An Introduction to the Volume Conjecture, III
Generalizations

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9th June, 2009

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Conjecture (Volume Conjecture, R. Kashaev, J. Murakami+H.M.)

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We may regard the left hand side as the definition of the Chern–Simons invariant for general knots.

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- If we deform the parameter $2\pi\sqrt{-1}$, does the limit corresponds to an incomplete hyperbolic structure?
- Let us consider the limit
$$\lim_{N\to\infty} \frac{\log J_N\Bigl(K; \exp\bigl((u + 2\pi\sqrt{-1})/N\bigr)\Bigr)}{N}$$
When $u = 0$, we have the (complexified) Volume Conjecture.

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$\exists \mathcal{O} \subset \mathbb{C}$: neighborhood of 0. If $u \in \mathcal{O} \setminus \pi\sqrt{-1}\mathbb{Q}$, the following limit exists

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(where the first argument of J_N is the knot )

Put

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  $\mathrm{Vol}($  $_u) + \sqrt{-1}\,\mathrm{CS}($  $_u)$

  $\equiv -\sqrt{-1}H(u) - \pi u + u\,v(u)\sqrt{-1}/4 - \pi\kappa(\gamma_u)/2 \pmod{\pi^2\sqrt{-1}\mathbb{Z}}.$

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- For $u \neq 0$, the hyperbolic structure is incomplete.

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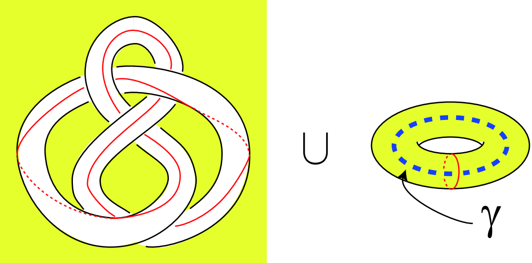

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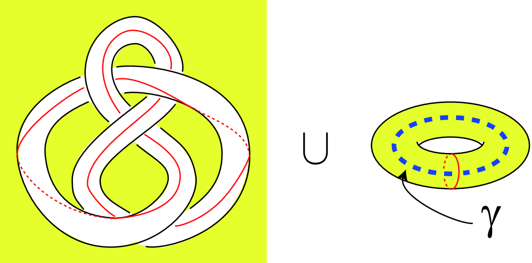

- $\kappa(\gamma_u) := \text{length}(\gamma_u) + \sqrt{-1}\,\text{torsion}(\gamma_u)$, where
  - length is its length,
  - torsion measures how the circle is twisted (mod 2π).

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$$J_N\left(\right.$$  $$\left. ; q\right) = \sum_{j=0}^{N-1} q^{jN} \prod_{k=1}^{j} \left(1 - q^{-N-k}\right)\left(1 - q^{-N+k}\right).$$

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Put

$$H(z,w) := \mathrm{Li}_2(z^{-1}w^{-1}) - \mathrm{Li}_2(zw^{-1}) + \log z \log w,$$

where

$$\mathrm{Li}_2(x) := -\int_0^x \frac{\log(1-t)}{t}\,dt.$$

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$$J_N\left(\text{(figure-eight knot)}; q\right) = \sum_{j=0}^{N-1} q^{jN} \prod_{k=1}^{j} \left(1 - q^{-N-k}\right)\left(1 - q^{-N+k}\right).$$

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If θ is near $2\pi\sqrt{-1} \in \mathbb{C}$ and not a rational multiple of $2\pi\sqrt{-1}$, then

$$\theta \lim_{N\to\infty} \frac{\log J_N\left(\text{(figure-eight knot)}; \exp(\theta/N)\right)}{N} = H\left(y, \exp(\theta)\right),$$

where y satisfies

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$$\frac{\log\left[\exp(\theta)+\exp(-\theta)-x-x^{-1}\right]}{x}=0.$$

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$(u + 2\pi\sqrt{-1}) \lim_{N\to\infty} \frac{J_N\Big(}{}$  $\frac{;\exp((u + 2\pi\sqrt{-1})/N)\Big)}{N} = H(u)$

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Note that this can be done rigorously.

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- Use the formula:

$$\operatorname{Vol}\bigl(\Delta(z)\bigr) = \operatorname{Im} \operatorname{Li}_2(z) + \log|z| \arg(1-z).$$

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Indeed, $\exp(v(u))$ corresponds to the longitude $z^2(1-z)^2$.

We will show:

$$\operatorname{length} \gamma_u = -\frac{1}{2\pi} \operatorname{Im}\left(u\overline{v(u)}\right).$$

Length of the geodesic γ_u (W. Neumann and D. Zagier)

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On $\partial\mathbb{H}^3 = S^2_\infty = \mathbb{C}\cup\{\infty\}$:

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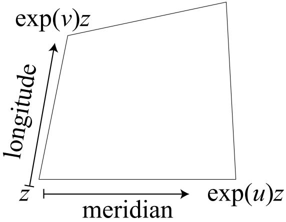

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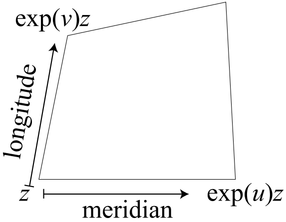

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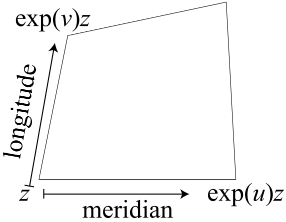

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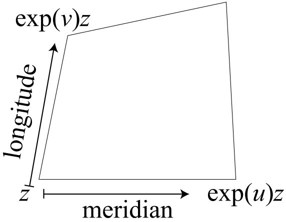

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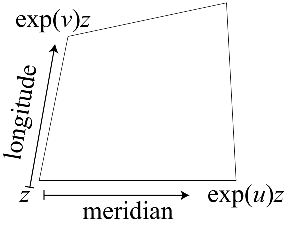

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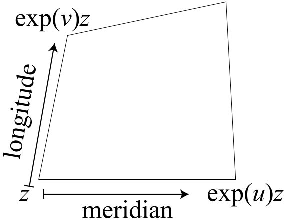

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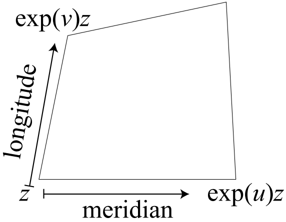

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  ($\because$ the meridian of the attached solid torus is identified with $p\mu + q\lambda$, and the meridian and γ_u make a basis of $H_1(\partial(S^3 \setminus$  $))$.)

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  (Here we choose the negative sign since $v = u \times \dfrac{|v|^2}{u\bar{v}}$ and the orientation of (u,v) should be positive on $\mathbb{C}$.)

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What happens between $2\pi\sqrt{-1}$ and 0?

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