Rough calculus

Lecture 4: Rough calculus for paths with finite p-th variation ($p > 2$)

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Outline

1. "Ito calculus without probability"
2. Ito-Föllmer calculus for functionals of paths with finite quadratic variation.
3. Properties of the pathwise integral: isometry and rough-smooth decomposition.
4. **Rough calculus for function(al)s of paths with finite p-th variation.**
5. The case of paths with fractional regularity (*)
6. $\mathcal{M}$-functionals and integral representations. (*)
7. Transport of measures along rough trajectories.
8. Rough dynamics on manifolds.

Lecture 3: Rough calculus for function(al)s of paths with finite p-th variation.

1. p-th variation along a sequence of partitions
2. Rough change of variable formula
3. Extension to vector-valued paths
4. Rough-smooth decomposition of regular functionals

Reference: (click on title to download)

- R Cont, N Perkowski (2019) Pathwise integration and change of variable formulas for continuous paths with arbitrary regularity, `Transactions of AMS`, 6:161-186.

p-th variation along a sequence of partitions

Let $p > 1$ and $\pi = (\pi_n)_{n\geq 1}$ be a sequence of partitions of $[0, T]$ with $|\pi_n| = \sup_{i=0..N(\pi_n)} |t^n_{i+1} - t^n_i| \to 0$.

Definition (p-th variation along a sequence of partitions)

$S \in C([0,T],\mathbb{R})$ is said to have (finite) p-th variation along $\pi = (\pi_n)_{n\geq 1}$if the sequence of measures

$$\mu^n = \sum_{[t_j,t_{j+1}]\in\pi_n} \delta(\cdot - t_j)|S(t_{j+1}) - S(t_j)|^p$$

converges weakly to a measure μ_S without atoms. We write $S \in V_p(\pi)$ and call

$$[S]^p(t) := \mu_S([0,t])$$

the *p-th variation* of S along π. $[S]^p$ is a continuous, increasing function.

p-th variation along a sequence of partitions

Lemma (Characterization)

Let $S \in C([0,T],\mathbb{R})$. $S \in V_p(\pi)$ if and only if there exists a continuous increasing function $[S]^p$ such that

$$\forall t \in [0,T], \qquad [S]_{\pi_n}(t) = \sum_{\substack{[t_j,t_{j+1}]\in\pi_n:\\ t_j\le t}} |S(t_{j+1}) - S(t_j)|^p \stackrel{n\to\infty}{\to} [S]^p(t).$$

The convergence is uniform.

Functions in $V_p(\pi)$ do not necessarily have finite p-**variation**:

$$\|S\|_{p-var} = \sup_{\tau\in\Pi(0,T)} [S]^p_\tau = \sum_\tau |S(u_{i+1}) - S(u_i)|^p \ge \lim_n [S]_{\pi_n}(T)$$

where $\Pi([0,T])$ = set of finite partitions of $[0,T]$.

Examples of processes with sample paths in $V_p(\pi)$

Fractional Brownian motion (fBM) with Hurst index $0 < H < 1$: real-valued Gaussian process $(B^H(t), t \in \mathbb{R})$ with

$$\mathbb{E}(B^H(t)) = 0 \qquad \mathbb{E}(B^H(t), B^H(s)) = \frac{|t|^{2H} + |s|^{2H} + |t-s|^{2H}}{2}$$

Proposition (Pratelli, 2011)

Let B^H be a fBM on $(\Omega, \mathcal{F}, \mathbb{P})$ with $H \in (0,1)$ and $\pi_n = \{kT/n : k = 0..n\}$. Then

$$\mathbb{P}(B_H \in V_{1/H}(\pi) \quad) = 1 \quad \text{and} \qquad [B_H]_\pi^{1/H}(t) = t\ \mathbb{E}[|B_H(1)|^{1/H}] \qquad \mathbb{P} - a.s.$$

while $\mathbb{P}(\|B_H\|_{p-var} = \infty) = 1$ for $p = 1/H$.

M Pratelli (2011) *Séminaire de Probabilités* XLIII, 215-219.

Typical sample paths of B^H lie in $C^{H-}([0, T])$ (Dudley 1981)

Example: heat equation with space-time white noise

J Swanson (2007) *Ann. Probability* 35:2122–2159.

$$\partial_t u(t,x) = \frac{1}{2}\partial_{xx} u(t,x) + \dot{w}(t,x) \qquad u(0,x) = 0$$

- $\dot{w}(t,x)$ space-time white noise on $[0,\infty) \times \mathbb{R}$.

$$u(t,x) = \int_{[0,t]\times\mathbb{R}} p(t-s, x-y)\dot{w}(s,y) \qquad p(t,x) = \frac{\exp(-\frac{x^2}{2t})}{\sqrt{2\pi t}}$$

- For a fixed x, $t \mapsto F(t) = u(t,x)$ is a Gaussian process with

$$\mathbb{E}(F(t)) = 0 \qquad \mathbb{E}(F(t)F(s)) = \frac{1}{\sqrt{2\pi}}\left(|t+s|^{1/2} - |t-s|^{1/2}\right)$$

- (Swanson 2007) $u(.,x) \in V_4([0,T])$: if $|\pi_n| \to 0$ then

$$\mathbb{E}\left(\sup_{t\in[0,T]} \Big| \sum_{\pi_n} |u(t_{i+1},x) - u(t_i,x)|^4 - \frac{6}{\pi}t\Big| \right) \stackrel{n\to\infty}{\to} 0$$

while at the same time: $\|u(.,x)\|_{4-var} = \infty$.

Example: Takagi-Landsberg functions

- $D = (D_n)$ dyadic partition sequence on $[0,1]$: $D_n = \{k/2^n, k = 0..2^n\}$.
- Faber-Schauder functions associated to D_n:

$$e_{0,0}(t) = (\min(t, 1-t))_+ \qquad e_{n,k}(t) = 2^{-n/2} e_{0,0}(2^n t - k), \quad k \in \mathbb{Z},\ n \in \mathbb{N}$$

$$S^H(t) = \sum_{m=0}^{\infty} 2^{m(\frac{1}{2}-H)} \sum_{k=0}^{2^m-1} \theta_{m,k} e_{m,k}(t) \qquad \theta_{m,k} \in \{-1,+1\}$$

- Theorem (Mishura & Schied 2019): For any choice of $\theta_{m,k} \in \{-1,+1\}$, $S^H \in V_p(\pi)$ for $p = 1/H$ and $[S^H]^p = c_H\ t$ where c_H is a constant.
- A Schied, Y Mishura (2019) On (signed) Takagi-Landsberg functions: pth variation, maximum, and modulus of continuity, `Journal of Mathematical Analysis and Applications`, 473:258-272.

Example: Takagi-Landsberg functions

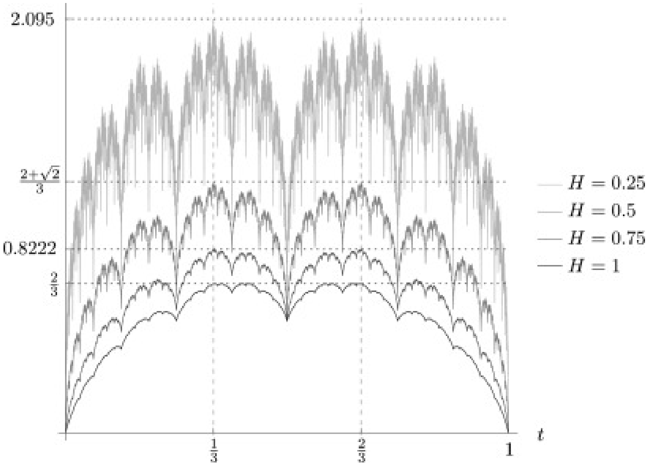

Figure: Takagi-Landsberg function: $\theta_{mk} = +1$

Example: random Takagi-Landsberg functions

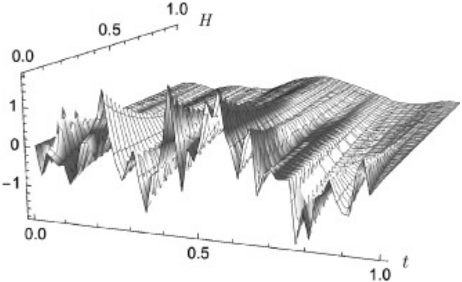

Figure: Random Takagi-Landsberg function: θ_{mk} IID Bernoulli variables

A Schied, Y Mishura (2019) On (signed) Takagi-Landsberg functions: pth variation, maximum, and modulus of continuity,
`Journal of Mathematical Analysis and Applications`, 473:258-272.

The rough change of variable formula

- Consider $S \in V_p(\pi) \cap C^0([0,T],\mathbb{R}^d)$, with $p \in \mathbb{N}$ and $f \in C^p(\mathbb{R})$.
- A Taylor expansion of order p yields

$$f(S(t^n_{i+1}))-f(S(t^n_i)) = \sum_{k=1}^{p} \frac{f^{(k)}(S(t_j))}{k!}(S(t_{j+1})-S(t_j)^k + r^n_j(S(t_{j+1})-S(t_j))^p$$

where $\sup_j r^n_j \to 0$ as $n \to \infty$ by uniform continuity of S.

- Separating the term of order p and summing across the partition we get

$$f(S(T) - f(S(0)) = \sum_{\pi_n}\sum_{k=1}^{p-1} \frac{f^{(k)}(S(t_j))}{k!}(S(t_{j+1}) - S(t_j)^k$$

$$+\sum_{\pi_n} \frac{f^{(p)}(S(t_j))}{p!}(S(t_{j+1}) - S(t_j)^p + r^n_j \sum_{\pi_n}(S(t_{j+1}) - S(t_j))^p$$

'Rough' Change of variable formula

Theorem (R.C- Perkowski (2019))

Let $p \in \mathbb{N}$, $p \geq 2$ and $S \in V_p(\pi)$. Then for every $f \in C^p(\mathbb{R}, \mathbb{R})$

$$f(S(t)) - f(S(0)) = \int_0^t < \nabla_{p-1} f(S), dS > + \frac{1}{p!} \int_0^t f^{(p)}(S(s)) d[S]^p(s),$$

where the integral is defined as a (pointwise) limit of compensated Riemann sums:

$$\int_0^t \nabla_{p-1} f \circ S.dS := \int_0^t < \nabla_{p-1} f(S)(u), dS(u) >$$

$$= \lim_{n \to \infty} \sum_{\pi_n} \sum_{k=1}^{p-1} \frac{f^{(k)}(S(t_j))}{k!} (S(t_{j+1} \wedge t) - S(t_j \wedge t))^k$$

Pathwise integral

The pathwise integral

$$\int_0^t < \nabla_{p-1} f \circ S, dS >:= \lim_n R_{p-1}(f, S, \pi_n)$$

is a pointwise limit of compensated Riemann sums

$$R_{p-1}(f, S, \pi_n) = \sum_{\pi_n} \sum_{k=1}^{p-1} \frac{f^{(k)}(S(t_j))}{k!} (S(t_{j+1} \wedge t) - S(t_j \wedge t))^k$$

It should be really seen as an integral of the $(p-1)-$**jet** $\nabla_{p-1} f$ of f

$$\nabla_{p-1} f(x) = (f^{(k)}(x), k = 0, 1, ..., p-1)$$

with respect to a differential structure of order $p-1$ constructed along $S \in V_p(\pi)$ using the powers of increments up to order $p-1$.
Note that *even after compensation* this limit cannot be defined as a Young integral!

Example: Fractional Brownian motion

Our result allows to define a pathwise Ito-type integral + change of variable formula for Fractional Brownian motion B^H with *any* Hurst exponent $1 > H > 0$.
Example: $H = 1/4$. Then $p = 4$, $[B^H]^4(t) = 3t$ and

$$\int_0^t \nabla_3 f \circ B^H . dB^H = \lim_{n\to\infty} \sum_{\pi_n} f'(B^H(t_j))\Delta_j B^H + \frac{f''(B^H(t_j))}{2}(\Delta_j B^H)^2 + \frac{f^{(3)}(B^H(t_j))}{6}(\Delta_j B^H)^3$$

where $\Delta_j B^H = B^H(t_{j+1}) - B^H(t_j)$

Example: $f(x) = x^4$

$$\int_0^t \nabla_3 f \circ B^H . dB^H = \lim_{n\to\infty} \sum_{\pi_n} 4\, B^H(t_j)^3 \Delta_j B^H + 6 B^H(t_j)^2 (\Delta_j B^H)^2 + 4 B^H(t_j)(\Delta_j B^H)^3$$

$$|B^H(t)|^4 = \int_0^t \nabla_3 (f \circ B^H).dS + \frac{t}{8}$$

The compensated Riemann sum converges pointwise but each term alone diverges.

Example: compensated exponential

Proposition

Let $X \in V_p(\pi) \cap C^0([0,T],\mathbb{R})$. There is a unique $Z = \mathcal{E}(X) \in C^0([0,T],\mathbb{R})$ satisfying

$$\forall t \geq 0, \quad Z(t) = 1 + \int_0^t Z(s).dX(s) \qquad \text{i.e.} \quad dZ(t) = Z(t).dX(t)$$

Z is given by

$$Z(t) = \mathcal{E}(X) = \exp\left(X(t) - \frac{[X](t)}{p!}\right)$$

$$\mathcal{E}(X)(t) = 1 + \lim_{n\to\infty} \sum_{\pi_n} e^{X(t_j) - [X]^p(t_j)/p!} \sum_{k=1}^{p-1} (\Delta_j X)^k$$

Example: B^H with $H = 1/4$. $\mathcal{E}(B^H) = \exp(B^H(t) - \frac{t}{8})$

Isometry formula for the pathwise integral

$p = 2$ (Ananova-C. 2017), $p \in 2\mathbb{N}$: (C.-Perkowski 2019)

Theorem (Isometry property of the pathwise integral)

Let $p \in 2\mathbb{N}$, (π_n) with $|\pi_n| \to 0$. If $S \in V_p(\pi) \cap C^\alpha([0,T],\mathbb{R})$ for some $\alpha > 0$ with $d[S]^p_\pi/dt > 0$, then for any $f \in C^p(\mathbb{R}^d)$,

$$f \circ S \in V_p(\pi) \qquad \int_0^{\cdot} (\nabla_{p-1} f \circ S)\, dS := \int_0^{\cdot} < \nabla_{p-1} f(S), dS > \in V_p(\pi)$$

$$[f(S)]^p(T) = [\int_0^{\cdot} (\nabla_{p-1} f \circ S)\, dS]^p(T) = \int_0^T |f'(S)|^p d[S]^p = \|f' \circ S\|^p_{L^p([0,T],d[S]^p)}.$$

Proof: $\int_{t_j}^{t_{j+1}} (\nabla_{p-1} f \circ S)\, dS = f'(S(t_j)).(S(t_{j+1}) - S(t_j)) + o(S(t_{j+1}) - S(t_j))$ so

$$|\int_{t_j}^{t_{j+1}} (\nabla_{p-1} f \circ S)\, dS|^p = |f'(S(t_j))|^p |(S(t_{j+1}) - S(t_j))|^p + \epsilon_n\, |(S(t_{j+1}) - S(t_j))|^p$$

Isometry formula: examples

- For $X \in V_p(\pi) \cap C^0([0,T],\mathbb{R})$, the compensated exponential $Z = \mathcal{E}(X)$ has finite $p-$the variation and

$$[\mathcal{E}(X)]^p_\pi(T) = \int_0^T |Z(t)|^p \quad d[X]^p_\pi = \int_0^T e^{pX - \frac{1}{(p-1)!}[X]^p} \quad d[X]^p$$

- Fractional Brownian motion with $H = 1/4$, $f \in C^4$. Then

$$[f(B^H)]^4(t) = [\int_0^. \nabla_3 f \circ B^H . dB^H]^4 = \int_0^t |f'(B^H(t))|^4 dt$$

$$[\mathcal{E}(B^H)]^4(T) = 3\int_0^T \exp\left(4B^H - \frac{t}{2}\right) \quad dt$$

Symmetric tensors

A symmetric p-tensor T on $\mathbb{R}^d$ is a p-tensor invariant under any permutation $\sigma \in \mathfrak{S}_p$ of its arguments: for $(v_1, v_2, \ldots, v_p) \in (\mathbb{R}^d)^p$

$$\sigma T(v_1, \ldots, v_p) := T(v_{\sigma 1}, v_{\sigma 2}, \ldots, v_{\sigma p}) = T(v_1, v_2, \ldots, v_p)$$

The space $\mathrm{Sym}_p(\mathbb{R}^d)$ of symmetric tensors of order p on $\mathbb{R}^d$ is naturally isomorphic to the dual of the space $\mathbb{H}_p[X_1, ..., X_d]$ of homogeneous polynomials of degree p on $\mathbb{R}^d$.

$$\mathbb{S}_p(\mathbb{R}^d) = \bigoplus_{k=0}^{p} \mathrm{Sym}_k(\mathbb{R}^d).$$

For any p-tensor T we define the *symmetric part*

$$\mathrm{Sym}(T) := \frac{1}{p!} \sum_{\sigma \in \mathfrak{S}_k} \sigma T \in \mathrm{Sym}_p(\mathbb{R}^d)$$

where $\mathfrak{S}_p$ of $\{1, \ldots, k\}$ is the group of permutations of $\{1, 2, ..., p\}$

Extension to vector functions

Consider now a continuous $\mathbb{R}^d$-valued path $S \in C([0,T],\mathbb{R}^d)$ and a sequence of partitions $\pi_n = \{t_0^n, \ldots, t_{N(\pi_n)}^n\}$ with $t_0^n = 0 < ... < t_k^n < ... < t_{N(\pi_n)}^n = T$. Then

$$\mu^n = \sum_{\pi_n} \underbrace{(S(t_{j+1}) - S(t_j)) \otimes ... \otimes (S(t_{j+1}) - S(t_j))}_{\text{p times}} \delta(\cdot - t_j)$$

defines a tensor-valued measure on $[0,T]$ with values in $\mathrm{Sym}_p(\mathbb{R}^d)$. This space of measures is in duality with the space $C([0,T],\mathbb{H}_p[X_1,...,X_d])$ of continuous functions taking values in homogeneous polynomials of degree $p =$ homogeneous polynomials of degree p with continuous time-dependent coefficients.

This motivates the following definition:

Definition (p-th variation of a vector-valued function)

Let $p \in 2\mathbb{N}$ be an (even) integer, and $S \in C([0,T],\mathbb{R}^d)$ a continuous path and $\pi = (\pi_n)_{n\geq 1}$ a sequence of partitions of $[0,T]$. $S \in C([0,T],\mathbb{R}^d)$ is said to have a p-th variation along $\pi = (\pi_n)_{n\geq 1}$ if $osc(S,\pi_n) \to 0$ and the sequence of tensor-valued measures

$$\mu_S^n = \sum_{\pi_n} (S(t_{j+1}) - S(t_j))^{\otimes p} \quad \delta(\cdot - t_j)$$

converges to a $\mathrm{Sym}_p(\mathbb{R}^d)$-valued measure μ_S without atoms in the following sense: $\forall f \in C([0,T],\mathbb{S}_{\mathbf{p}}(\mathbb{R}^{\mathbf{d}}))$,

$$< f, \mu_n >= \sum_{\pi_n} < f(t_j), (S(t_{j+1}) - S(t_j))^{\otimes p} > \overset{n\to\infty}{\to} < f, \mu_S > .$$

We write $S \in V_p(\pi)$ and call $[S]^p(t) := \mu([0,t])$ the *p-th variation* of S.

Theorem (Rough change of variable formula: vector case)

Let $p \in 2\mathbb{N}$ be an even integer, let (π_n) be a sequence of partitions of $[0, T]$ and $S \in V_p(\pi) \cap C([0, T], \mathbb{R}^d)$. Then for every $f \in C^p(\mathbb{R}, \mathbb{R})$ the limit of compensated Riemann sums

$$\int_0^t < \nabla_{p-1} f \circ S, dS > := \lim_{n \to \infty} \sum_{\pi_n} \sum_{k=1}^{p-1} \frac{1}{k!} < \nabla^k f(S(t_j)), (S(t_{j+1} \wedge t) - S(t_j \wedge t))^{\otimes k} >$$

exists for every $t \in [0, T]$ and satisfies

$$f(S(t)) - f(S(0)) = \int_0^t < \nabla_{p-1} f \circ S, dS > + \frac{1}{p!} \int_0^t < \nabla^p f(S(t))), d[S]^p(u) > .$$

Extension to non-anticipative functionals

We now consider *non-anticipative* functionals i.e. maps $F : [0, T] \times D([0, T], \mathbb{R}^d) \mapsto \mathbb{R}$ such that $F(t, x) = F(t, x(t \wedge .))$

Definition (Horizontal and vertical derivatives)

A non-anticipative functional F is said to be:

- horizontally differentiable at $(t, x) \in \Lambda_T^d$ if the finite limit exists

$$\mathcal{D}F(t, x) := \lim_{h \to 0+} \frac{F(t + h, x_t) - F(t, x_t)}{h}.$$

- vertically differentiable at $(t, x) \in \Lambda_T^d$ if the map

$$\mathbb{R}^d \to \mathbb{R}, \; e \mapsto F(t, x(t \wedge .) + e1_{[t,T]})$$

  is differentiable at 0; its gradient at 0 is denoted by $\nabla_x F(t, x)$.

Smooth functionals

Definition ($\mathbb{C}_b^{1,p}(\Lambda_T^d)$ functionals)

We denote by $\mathbb{C}_b^{1,p}(\Lambda_T^d)$ the set of non-anticipative functionals $F \in \mathbb{C}_l^{0,0}(\Lambda_T^d)$, such that

- F is horizontally differentiable with $\mathcal{D}F$ continuous at fixed times,
- F is p times vertically differentiable with $\nabla_x^j F \in \mathbb{C}_l^{0,0}(\Lambda_T^d)$ for $j = 1..p$
- $\mathcal{D}F, \nabla_x^j F \in \mathbb{B}(\Lambda_T^d)$ for $j = 1..p$.

Expansions for smooth functionals

Smooth functionals may be used to obtain Taylor-type expansions:

Lemma (Lemma 2.2 in C-Ananova 2017)

Let $S \in C^{\alpha}([0,T],\mathbb{R})$ *for some* $\alpha > 0$ *and* $F \in \mathbb{C}_b^{1,2}(\Lambda_T)$ *be a Lipschitz map such that* $\nabla_\omega F \in \mathbb{C}_b^{1,1}(\Lambda_T)$*. Define*

$$R^F_{t,t+h}(S) = F(t+h,S) - F(t,S) - \nabla_\omega F(t,S).(S(t+h)-S(t))$$

There exists $C(F,T,\|S\|_\alpha) > 0$ *which only depends on* $(F,T,\|S\|_\alpha)$ *such that*

$$\|R^F_{t,t+h}(S)\| \leq C(F,T,\|S\|_\alpha) \quad |h|^{\alpha^2+\alpha}$$

- Unlike the Taylor expansion for functions we have $\alpha^2 + \alpha < 2\alpha$: there is loss of regularity due to piecewise-constant approximation of the path S.
- Typical examples of $S \in V_p(\pi)$ will have Hölder regularity $a = p - \epsilon$. $\alpha^2 + \alpha > 1/p$ if $\alpha > ((1+\frac{4}{p})^{1/2} - 1)/2$. This latter bound is $< 1/p$ so typical examples of $S \in V_p(\pi)$ will satisfy this condition.

Functional change of variable formula: general case

Theorem (C.- Perkowski, 2019)

Let $p \in 2\mathbb{N}$ $F \in \mathbb{C}_b^{1,p}(\Lambda_T)$, *and* $S \in V_p(\pi)$ *for a sequence of partitions* (π_n) *with* $|\pi_n| \to 0$. *Then the limit of compensated Riemann sums*

$$\int_0^t \mathbb{T}_{p-1}F(S).dS = \lim_{n\to\infty} \sum_{[t_j,t_{j+1}]\in\pi_n} \sum_{k=1}^{p-1} \frac{1}{k!} \nabla_\omega^k F(t_j, S^n_{t_j-})(S(t_{j+1}\wedge t) - S(t_j \wedge t))^k$$

exists and

$$F(t,S_t) = F(0,S_0) + \int_0^t \mathbb{T}_{p-1}F(S).dS + \int_0^t \mathcal{D}F(u,S_u)du + \frac{1}{p!}\int_0^t \nabla_x^p F(.,S).d[S]^p$$

This extends the pathwise integral to all ‘exact forms’:
$\mathbb{T}_{p-1}\mathbb{C}_b^{1,p} := \{\mathbb{T}_{p-1}F,\ F \in \mathbb{C}_b^{1,p}(\Lambda_T)\}$

Pathwise isometry formula: $p \in 2\mathbb{N}$

Theorem (Pathwise Isometry formula: general case)

Let $p \in \mathbb{N}$ be an even integer, (π_n) a sequence of partitions with mesh size going to zero, and $S \in V_p(\pi) \cap C^\alpha([0,T],\mathbb{R})$ with $\alpha > ((1+\frac{4}{p})^{1/2}-1)/2$. Let $F \in \mathbb{C}_b^{1,p}(\Lambda_T) \cap \mathrm{Lip}(\Lambda_T, d_\infty)$ be such that $\nabla_x F \in \mathbb{C}_b^{1,1}(\Lambda_T)$. Then

$$F(S) \in V_p(\pi), \qquad \int_0^{\cdot} (\mathbb{T}_{p-1} F \circ S) dS \in V_p(\pi) \qquad \text{and}$$

$$[\int_0^{\cdot} (\mathbb{T}_{p-1} F \circ S).dS]^p(t) = \int_0^t |\nabla_x F \circ S|^p d[S]^p = \|\nabla_x F(S)\|^p_{L^p([0,t],d[S]^p)}.$$

Denoting $J_p = \mathbb{T}_{p-1}\left(\mathbb{C}_b^{1,p}(\Lambda_T) \cap \mathrm{Lip}(\Lambda_T, d_\infty)\right)$, the pathwise integral thus defines an isometry

$$I_S : \phi \in J_p \subset L^p([0,T], d[S]^p) \mapsto \int_0^{\cdot} \phi.dS \in V_p(\pi)$$

Isometry formula: proof

$$|R_F(s,t)| := |F(t,S_t) - F(s,S_s) - \nabla_\omega F(s,S_s)(S(t)-S(s))| \le C|t-s|^{\alpha+\alpha^2}. \quad (1)$$

Let $\gamma_F(s,t) := \nabla_\omega F(s,S_s)(S(t)-S(s))$. Then

$$\sum_{\substack{[t_j,t_{j+1}]\in\pi_n:\\ t_{j+1}\le t}} |F(t_{j+1},S_{t_{j+1}}) - F(t_j,S_{t_j})|^p = \sum_{\substack{[t_j,t_{j+1}]\in\pi_n:\\ t_{j+1}\le t}} |R_F(t_j,t_{j+1}) + \gamma_F(t_j,t_{j+1})|^p$$

$$= \sum_{\substack{[t_j,t_{j+1}]\in\pi_n:\\ t_{j+1}\le t}} |\gamma_F(t_j,t_{j+1})|^p + \sum_{k=1}^{p}\binom{p}{k} \sum_{\substack{[t_j,t_{j+1}]\in\pi_n:\\ t_{j+1}\le t}} R_F(t_j,t_{j+1})^k \gamma_F(t_j,t_{j+1})^{p-k}. \quad (2)$$

Since $S \in V_p(\pi)$ we have

$$\lim_{n\to\infty} \sum_{\substack{[t_j,t_{j+1}]\in\pi_n:\\ t_{j+1}\le t}} |\gamma_F(t_j,t_{j+1})|^p = \int_0^t |\nabla_\omega F(s,S(s))|^p \mathrm{d}[S]^p(s). \quad (3)$$

We need show that the double sum on the right hand side of (2) vanishes. Let $k \in \{1, \ldots, p\}$ and write $q_k := p/(p-k) \in [1, \infty]$ and let $q_k' = p/k$ be its conjugate exponent. Hölder's inequality yields

$$\Big| \sum_{\substack{[t_j, t_{j+1}] \in \pi_n: \\ t_{j+1} \le t}} R_F(t_j, t_{j+1})^k \gamma_F(t_j, t_{j+1})^{p-k} \Big|$$

$$\le \Big(\sum_{\substack{[t_j, t_{j+1}] \in \pi_n: \\ t_{j+1} \le t}} |R_F(t_j, t_{j+1})|^{k q_k'} \Big)^{1/q_k'} \Big(\sum_{\substack{[t_j, t_{j+1}] \in \pi_n: \\ t_{j+1} \le t}} |\gamma_F(t_j, t_{j+1})|^{(p-k) q_k} \Big)^{1/q_k}$$

$$= \Big(\sum_{\substack{[t_j, t_{j+1}] \in \pi_n: \\ t_{j+1} \le t}} |R_F(t_j, t_{j+1})|^{p} \Big)^{k/p} \Big(\sum_{\substack{[t_j, t_{j+1}] \in \pi_n: \\ t_{j+1} \le t}} |\gamma_F(t_j, t_{j+1})|^{p} \Big)^{(p-k)/p}.$$

By (1) the first sum on the right hand side is bounded by

$$\Big(\sum_{\substack{[t_j, t_{j+1}] \in \pi_n: \\ t_{j+1} \le t}} |R_F(t_j, t_{j+1})|^{p} \Big)^{k/p} \le c \Big(\sum_{\substack{[t_j, t_{j+1}] \in \pi_n: \\ t_{j+1} \le t}} |t_{j+1} - t_j|^{p(\alpha+\alpha^2)} \Big)^{k/p}$$
$$\le (t \times \max\{|t_{j+1} - t_j|^{p(\alpha+\alpha^2)-1} : [t_j, t_{j+1}] \in \pi_n, t_{j+1} \le t\})^{k/p}, \qquad (4)$$

which converges to zero for $n \to \infty$ because for $\alpha > (\sqrt{1 + \frac{4}{p}} - 1)/2$ we have $p(\alpha + \alpha^2) > 1$ and $k > 0$. Moreover, by (3) the sum over $|\gamma_F(t_j, t_{j+1})|^p$ is bounded and this concludes the proof.

Rough-smooth decomposition

'Signal+noise' decomposition for smooth functionals of a rough process:

Theorem (Rough-smooth decomposition: general case)

Let $p \in \mathbb{N}$ be an even integer, let $\alpha > ((1+\frac{4}{p})^{1/2}-1)/2$, and let $S \in V_p(\pi) \cap C^\alpha([0,T],\mathbb{R})$ be a path with strictly increasing p-th variation $[S]^p$ along (π_n). Then any $X \in \mathbb{C}_b^{1,p}(S)$ admits a unique decomposition

$$\exists! \phi \in \mathbb{T}_{p-1}\mathbb{C}_b^{1,p}, \qquad X = X(0) + A + \int_0^t < \phi \circ S, dS >$$

where $\phi = \mathbb{T}_{p-1}F$ is an exact form and $[A]^p = 0$.

- For S martingale, $p = 2$ this coincides with the semimartingale decomposition. Here: strictly pathwise/ non-probabilistic.
- Such decompositions were obtained in the rough path setting by Hairer & Pillai (2013), Friz & Shekhar (2013). Here we do not require any rough path machinery, nor any extension of the path S: our constructions are 'canonical' and pathwise.

Rough-smooth decomposition: proof

Existence is a consequence of the change of variable formula. Consider two such decompositions $X - X_0 = A + M = \tilde{A} + \tilde{M}$. Since $[A]^p = [\tilde{A}]^p = 0$ and

$$|(A - \tilde{A})(t) - (A - \tilde{A})(s)|^p \lesssim |A(t) - A(s)|^p + |\tilde{A}(t) - \tilde{A}(s)|^p,$$

we get $A - \tilde{A} \in V_p(\pi)$ and $[A - \tilde{A}]^p \equiv 0$. But then also $[M - \tilde{M}]^p = [A - \tilde{A}]^p \equiv 0$. Now

$$M(t) = \int_0^t \nabla_\omega F(s, S_s) \mathrm{d}S(s), \qquad \tilde{M}(t) = \int_0^t \nabla_\omega \tilde{F}(s, S_s) \mathrm{d}S(s)$$

for some $F, \tilde{F} \in C_b^{1,p}(\Lambda_T)$, and by Theorem 13 we have

$$0 = [M - \tilde{M}]^p(T) = \int_0^T |\nabla_\omega (F - \tilde{F})(s, S_s)|^p \mathrm{d}[S]^p(s).$$

Since $(F - \tilde{F})(s, S_s)$ is continuous in s and $[S]^p$ is strictly increasing we have $\nabla_\omega (F - \tilde{F})(\cdot, S) \equiv 0$. This means that $M - \tilde{M} \equiv 0$, and then also $A - \tilde{A} \equiv 0$.

Relation with 'rough path integration'

Define a *control function* as a continuous map $c\colon \Delta_T \to \mathbb{R}_+$ such that $c(t,t) = 0$ and $c(s,u) + c(u,t) \leq c(s,t)$.

Definition (Reduced rough path of order p)

Let $p \geq 1$. A *reduced rough path* of finite p-variation is a map $\mathbb{X} = (1, \mathbb{X}^1, \ldots, \mathbb{X}^{\lfloor p \rfloor})\colon \Delta_T \longrightarrow \mathbb{S}_{\lfloor p \rfloor}(\mathbb{R}^d)$,
such that

$$\sum_{k=1}^{\lfloor p \rfloor} |\mathbb{X}^k_{s,t}|^{p/k} \leq c(s,t), \qquad (s,t) \in \Delta_T;$$

for some control function c and the *reduced Chen relation* holds

$$\mathbb{X}_{s,t} = \mathrm{Sym}(\mathbb{X}_{s,u} \otimes \mathbb{X}_{u,t}), \qquad (s,u),(u,t) \in \Delta_T.$$

A canonical reduced rough path for $S \in V_p(\pi)$

Lemma

Let $p \geq 1$, $S \in C([0,T],\mathbb{R}^d) \cap V_p(\pi)$ *where*

$$\pi_n = (t_k^n), \qquad t_0^n = 0, \qquad t_{k+1}^n = \inf\{t \in [t_k^n, T], \quad |S(t) - S(t_k^n)| \geq 2^{-n}\}.$$

Then for any $q > p$ *with* $\lfloor q \rfloor = \lfloor p \rfloor$ *we obtain a reduced rough path of finite q-variation by setting* $\mathbb{X}_{s,t}^0(S) := 1$,

$$\mathbb{X}_{s,t}^k(S) := \frac{1}{k!}(S(t) - S(s))^{\otimes k}, \qquad k = 1, \ldots, \lfloor p \rfloor - 1,$$
$$\mathbb{X}_{s,t}^{\lfloor p \rfloor}(S) := \frac{1}{\lfloor p \rfloor !}(S(t) - S(s))^{\otimes \lfloor p \rfloor} - \frac{1}{\lfloor p \rfloor !}([S]^p(t) - [S]^p(s)).$$

Furthermore $\mathbb{X} : S \mapsto \mathbb{X}(S)$ *is a non-anticipative functional.*

Proposition

Let $p \geq 1$, let $\mathbb{X}$ be a reduced rough path of finite p-variation and let $Y \in \mathcal{D}_{\mathbb{X}}^{\lfloor p \rfloor / p}([0, T])$. Then the 'rough path integral'

$$I_{\mathbb{X}}(Y)(t) = \int_0^t \langle Y(s), \mathrm{d}\mathbb{X}(s) \rangle = \lim_{\substack{\pi \in \Pi([0,t]) \\ |\pi| \to 0}} \sum_{[t_j, t_{j+1}] \in \pi} \sum_{k=1}^{\lfloor p \rfloor} \langle Y^k(t_j), \mathbb{X}^k_{t_j, t_{j+1}} \rangle,$$

defines a function in $C([0, T], \mathbb{R})$, and it is the unique function with $I_{\mathbb{X}}(Y)(0) = 0$ for which there exists a control function c with

$$\Big| \int_s^t \langle Y(r), \mathrm{d}\mathbb{X}(r) \rangle - \sum_{k=1}^{\lfloor p \rfloor} \langle Y^k(s), \mathbb{X}^k_{s,t} \rangle \Big| \lesssim c(s, t)^{\frac{\lfloor p \rfloor + 1}{p}}, \qquad (s, t) \in \Delta_T.$$

Pathwise integral as canonical rough integral

Proposition (C- Perkowski 2019)

Let $p \in 2\mathbb{N}$ be an even integer, $S \in V_p(\pi)$ and $\mathbb{X}$ the canonical reduced rough path of order p associated to S, defined above. Then

$$\underbrace{\int_0^t \langle \nabla f(S(s)), \mathrm{d}\mathbb{X}(s) \rangle}_{\text{Rough integral}} = \underbrace{\int_0^t \langle \nabla_{p-1} f(S), \mathrm{d}S \rangle}_{\text{Pathwise integral}},$$

where the right hand side is the pathwise integral defined as a limit of compensated Riemann sums.

Note that, unlike the typical rough path construction:

- The construction is canonical: only the path S itself is used to construct $\mathbb{X}$
- We do NOT need S to have finite p-variation. In fact in all our examples above $S \in V_p(\pi)$ but $\|S\|_{p-var} = \infty$.