \underset{N\to\infty}{\approx} \frac{N}{2\pi\sqrt{-1}}\left[\mathrm{Li}_2(1) - \mathrm{Li}_2(\zeta_N^k)\right].$$

- $$(\zeta_N)_{k^\pm} \underset{N\to\infty}{\approx} \exp\left[-\frac{N}{2\pi\sqrt{-1}}\,\mathrm{Li}_2(\zeta_N^{\pm k})\right].$$

Approximation of the colored Jones polynomial by dilogarithm

$$J_N(K;\zeta_N) \underset{N\to\infty}{\approx} \sum_{\text{labellings}} (\text{polynomial of } N) \times (\text{power of } \zeta_N)$$

$$\exp\left[\frac{N}{2\pi\sqrt{-1}} \sum_{\text{crossings}} \left\{\mathrm{Li}_2(\zeta_N^m) + \mathrm{Li}_2(\zeta_N^{\pm i}) + \mathrm{Li}_2(\zeta_N^{\mp j}) + \mathrm{Li}_2(\zeta_N^{\pm k}) + \mathrm{Li}_2(\zeta_N^{\mp l}) + \text{log terms}\right\}\right]$$

where a log term comes from powers of ζ_N. For example

$$q^{k^2} = \exp\left(\frac{N}{2\pi\sqrt{-1}}\left(\frac{2\pi\sqrt{-1}k}{N}\right)^2\right) = \exp\left[\frac{N}{2\pi\sqrt{-1}}(\log\zeta_N^k)^2\right].$$

Approximation of the colored Jones polynomial by integral

$$J_N(K;\zeta_N) \underset{N\to\infty}{\approx} \sum_{i_1,\dots,i_c} (\text{polynomial of } N) \exp\left[\frac{N}{2\pi\sqrt{-1}} V(\zeta_N^{i_1},\dots,\zeta_N^{i_c})\right]$$

(ignore polynomials since exp grows much bigger)

$$\underset{N\to\infty}{\approx} \sum_{i_1,\dots,i_c} \exp\left[\frac{N}{2\pi\sqrt{-1}} V(\zeta_N^{i_1},\dots,\zeta_N^{i_c})\right]$$

$$\underset{N\to\infty}{\approx} \int_{J_1}\cdots\int_{J_c} \exp\left[\frac{N}{2\pi\sqrt{-1}} V(z_1,\dots,z_c)\, dz_1\cdots dz_c\right],$$

where

- $i_1,\dots,i_c$: labellings on arcs.

  $$V(\zeta_N^{i_1},\dots,\zeta_N^{i_c}) := \sum_{\text{crossings}} \left\{\mathrm{Li}_2(\zeta_N^m) + \mathrm{Li}_2(\zeta_N^{\pm i}) + \mathrm{Li}_2(\zeta_N^{\mp j}) + \mathrm{Li}_2(\zeta_N^{\pm k}) + \mathrm{Li}_2(\zeta_N^{\mp l})\right\}.$$

- $J_1,\dots,J_c$: contours.

Saddle point method

$V(x_1,\dots,x_c)$: the 'maximum' of $\{\operatorname{Im} V(z_1,\dots,z_c)\}_{(z_1,\dots,z_c)\in J_1\times\cdots\times J_c}$ to find the maximum of $\left|\exp\left[\frac{N}{2\pi\sqrt{-1}}V(z_1,\dots,z_c)\,dz_1\cdots dz_c\right]\right|$.

$\Rightarrow$

$$J_N(K;\zeta_N) \underset{N\to\infty}{\approx} \exp\left[\frac{N}{2\pi\sqrt{-1}}V(x_1,\dots,x_c)\right],$$

By the saddle point method, $(x_1,\dots,x_c)$ satisfies the following.

$$\frac{\partial V}{\partial z_k}(x_1,\dots,x_c)=0 \quad (k=1,\dots,c)$$

$\Rightarrow$

$$2\pi\sqrt{-1}\lim_{N\to\infty}\frac{J_N(K;\zeta_N)}{N}=V(x_1,\dots,x_c)$$

Difficulties

Difficulties so far:

- Replacing the summation into an integral

$$\sum_{i_1,\dots,i_c} \exp\left[\frac{N}{2\pi\sqrt{-1}} V(\zeta_N^{i_1},\dots,\zeta_N^{i_c})\right] \underset{N\to\infty}{\approx} \int_{J_1}\cdots\int_{J_c} \exp\left[\frac{N}{2\pi\sqrt{-1}} V(z_1,\dots,z_c)\,dz_1\cdots dz_c\right].$$

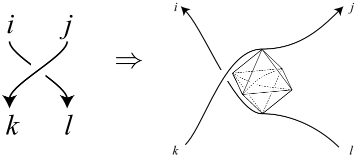
- How to apply the saddle point method. In particular, which saddle point to choose. In general, we have many solutions to the system of equations.

$$\int_{J_1}\cdots\int_{J_c} \exp\left[\frac{N}{2\pi\sqrt{-1}} V(z_1,\dots,z_c)\,dz_1\cdots dz_c\right] \underset{N\to\infty}{\approx} \exp\left[\frac{N}{2\pi\sqrt{-1}} V(x_1,\dots,x_c)\right].$$

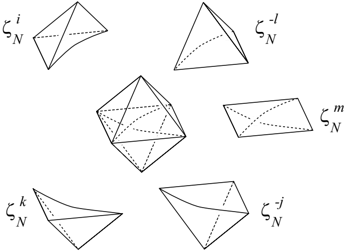
Decomposition into octahedra (by D. Thurston)

Decompose the knot complement into (topological, truncated) tetrahedra.

- Around each crossing, put an octahedron:

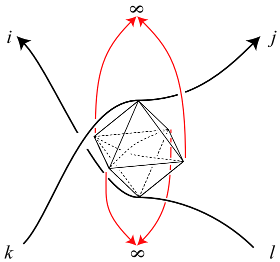


- Decompose the octahedron into five tetrahedra:



Decomposition into topological tetrahedra

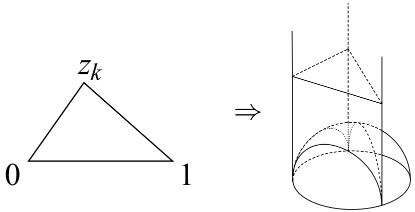
- Pull the vertices to the point at infinity:



- $S^3 \setminus K$ is now decomposed into topological, truncated tetrahedra, decorated with complex numbers $\zeta_N^{i_k}$.

Decomposition into hyperbolic tetrahedra

- Each topological, truncated tetrahedron is decorated with a complex number $\zeta_N^{i_k}$.
- We want to regard it as a hyperbolic, ideal tetrahedron.
- Recall that we have replaced a summation over i_k into an integral over z_k.
- Replace $\zeta_N^{i_k}$ with a complex variable z_k.
- Regard the tetrahedron decorated with z_k as an hyperbolic, ideal tetrahedron parametrized by z_k.



Hyperbolic structure on the knot complement

- Now the knot complement is decomposed into ideal, hyperbolic tetrahedra parametrized by $z_1, \dots, z_c$.
- Choose $z_1, \dots, z_c$ so that we can glue these tetrahedra well, that is,
  - around each edge, the sum of angles is 2π,
  - the triangles that appear in the boundary torus make the torus Euclidean.
- These conditions are the same as the system of equations that we used in the saddle point method!

$$\frac{\partial V}{\partial z_k}(x_1, \dots, x_c) = 0 \quad (k = 1, \dots, c)$$

- $\Rightarrow (x_1, \dots, x_c)$ gives the complete hyperbolic structure.
- Then, what does $V(x_1, \dots, x_c)(= 2\pi\sqrt{-1} \lim_{N\to\infty} \frac{J_N(K, \zeta_N)}{N})$ mean?

Geometric meaning of the limit

Recall: $V(x_1,\dots,x_c)$ is the sum of $\mathrm{Li}_2(x_k)$ (and log), where x_k defines an ideal hyperbolic tetrahedron. We use the following formula:

$$\mathrm{Vol}(\text{tetrahedron parametrized by } z) = \operatorname{Im}\mathrm{Li}_2(z) - \log|z|\arg(1-z).$$

Therefore we finally have

$$\operatorname{Im}\left(2\pi\sqrt{-1}\lim_{N\to\infty}\frac{J_N(K,\zeta_N)}{N}\right) = \mathrm{Vol}(S^3\setminus K).$$

$\Rightarrow$

$$2\pi\lim_{N\to\infty}\frac{|J_N(K,\zeta_N)|}{N} = \mathrm{Vol}(S^3\setminus K),$$

which is the Volume Conjecture.