Automorphic Forms Learning Seminar Notes

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These notes are essentially a summary of Goldfeld's *Automorphic Forms and L-functions for the Group GL*$(n, \mathbb{Z})$.

1 Discrete Group Actions

1.1 Action of a Topological Space

- Left group action of G on X: continuous if $x \to g \circ x$ is continuous for all g. We denote the set of orbits $G \backslash X$ (right cosets).
- $\Gamma \subseteq G$ is discrete if for any compact K, there exists finitely many $\gamma \in \Gamma$ such that $\gamma K \cap K \neq \emptyset$.
- $\mathrm{SL}(2, \mathbb{Z})$ is a discrete subgroup of $\mathrm{SL}(2, \mathbb{R})$.
- $\Gamma_\infty = \left\{ \begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix} | m \in \mathbb{Z} \right\}$.
- Proof that $\mathrm{SL}(2, \mathbb{Z})$ is discrete: Finitely many $\gamma \in \Gamma_\infty \backslash \mathrm{SL}(2, \mathbb{Z})$ such that a rectangle intersects itself after translation by Γ. Multiplication by something in Γ_∞ corresponds to translation, and only finitely many possible translate can hit the same rectangle.
- Standard action of $\mathrm{SL}(2, \mathbb{Z})$ on $\mathbb{H}$, and fundamental domain.

1.2 Iwasawa Decomposition

- Iwasawa decompostion for $\mathrm{GL}(2, \mathbb{R})$: We can express

$$g = zkd,$$

where z is upper triangular with 1 in the lower right corner, k is orthogonal, and d is diagonal with the same entry along the diagonal. k and d are unique up to multiplication by $\pm I$, and z is unique.

- Generalized upper half-plane: $\mathfrak{h}^n$ is the set of all matrices in $\mathrm{GL}(n, \mathbb{R})$ of the form xy, where x is upper triangular with 1s on the diagonal, and

$$y = \begin{pmatrix} y_1 y_2 \cdots y_{n-1} & & & & \\ & y_1 y_2 \cdots y_{n-2} & & & \\ & & \ddots & & \\ & & & y_1 & \\ & & & & 1 \end{pmatrix}.$$

- $\mathfrak{h}^3$ does not have complex structure, compared to $\mathfrak{h}^2$. This is what makes $\mathrm{GL}(n)$ automorphic forms different.
- Iwasawa decomposition for $\mathrm{GL}(n)$: We have that

$$\mathrm{GL}(n, \mathbb{R}) = \mathfrak{h}^n O(n, R) Z_n,$$

where Z_n is the center of $\mathrm{GL}(n, \mathbb{R})$, i.e. diagonal matrices with everything the same along the diagonal. Letting $g = zkd$ be the decomposition, k and d are unique up to multiplication by $\pm I$, and z is unique. Hence

$$\mathfrak{h}^n \cong \mathrm{GL}(n, \mathbb{R}) / (O(n, \mathbb{R}) \mathbb{R}^*),$$

defining an action of $\mathrm{GL}(n, \mathbb{R})$ (and $\mathrm{GL}(n, \mathbb{Z})$) on $\mathfrak{h}^n$.

- Proof of decomposition: explicit computation involving factoring gg^T in terms of upper and lower triangular matrices.

1.3 Siegel Sets

- Siegel set: $\Sigma_{a,b} \subseteq \mathfrak{h}^n$ is the set of $z = x \cdot y \in \mathfrak{h}^n$ such that $|x_{i,j}| \leq b$ and $y_i > a$.
- $\Gamma^n = \mathrm{GL}(n,\mathbb{Z})$ acts discretely on $\mathfrak{h}^n$. In particular, for any $z \in \mathfrak{h}^n$, there are only finitely many $g \in \Gamma^n$ such that $gz \in \Sigma_{\sqrt{3}/2,1/2}$. In fact, we can write

$$\mathrm{GL}(n,\mathbb{R}) = \bigcup_{g \in \Gamma^n} g\Sigma_{\sqrt{3}/2,1/2}.$$

  Hence $\Sigma_{\sqrt{3}/2,1/2}$ serves as a "good approximation" for a fundamental domain for $\mathfrak{h}^n$.

- Proof idea: Reduce to $\mathrm{SL}(n,\mathbb{R})$ and $\mathrm{SL}(n,\mathbb{Z})$. Show that if $\phi(\gamma z)$ is minimized for $\gamma \in \mathrm{SL}(n,\mathbb{Z})$, where ϕ is the norm of the last row (which exists because $\mathrm{SL}(n,\mathbb{Z})$ is a lattice) then $\gamma z \in \Sigma^*_{\sqrt{3}/2,1/2}$ (determinant 1 version). This proves the cover of $\mathrm{GL}(n,\mathbb{R})$ by elements of $\Sigma_{\sqrt{3}/2,1/2}$.
- Proof of discreteness of action: We show that there are finitely many $\gamma \in \mathrm{SL}(n,\mathbb{Z})$ such that $\gamma z \in \Sigma^*_{\sqrt{3}/2,1/2}$. (This is good enough because $\mathrm{GL}(n,\mathbb{Z})/\mathrm{SL}(n,\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}$.) Define $\phi_i(z) = \|e_i\gamma z\|$, well-defined on $\mathrm{SL}(n,\mathbb{R})/\mathrm{SO}(n,\mathbb{R})$. Letting $z = xy$ and explicitely computing $\phi_i(z)$ shows that $\gamma z \in \Sigma^*_{\sqrt{3}/2,1/2} \implies \phi_i(z)$ bounded, hence since $e_i\gamma z$ has lattice structure, there are only finitely many ith rows of γ so that γz lies in $\Sigma^*_{\sqrt{3}/2,1/2}$, and hence finitely many γ.

1.4 Haar Measure

- Topological group: A topological space G such that G is a group, and

$$(g,h) \mapsto g \cdot h^{-1}$$

  is continuous in both variables; i.e. multiplication and inversion is continuous.

- Locally compact: every point has compact neighborhood
- Hausdorff: distinct elements can be separated by opens
- In particular, $\mathrm{GL}(n,\mathbb{R})$ is a locally compact Hausdorff topological group, coming from the subspace topology of $\mathrm{GL}(n,\mathbb{R}) \subseteq \mathrm{Mat}(n,\mathbb{R}) = \mathfrak{gl}(n,\mathbb{R})$.
- (left) Haar measure: For locally compact Hausdorff topological group, we want a positive Borel measure μ on G, left invariant on the action by G, i.e. $\mu(gE) = \mu(E)$. Same for right. If left invariant measure means right invariant measure on G, G is unimodular.

  Can define differential one form, such that that we have integrals for compactly supported $f : G \to \mathbb{C}$

$$\int_G f(g)\,\mathrm{d}\mu(g)\,,$$

  and

$$\int_E \mathrm{d}\mu(g) = \mu(E).$$

  This $\mathrm{d}\mu(g)$ is the Haar measure.

- Key theorem: For any locally compact Hausdorff topological group, there exists a unique left Haar measure on G, up to positive real multiples. Proof of uniquness: Fubini.

- Haar measure on $\mathrm{GL}(n,\mathbb{R})$: For $g=(g_{i,j})_{i,j}\in\mathrm{GL}(n,\mathbb{R})$, the unique left-right invariant measure on $\mathrm{GL}(n,\mathbb{R})$ is
$$\mathrm{d}\mu(g)=\frac{\prod_{1\le i,j\le n}\mathrm{d}g_{i,j}}{\det(g)^n}.$$
Proof: Decompose $\mathrm{GL}(n,\mathbb{R})$ into Z_n (center of $\mathrm{GL}(n,\mathbb{R})$) and elements that are 1 on the diagonal and $x_{r,s}$ and (r,s), and do casework.

1.5 Invariant measure on coset spaces

- Let G be a locally compact Hausdorff topological group, and H a compact subgroup of G, with corresponding Haar measure μ and ν, respectively. Then there exists a unique (up to scalar multiple) quotient measure $\widetilde{\mu}$ on G/H such that
$$\int_G f(g)\,\mathrm{d}\mu(g)=\int_{G/H}\left(\int_H f(gh)\,\mathrm{d}\nu(h)\right)\mathrm{d}\widetilde{\mu}(gH)\,.$$
- $\mathfrak{h}^n$ and $\mathrm{GL}(n,\mathbb{R})$: The measure left invariant $\mathrm{GL}(n,\mathbb{R})$ measure on $\mathfrak{h}^n$ can be expressed as
$$\mathrm{d}^*z=\mathrm{d}^*x\mathrm{d}^*y,$$
with
$$\mathrm{d}^*x=\prod_{1\le i<j\le n}\mathrm{d}x_{i,j}$$
and
$$\mathrm{d}^*y=\prod_{1\le i\le k}y_k^{-k(n-k)-1}\,\mathrm{d}y_k\,.$$
Proof: Check invariance under diagonal matrices, upper triangular matrices with 1s on diagonal, and transpositions.

1.6 Volume of $\mathbf{SL}(n,\mathbb{Z})\backslash\mathbf{SL}(n,\mathbb{R})/\mathbf{SO}(n,\mathbb{R})$

- Note that $\mathrm{SL}(n,\mathbb{R})/\mathrm{SO}(n,\mathbb{R})\cong\mathrm{GL}(n,\mathbb{R})/(\mathrm{O}(n,\mathbb{R})\cdot\mathbb{R}^*)$.
- The volume of $\mathrm{SL}(n,\mathbb{Z})\backslash\mathrm{SL}(n,\mathbb{R})/\mathrm{SO}(n,\mathbb{R})$ can be explicitly computed to be
$$n2^{n-1}\prod_{\ell=2}^{n}\frac{\zeta(\ell)}{\mathrm{Vol}(S^{\ell-1})}.$$
- Proof idea: Induction.
- Base case ($n=2$): Can directly integrate using the fundamental domain. Or use the technique from the general case.
- General case: Define a test function f, then create a periodic function
$$F(z)=\sum_{m\in\mathbb{Z}^n}f(m\cdot z).$$
Split sum in casework by last row: take out common factor ℓ, then treat as coset of $\mathbb{P}_n\backslash\mathfrak{h}^n$, where P_n is anything with e_n as the last row. Integrate over a fundamental domain $\Gamma_n\backslash\mathfrak{h}^n$. Now break up $\ell e_n\cdot z$ via the Iwasawa decomposition into three components; one that is integrating over $\mathrm{SL}(n-1,\mathbb{Z})$, one over $(\mathbb{R}/\mathbb{Z})^{n-1}$ (corresponding to $x_{j,n}$), and one integrating over $(0,\infty)$; corresponding to $t=\left(\prod_{i=1}^{n-1}y_i^{n-i}\right)^{-1/n}$. Applying spherical integration techniques, this can be computed in terms of $\widehat{f}(0)$. Now, applying Poisson summation, replace f by $\widehat{f}$, and get the same formula but with $\widehat{f}$ and f switched. Choosing an appropriate f, this gives the desired result.

2 Invariant differential operators

- The periodic functions $e^{2\pi inx}$ on $\mathcal{L}^2(\mathbb{Z}\backslash\mathbb{R})$ are precisely the eigenfunctions for the Laplacian operator $\frac{\mathrm{d}^2}{\mathrm{d}x^2}$, with eigenvalue $-4\pi^2n^2$. This directly leads to Fourier theory.
- Thus, we are motivated to consider differential operators invariant under the discrete group, and their eigenvalues/functions.

2.1 Lie algebra

- Associative algebra: Associative algebra A over field K is a vector space over K with an associative product closed in A satisfying the distributive law.
- Lie algebra: Vector space over K with bilinear map $[\cdot,\cdot] : L \times L \to L$ such that
  - $[a, \beta b + \gamma c] = \beta[a, b] + \gamma[a, c]$
  - $[a, a] = 0$
  - $[a, b] = -[b, a]$
  - $[a, [b, c]] + [b, [c, a]] + [c, [a, b]] = 0$.
- Given an associative algebra A, the associated Lie algebra $\mathrm{Lie}(A)$ is A equipped with the bracket

  $$[a, b] = ab - ba.$$

- Universal enveloping algebra: For any Lie algebra $\mathfrak{L}$ over K, consider the tensor algebra

  $$T(\mathfrak{L}) = \oplus_{k=0}^{\infty} \otimes^k \mathfrak{L},$$

  where the tensor product is taken over K. Let $I(\mathfrak{L})$ be the two-sided ideal generated by $X \otimes Y - Y \otimes X - [X, Y]$. Then the universal enveloping algebra is precisely

  $$U(\mathfrak{L}) = T(\mathfrak{L})/I(\mathfrak{L}),$$

  an associative algebra with the product $X \circ Y = X \otimes Y \pmod{I(\mathfrak{L})}$. In particular, by definition,

  $$\mathfrak{L} \subseteq \mathrm{Lie}(U(\mathfrak{L}))$$

  with the inclusion respecting the bracket.

2.2 Universal enveloping algebra of $\mathfrak{gl}(n, \mathbb{R})$

- $\mathfrak{gl}(n, \mathbb{R})$: Precisely $\mathrm{Mat}(n, \mathbb{R})$, with Lie bracket

  $$[\alpha, \beta] = \alpha\beta - \beta\alpha.$$

- We have the (left invariant) differential operators D_α, for $\alpha \in \mathfrak{gl}(n, \mathbb{R})$, acting on the set of smooth functions $\mathrm{GL}(n, \mathbb{R}) \to \mathbb{C}$, via

  $$D_\alpha F(g) = \frac{\partial}{\partial t} F(g \exp(t\alpha))|_{t=0} = \frac{\partial}{\partial t} F(g + tg\alpha)|_{t=0}.$$

  Denote $\mathcal{D}^n$ to be the associative algebra generated by the D_α, where the multiplication is composition.

- Some properties of the differential operators:
  - $D_\alpha(FG) = D_\alpha F \cdot G + F \cdot D_\alpha G$
  - $D_\alpha(F(G(g))) = (D_\alpha F)(G(g))D_\alpha(G)(g)$

  - $D_{\alpha+\beta} = D_\alpha + D_\beta$
  - $D_\alpha \circ D_\beta - D_\beta \circ D_\alpha = D_{[\alpha,\beta]}$.
  - $D_\alpha \circ D_\beta = D_\beta \circ D_\alpha \implies D_{\alpha\beta} = D_{\alpha\beta}$.

  In particular, $\mathcal{D}^n$ can be realized as the universal enveloping algebra of $\mathfrak{gl}(n,\mathbb{R})$. Letting $[D, D'] = D \circ D' - D' \circ D$ be the bracket for the induced Lie algebra (from the universal enveloping algebra), we have that $[D_\alpha, D_\beta \circ D] = [D_\alpha, D_\beta] \circ D + D_\beta \circ [D_\alpha, D]$.

  Proof of properties: Direct calculation using multivariate chain rule.

- If $f : \mathrm{GL}(n,\mathbb{R}) \to \mathbb{C}$ is left-invariant by $\mathrm{GL}(n,\mathbb{Z})$ and right-invariant by Z_n, then for all $D \in \mathcal{D}^n$, Df is also left-invariant by $\mathrm{GL}(n,\mathbb{Z})$ and right-invariant by Z_n.

2.3 The center of the universal enveloping algebra of $\mathfrak{gl}(n,\mathbb{R})$

- Denote $\mathfrak{D}^n$ to be the center of the universal enveloping algebra $\mathcal{D}^n$.
- If $D \in \mathfrak{D}^n$, and f is a smooth function

$$f : \mathrm{GL}(n,\mathbb{Z})\backslash\mathrm{GL}(n,\mathbb{R})/(O(n,\mathbb{R})Z_n) \to \mathbb{C},$$

  then Df is also well defined over $\mathrm{GL}(n,\mathbb{Z})\backslash\mathrm{GL}(n,\mathbb{R})/(O(n,\mathbb{R})Z_n)$; i.e. it is left-invariant by $\mathrm{GL}(n,\mathbb{Z})$ and right invariant by $O(n,\mathbb{R})Z_n$.

- Proof idea: Use the fact that $SO(n,\mathbb{R})$ is generated by exp of skew-symmetric matrices, and that exp commutes well with the definition of D. Then use that D lies in the center, and f is right-invariant by $O(n,\mathbb{R})$ (including δ_1!).
- Casimir operators: Let $E_{i,j}$ be the matrix with 1 at i,j and 0 elsewhere, and let $D_{i,j} = D_{E_{i,j}}$. Then for each $m \geq 2$, we have the Casimir operator

$$\sum_{i_1=1}^{n}\sum_{i_2=1}^{n}\cdots\sum_{i_m=1}^{n} D_{i_1,i_2} \circ D_{i_2,i_3} \circ \cdots \circ D_{i_m,i_1},$$

  which lies in $\mathfrak{D}^n$.

- For $\mathfrak{gl}(n,\mathbb{R})$: The center is a rank n algebra. Any element in the center can be expressed as a polynomial in $\mathbb{R}$ in the Casimir operators defined before and D_{I_n}. Moreover, D_{I_n} annihilates any function invariant under Z_n.

2.4 Eigenfunctions of invariant differential operators

- We want a smooth function $f : \mathfrak{h}^n \to \mathbb{C}$ that is an eigenfunction for all $D \in \mathfrak{D}^n$; i.e. we want

$$Df(z) = \lambda_D f(z)$$

  for all D in the center of the universal enveloping algebra and $z \in \mathfrak{h}^n$.

- Power function: We have the I_s function, a generalization of the imaginary part function raised to the power s;

$$I_s(z) = \prod_{i=1}^{n-1}\prod_{j=1}^{n-1} y_i^{b_{i,j}s_j},$$

  where

$$b_{i,j} = \begin{cases} ij & i+j \leq n \\ (n-i)(n-j) & i+j > n \end{cases}$$

- On $\mathrm{GL}(2,\mathbb{R})$, this is just y^s. We have that $\Delta = y^2(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2})$ generates $\mathfrak{D}^2$ for functions over $\mathfrak{h}^2$ (we can ignore D_{I_2}, as all functions are right-invariant by the center). In particular, note that

$$\Delta I_s(z) = s(s-1)I_s(z).$$

- Claim: $D_{i,j}^k I_s(z) = \left(\sum_{k=1}^{n-i} ks_{n-k} - \sum_{k=1}^{i-1} ks_k\right)^k I_s(z)$ when $i = j$, and 0 otherwise. Proof: intensive computation. (Remark: I believe the proof in Goldfeld for this has some minor errors (computation when $i < j$), and the proposition doesn't imply the desired claim)
- In particular, $I_s(z)$ is such an eigenfunction for all $D \in \mathfrak{D}^n$.
- Theme: Any function in just ys is an eigenfunction.

3 Automorphic forms and L-functions for $\mathrm{SL}(2,\mathbb{Z})$

- Key idea: Automorphicity is equivalent to existence of functional equation for certain L-functions - this is the idea of converse theorems.
- Hecke operators: Simultaneous eigenfunction of all Hecke operators corresponds to Euler product for L-function.

3.1 Eisenstein series

- Hyperbolic Laplacian:

$$\Delta = -y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right),$$

  invariant under the action by $\mathrm{GL}(2,\mathbb{R})^+$.
- y^s is an eigenfunction of this operator, with eigenvalue $s(1-s)$.
- Automorphic function: Smooth function $\mathrm{SL}(2,\mathbb{Z})\backslash\mathfrak{h} \to \mathbb{C}$.
- To construct automorphic function, we average over the group to get the Eisenstein series:

$$E(z,s) = \sum_{\gamma\in\Gamma_\infty\backslash \mathrm{SL}(2,\mathbb{Z})} \frac{I_s(\gamma z)}{2} = \frac{1}{2}\sum_{\substack{c,d\in\mathbb{Z}\\ \gcd(c,d)=1}} \frac{y^s}{|cz+d|^{2s}}$$

- $E(z,s)$ converges absolutely and uniformly on compact subset for $z \in \mathfrak{h}^2$ and Re $(s) > 1$.
- Real analytic in z and complex analytic in s.
- More properties:
  - $|E(z,s) - y^s| \le c(\varepsilon)y^{-\varepsilon}$ for $\sigma \ge 1+\varepsilon > 1$.
  - $E(\gamma z, s) = E(z,s)$ for $\gamma \in \mathrm{SL}(2,\mathbb{Z})$.
  - $\Delta E(z,s) = s(1-s)E(z,s)$.
- Bessel function:

$$K_s(y) = \frac{1}{2}\int_0^\infty e^{-\frac{1}{2}y(u+1/u)}u^s\frac{\mathrm{d}u}{u}.$$

  In particular, $K_s(y) = K_{-s}(y)$.

- Fourier coefficients of Eisenstein series: We have that
$$E(z,s) = y^s + \phi(s)y^{1-s} + \frac{2\pi^s\sqrt{y}}{\Gamma(s)\zeta(2s)}\sum_{n\neq 0}\sigma_{1-2s}(n)|n|^{s-1/2}K_{s-1/2}(2\pi|n|y)e^{2\pi inx},$$
where
$$\phi(s) = \sqrt{\pi}\frac{\Gamma(s-1/2)}{\Gamma(s)}\frac{\zeta(2s-1)}{\zeta(2s)}.$$
- Idea of proof: Integral calculation. A little bit of Ramanujan sums. Some identities from Gamma integrals involve rewriting the Gamma integral then performing a change of variable.
- Properties of ϕ:
  - $\phi(s)\phi(1-s) = 1$,
  - $E(z,s) = \phi(s)E(z,1-s)$.
- We have the modified function $E^*(z,s) = \pi^{-s}\Gamma(s)\zeta(2s)E(z,s)$. It is meromorphic, with simple poles at $s = 0, 1$. It has functional equation
$$E^*(z,s) = E^*(z,1-s)$$
(which follows by examining the Fourier coefficients) and has
$$\mathrm{Res}_{s=1} E(z,s) = \frac{3}{\pi}$$
for all $z \in \mathfrak{h}^2$.
- Why do we care? Useful in the Rankin-Selberg method (discussed previously) to get functional equations for L-functions. Will also arise the in the Selberg spectral decomposition of $\mathcal{L}^2(\mathrm{SL}(2,\mathbb{Z})\backslash\mathfrak{h}^2)$ functions.

3.2 Hyperbolic Fourier expansion of Eisenstein series

- Idea: We can use a hyperbolic Fourier expansion of the Eisenstein series to recover the functional equation for the Hecke L-function associated to $\mathbb{Q}(\sqrt{D})$, where this is a real quadratic field (also for D of specific form).
- Let $\rho = \begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$. be a hyperbolic element; i.e. $\gamma > 0$ and $|\alpha+\delta| > 2$. This has two fixed points
$$\omega = \frac{\alpha-\delta+\sqrt{D}}{2\gamma}$$
and
$$\omega' = \frac{\alpha-\delta-\sqrt{D}}{2\gamma},$$
where $D = (\alpha+\delta)^2 - 4$.
- Let
$$\kappa = \begin{pmatrix}1 & -\omega\\ 1 & -\omega'\end{pmatrix}.$$
Then $\kappa\rho\kappa^{-1}$ (the diagonalization) is equivalent to
$$\begin{pmatrix}\varepsilon & 0\\ 0 & \varepsilon^{-1}\end{pmatrix},$$
where
$$\varepsilon = (\alpha+\delta-\sqrt{D})/2.$$
so that $\varepsilon + \varepsilon^{-1} = \alpha+\delta$. Moreover, it is a unit in $\mathbb{Q}(\sqrt{D})$. Since $\mathbb{Q}(\sqrt{D})$ is a quadratic extension of $\mathbb{Q}$, the ring of integers is rank 1, and we suppose that ε is a fundamental unit of the group of units.

- In particular, we have that
$$E(\kappa^{-1}z, s) = E(\kappa^{-1}(\varepsilon^2 z), s).$$
Consider this series as a function of v, where $z = iv$. We get a Fourier expansion
$$\zeta(2s)E(\kappa^{-1}(iv), s) = \sum_{n\in\mathbb{Z}} b_n(s) v^{\frac{\pi i n}{\log \varepsilon}},$$
with
$$b_n(s) = \frac{1}{2\log\varepsilon}\int_1^{\varepsilon^2} \zeta(2s)E(\kappa^{-1}(iv), s) v^{\frac{\pi i n}{\log\varepsilon}} \frac{\mathrm{d}v}{v}$$
- After some tedious calculation, you get that
$$b_n(s) = \frac{(\omega-\omega')^s}{4\log\varepsilon}\sum_{\beta\neq 0} N(\beta)^{-s}\left|\frac{\beta}{\beta'}\right|^{\frac{-\pi i n}{\log\varepsilon}} \int_{|\beta'/\beta|}^{\varepsilon^2|\beta'/\beta|}\left(\frac{v}{v^2+1}\right)^s v^{-\pi i n/\log\varepsilon}\frac{\mathrm{d}v}{v}$$
Note that there are typos in the book: extra factor of 1/2, and inside term is v and not v^2.
- The $\beta = c\omega + d$ lie in an (fractional) ideal $\mathfrak{b}$ such that $N(\mathfrak{b}) = \frac{1}{\gamma}$, so using the definition of two principal ideals being equal (using that ε is a fundamental unit) and an integral calculation similar to Bump Proposition 1.9.1, we have that
$$b_n(s) = \frac{\Gamma\left(\frac{s-\frac{\pi i n}{\log\varepsilon}}{2}\right)\Gamma\left(\frac{s+\frac{\pi i n}{\log\varepsilon}}{2}\right)}{\Gamma(s)}\frac{(N(\mathfrak{b})\sqrt{D})^s}{8\log\varepsilon}\sum_{\mathfrak{b}|(\beta)\neq 0}\left|\frac{\beta}{\beta'}\right|^{-\pi i n/\log\varepsilon} N(\beta)^{-s}.$$
Note: There is another factor of 1/2 here compared to the book from the gamma integral.
- We have the Hecke grossencharakter
$$\psi((\beta)) = \left|\frac{\beta}{\beta'}\right|^{-\pi i n/\log\varepsilon}$$
and Hecke L-function
$$L_{\mathfrak{b}}(s, \psi^n) = \sum_{\mathfrak{b}|(\beta)\neq 0}\psi^n((\beta))N(\beta)^{-s}.$$
- Hence the expansion for the Eisenstein series invovles the Hecke L-function:
$$E^*(\kappa^{-1}(iv), s) = \frac{(N(\mathfrak{b})\sqrt{D})^s}{8\pi^s\log\varepsilon}\sum_{n\in\mathbb{Z}}\Gamma\left(\frac{s-\frac{\pi i n}{\log\varepsilon}}{2}\right)\Gamma\left(\frac{s+\frac{\pi i n}{\log\varepsilon}}{2}\right)L_{\mathfrak{b}}(s,\psi^n)v^{\pi i n/\log\varepsilon}.$$
- The functional equation for the Eisenstein series hence gives the functional equation for the Hecke L-function: $L_{\mathfrak{b}}(s, \psi^n)$ has meromorphic continuation except a simple pole at $s = 1$, and letting
$$\Lambda_{\mathfrak{b}}^n(s) = \frac{(N(\mathfrak{b})\sqrt{D})^s}{\pi^s}\Gamma\left(\frac{s-\frac{\pi i n}{\log\varepsilon}}{2}\right)\Gamma\left(\frac{s+\frac{\pi i n}{\log\varepsilon}}{2}\right)L_{\mathfrak{b}}(s,\psi^n),$$
we have $\Lambda_{\mathfrak{b}}^n(s) = \Lambda_{\mathfrak{b}}^n(1-s)$.

3.3 Maass forms

- We have a Hilbert space of $\mathcal{L}^2(\mathrm{SL}(2,\mathbb{Z})\backslash\mathfrak{h}^2)$ with the inner product given by the Petersson inner product:

$$\int_{\mathrm{SL}(2,\mathbb{Z})\backslash\mathfrak{h}^2} f(z)\overline{g(z)}\frac{\mathrm{d}x\,\mathrm{d}y}{y^2}.$$

- We define a Maass form of type v to be a function $f \in \mathcal{L}^2(\mathrm{SL}(2,\mathbb{Z})\backslash\mathfrak{h}^2)$ such that
  - $f(\gamma z) = f(z)$ for $\gamma \in \mathrm{SL}(2,\mathbb{Z})$.
  - $\Delta f = v(1-v)f$
  - $\int_0^1 f(z)\,\mathrm{d}x = 0$.

  (In other sources, the last condition is for a Maass cuspform.)

- Δ is a self-adjoint operator. Hence $v(1-v)$ is real and nonnegative. Proof idea: Use that

$$\int_{\Gamma\backslash\mathfrak{h}^2}\left(\frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y^2}\right)\overline{f} = \int_{\Gamma\backslash\mathfrak{h}^2}\left|\frac{\partial f}{\partial x}\right|^2+\left|\frac{\partial f}{\partial y}\right|^2$$

  for f a Maass form. This follows from Green's theorem.

- Maass form of types 0 or 1 must be constant. Follows from properties of harmonic functions. (Why is f bounded as the imaginary part of $z \to \infty$? Answer: Follows from Fourier expansion; we show later that each of the Fourier expansions has rapid decay, so the function is hence bounded. See: `https://math.stackexchange.com/questions/4980702/a-question-on-properties-of-mass-forms`)

3.4 Whittaker expansions and multiplicity one for $\mathbf{GL}(2,\mathbb{R})$

- For a Maass form f, using the transformation property gives a Fourier expansion

$$f(z) = \sum_{m\in\mathbb{Z}} A_m(y)e^{2\pi imx},$$

  and $A_m(y)e^{2\pi imx}$ satsifies the two properties
  - $\Delta W_m(z) = v(1-v)W_m(z)$
  - $W_m\left(\begin{pmatrix}1 & u\\ 0 & 1\end{pmatrix}z\right) = W_m(z)e^{2\pi imu}$

  This motivates the following definition.

- A Whittaker function of type v with additive character $\psi : R \to S^1$ is a smooth nonzero function $W : \mathfrak{h}^2 \to \mathbb{C}$ such that
  - $\Delta W(z) = v(1-v)W(z)$
  - $W\left(\begin{pmatrix}1 & u\\ 0 & 1\end{pmatrix}z\right) = W_m(z)\psi(u)$

- On $\mathrm{GL}(2,\mathbb{R})$, we can construct these functions explicitely. We can check that

$$W(z,v,\psi_m) = \sqrt{2}\frac{(\pi|m|)^{v-1/2}}{\Gamma(v)}\sqrt{2\pi y}K_{v-1/2}(2\pi|m|y)e^{2\pi imx}.$$

  where

$$K_v(y) = \frac{1}{2}\int_0^\infty e^{-1/2y(u+1/u)}u^v\frac{\mathrm{d}u}{u}.$$

- Multiplicity 1: For $\mathrm{SL}(2,\mathbb{Z})$-Whittaker functions of type not 0 or 1 with rapid decay at infinity, it must be a constant multiple of the W computed before. In particular, if $\psi = 1$, then $a = 0$. Moreover, if ψ is non-trivial, we can assume W has polynomial growth.
- Proof follows from differential equation theory. In the nontrivial case, there are two solutions; one has rapid decay ($K_v(y)$) and one has rapid growth, and it is precisely the function defined before.

3.5 Fourier-Whittaker expansions on $\mathbf{GL}(2,\mathbb{R})$

- Corollary of Multiplicity One theorem: Every nonconstant Maass form of type v (i.e. type not 0 or 1) has Whittaker expansion of the form

$$f(z) = \sum_{n\neq 0} a_n \sqrt{2\pi y} K_{v-1/2}(2\pi|n|y)e^{2\pi i n z}.$$

- Proof: The integral condition requires that the $e^{2\pi i 0}$ coefficient is 0. Maass forms being $\mathcal{L}^2$ implies that it has polynomial growth. Hence all of the Whittaker functions corresponding to each Fourier coefficient (corresponding not to 0) must be at worst polynomial growth. By multiplicity one, this forces every Fourier coefficient to be of the form above.

3.6 Ramunujan-Petersson Conjecture

- For holomorphic modular cuspforms of weight k, we have the Ramunujan-Petersson conjecture

$$|a_n| = O(n^{(k-1)/2}d(n))$$

  where $d(n)$ is the number of divisors of n.
- Idea: Non-constant Maass forms like holomorphic modular forms of weight 0.
- This leads to Ramanujan-Petersson conjecture for Maass forms:

$$|a_n| = O(d(n)),$$

  with constant only dependent on the Petersson norm of f.
- What we can show: If f has Petersson norm 1 and is of type v, we have that

$$|a_n| = O_v(\sqrt{|n|}).$$

  Proof idea: Integrate from $x \in [0,1]$, $y \in [Y,\infty)$ of $|f(z)|^2$, then isolate the a_n. Then use a change of variable $y \mapsto Yy$ to get a $\frac{1}{Y}$ factor times the area of $|f|^2$ over the fundamental domain.

3.7 Selberg eigenvalue conjecture

- We know that for a non-constant Maass form f, $\Delta f = v(1-v)f$, where $\lambda = v(1-v)$ is real and positive. How small can λ be?
- Maass form for a congruence subgroup Γ: Smooth on $\mathfrak{h}^2$, automorphic on Γ, lies in $\mathcal{L}^2(\Gamma\backslash\mathfrak{h}^2)$, constant terms of Fourier expansions at cusps vanish, and $\Delta f = v(1-v)f$.
- Selberg conjecture: If f is a Maass form of type v for a congruence subgroup Γ, then $v(1-v) \geq 1/4$; i.e. Re $(v) = 1/2$.
- For Maass forms on $\mathrm{SL}(2,\mathbb{Z})$, can prove (according to M-F Vigneras) that $v(1-v) \geq \frac{3\pi^2}{2}$.
- Proof idea: Use again that

$$\int_{\Gamma\backslash\mathfrak{h}^2} \left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}\right)\overline{f} = \int_{\Gamma\backslash\mathfrak{h}^2} \left|\frac{\partial f}{\partial x}\right|^2 + \left|\frac{\partial f}{\partial y}\right|^2$$

  for f a Maass form. This follows from Green's theorem.

3.8 Finite dimensionality of the eigenspaces

- Let $\mathfrak{S}_\lambda$ be the space of Maass forms of eigenvalue $\Lambda = v(1-v)$ under Δ. This space is finite dimensional.
- Idea of proof: If $a_n = 0$ for $n \le n_0$, for n_0 sufficiently large, then f itself must be 0. Get bound using that $a_n = O(\sqrt{|n|})$ and use that $K_v(y) \asymp \frac{e^{-y}}{\sqrt{y}}$.

3.9 Even and odd Maass forms

- T_{-1}: Operator such that
$$T_{-1}f(x+iy) = f(-x+iy).$$
Notation is written to match Hecke operators later. This sends Maass forms of type v to Maass forms of type v.
- In particular, the eigenvalues of T_{-1} must be ± 1, since $T_{-1}^2 = I$.
- If $T_{-1}f = f$, then f is even. If $T_{-1}f = -f$, then f is odd.
- If f is even, then $a_n = a_{-n}$. If f is odd, then $a_n = -a_{-n}$. Proof: Clear from Fourier inversion after making substitution $x \mapsto -x$.
- Any Maass form of type v can be expressed as the sum of an even and odd Maass form:
$$f = \frac{1}{2}\left(f + T_{-1}f\right) + \frac{1}{2}\left(f - T_{-1}f\right);$$
the left is an even Maass form and the right is an odd Maass form.

3.10 Hecke operators

We will show this in more generality, then later apply to the case of $\Gamma = \mathrm{SL}(2,\mathbb{Z})$ and $X = \mathfrak{h}^2$.

- G is a group acting continuously on topological space X, Γ is a discrete subgroup of G, $\Gamma\backslash X$ as left Γ-invariant measure $\mathrm{d}x$.
- We have the commensurator of Γ
$$C_G(\Gamma) = \left\{g \in G | (g^{-1}\Gamma g) \cap \Gamma \text{ has finite index in both } \Gamma \text{ and } g^{-1}\Gamma g\right\}.$$
- For any $g \in C_G(\Gamma)$, we have the decomposition
$$\Gamma = \cup_{i=1}^d \left((g^{-1}\Gamma g) \cap \Gamma\right)\delta_i,$$
giving double coset decomposition
$$\Gamma g \Gamma = \cup_{i=1}^d \Gamma g \delta_i$$
for some representatives $\delta_i \in \Gamma$, where $d = [\Gamma : (g^{-1}\Gamma g) \cap \Gamma]$.
- We define the Hecke operator
$$T_g : \mathcal{L}^2(\Gamma\backslash X) \to \mathcal{L}^2(\Gamma\backslash X)$$
by
$$T_g(f(x)) = \sum_{i=1}^d f(g\delta_i x).$$
- This is well-defined; the choice of δ_i is preserved because f is invariant under left-multiplication under Γ, and $T_g(f(\gamma x)) = T_g(f(x))$ for $\gamma \in \Gamma$ because $\delta_i\gamma = \gamma_i'\delta_{\sigma(i)}$ for $\gamma_i' \in g^{-1}\Gamma g \cap \Gamma$, and so
$$g\delta_i\gamma = g\gamma_i'\delta_{\sigma(i)} \in \Gamma g \delta_{\sigma_i},$$
and then we invoke left invariance of Γ.

- We get the Hecke ring by considering formal sums
$$\sum_k m_k T_{g_k}.$$
- For multiplication, we consider the multiplication of the double cosets:
$$(\Gamma g \Gamma)(\Gamma h \Gamma) = \cup_j \Gamma g \Gamma \beta_j = \cup_{i,j} \Gamma \alpha_i \beta_j = \cup_{\Gamma w \subseteq \Gamma g \Gamma h \Gamma} \Gamma w = \cup_{\Gamma w \Gamma \subseteq \Gamma g \Gamma h \Gamma} \Gamma w \Gamma.$$
- Then
$$T_g T_h = \sum_{\Gamma w \Gamma \subseteq \Gamma g \Gamma h \Gamma} m(g,h,w) T_w,$$
where $m(g,h,w)$ is the number of i,j such that $\Gamma \alpha_i \beta_j = \Gamma w$. This product ends up being associative.
- Let Δ be a semigroup such that $\Gamma \subseteq \Gamma \subseteq C_G(\Gamma)$. The Hecke ring $\mathcal{R}_{\Gamma,\Delta}$ is the set of all formal sums
$$\sum_k c_k T_{g_k}$$
with $c_k \in \mathbb{Z}$ and $g_k \in \Delta$.
- Antiautomorphism: $g \mapsto g^*$ such that $(gh)^* = h^* g^*$. For example, transpose of matrix, which is what we care about.
- Commutativity of Hecke ring: If there exists antiautomorphism $g \mapsto g^*$ of $C_G(\Gamma)$ such that $\Gamma^* = \Gamma$ and $(\Gamma g \Gamma)^* = \Gamma g \Gamma$ for all $g \in \Delta$, then $\mathcal{R}_{\Gamma,\Delta}$ is a commutative ring.
- Proof: Idea: Use the antiautomorphism to show that left and right coset decompositions are basically the same. Then use antiautomorphism to show that products should come out to the same thing.

3.11 Hermite and Smith normal forms

- Hermite normal form: Every matrix $A \in \mathrm{GL}(n,\mathbb{Z})^+$ is left-equivalent under $\mathrm{SL}(n,\mathbb{Z})$ to a matrix B, i.e. $B = \gamma A$, with $\gamma \in \mathrm{SL}(n,\mathbb{Z})$, of the form
$$\begin{pmatrix} d_1 & \alpha_{2,1} & \dots & \alpha_{n,1} \\ 0 & d_2 & \dots & \alpha_{n,2} \\ 0 & 0 & \ddots & \vdots \\ 0 & 0 & \dots & d_n \end{pmatrix}$$
where the d_i are positive integers and $0 \leq \alpha_{k,j} < d_k$.
- Idea of proof: You can get this form by performing row operations that preserve the determinant, which is equivalent to left-multiplication by $\gamma \in \mathrm{SL}(n,\mathbb{Z})$.
- Smith normal form: Every matrix $A \in \mathrm{GL}(n,\mathbb{Z})^+$ is left-right equivalent under $\mathrm{SL}(n,\mathbb{Z})$ to a matrix D; i.e. $D = \gamma_1 A \gamma_2$, where D is a diagonal matrix, with d_n in the top left and d_1 in the bottom right, such that $d_i \mid d_{i+1}$, and the $d_i > 0$.
- Idea of proof: Same idea, but now with both row and column operations. Uniqueness: GCD of all $k \times k$ components determines d_k.

3.12 Hecke operators for $\mathcal{L}^2(\mathbf{SL}(2,\mathbb{Z})\backslash\mathfrak{h}^2)$

- In this case, we have $G = \mathrm{GL}(2,\mathbb{R})$, $\Gamma = \mathrm{SL}(2,\mathbb{Z})$, and $X = \mathfrak{h}^2$.
- The matrix
$$\begin{pmatrix} n_0 n_1 & 0 \\ 0 & n_0 \end{pmatrix}$$
for integers $n_0, n_1 \geq 1$ lies in $C_G(\Gamma)$, we can let Δ be the semigroup generated by Γ and these matrices.
- For this Δ, we have the antiautomorphism of transposition. We have that
$$\Gamma^t = \Gamma$$
and
$$(\Gamma g \Gamma)^t = \Gamma g \Gamma$$
for $g \in \Delta$, as g is generated by diagonal matrices and elements of Γ, so the Hecke ring $\mathcal{R}_{\Gamma,\Delta}$ is commutative.
- Let
$$S_n = \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} | ad = n, 0 \leq b < d \right\}$$
then
$$\cup_{m_0^2 m_1 = n} \Gamma \begin{pmatrix} m_0 m_1 & 0 \\ 0 & m_0 \end{pmatrix} \Gamma = \cup_{\alpha \in S_n} \Gamma \alpha$$
is a disjoint decomposition.
- Proof idea: Basically follows from Hermite/Smith normal forms.
- Thus, we use the double coset disjoint union on the right to define the Hecke operator
$$T_n f(z) = \frac{1}{n} \sum_{\substack{ad=n \\ 0 \leq b < d}} f\left(\frac{az+b}{d}\right),$$
where $1/\sqrt{n}$ is a normalization factor to help with formulas later.
- The Hecke operators are self-adjoint wrt the Petersson inner product:
$$\langle T_n f, g \rangle = \langle f, T_n g \rangle .$$
Proof idea: Use that diagonal matrices are invariatn under transposition, and that
$$S\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1}\right)^T S^{-1} = \frac{1}{ad}\begin{pmatrix} a & b \\ c & d \end{pmatrix},$$
and acting on z this is the same as $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$.
- In particular, one can check that the Hecke operators, T_{-1}, and Δ all commute.
- Hence, we can simulatenously diagonalize with respect to all of the operators, giving Maass Hecke-eigenforms. These must be either even or odd.
- In particular, letting
$$f(z) = \sum_{n \neq 0} a_n \sqrt{2\pi y} K_{v-1/2}(2\pi |n| y) e^{2\pi i n x}$$
be the Fourier-Whittaker decomposition, we have that for a Maass eigenform of type v $a(1) = 0 \implies f = 0$. If f is nonzero and we normalize such that $a(1) = 1$, then we have the following properties:

  - $T_n f = a_n f$
  - $a_m a_n = a_{mn}$, $\gcd(m,n) = 1$
  - $a_m a_n = \sum_{d|(m,n)} a_{mn/d^2}$
  - $a_{p^{r+1}} = a_p a_{p^r} - a_{p^{r-1}}$

  for all primes p and $r \geq 1$.

- Proof idea: Direct computation using the definition of the Hecke operators.

3.13 L-functions associated to Maass forms

- Let f be a Maass Hecke eigenform of type v that is also an eigenfunction for T_{-1}. We have the L-function associated to f

$$L_f(s) = \sum_{n=1}^{\infty} a_n n^{-s}.$$

  Since we know that $a_n = O(\sqrt{n})$, this series is absolutely convergent for Re $(s) > 3/2$.

- Since the a_n are multiplicative, we have the Euler product

$$L_f(s) = \prod_p \left(\sum_{\ell=0}^{\infty} \frac{a_{p^\ell}}{p^{\ell s}} \right).$$

- Using the previous formulas for the a_{p^r} gives that

$$L_f(s) = \prod_p (1 - a_p p^{-s} + p^{-2s})^{-1} = \prod_p (1 - \alpha_p p^{-s})^{-1} (1 - \beta_p p^{-s})^{-1},$$

  where $\alpha_p \beta_p = 1$ and $\alpha_p + \beta_p = a_p$.

- We have the following holomorphic continuation and functional equation for $L_f(s)$: Let $\varepsilon = 0, 1$ be such that $T_{-1} f = (-1)^\varepsilon f$. Then the completed L-function is

$$\Lambda_f(s) = \pi^{-s} \Gamma\left(\frac{s + \varepsilon - 1/2 + v}{2} \right) \Gamma\left(\frac{s + \varepsilon + 1/2 - v}{2} \right) L_f(s),$$

  and we have the functional equation

$$\Lambda_f(s) = (-1)^\varepsilon \Lambda_f(1-s).$$

- Proof: Consider $x = 0$, and consider as function of y for $y > 0$. Take the Mellin transform of the function. The two gamma factors arise out of the Mellin transform of Bessel functions. If f is even, use that $a_n = a_{-n}$. If f is odd, instead take the Mellin transform of $\frac{\partial}{\partial x} f$.

3.14 L-functions associated to Eisenstein series

- Recall that for Re $(w) > 1$, we had the Eisenstein series

$$E(z,w) = \frac{1}{2} \sum_{\substack{c,d \in \mathbb{Z} \\ \gcd(c,d)=1}} \frac{y^w}{|cz+d|^{2w}}$$

  with Fourier-Whittaker expansion

$$E(z,w) = y^w + \phi(w) y^{1-w} + \frac{2^{1/2} \pi^{w-1/2}}{\Gamma(w)\zeta(2w)} \sum_{n \neq 0} \sigma_{1-2w}(n) |n|^{w-1} \sqrt{2\pi |n| y} K_{w-1/2}(2\pi |n| y) e^{2\pi i n x}.$$

  Note that the $\sigma_{1-2w}(n)|n|^{w-1/2}$ are analogous to the a_n for Maass forms.

- Hence we define the L-function associated to $E(z,w)$ by

$$L_{E(*,w)}(s) = \sum_{n=1}^{\infty} \sigma_{1-2w}(n) n^{w-1/2-s}.$$

- It turns out that

$$L_{E(*,w)}(s) = \zeta(s+w-1/2)\zeta(s-w+1/2),$$

so letting (completing in the natural way for each zeta)

$$\Lambda_{E(*,w)}(s) = \pi^{-s}\Gamma\left(\frac{s+w-1/2}{2}\right)\Gamma\left(\frac{s-w+1/2}{2}\right)L_{E(*,w)}(s),$$

we get the functional equation

$$\Lambda_{E(*,w)}(s) = \Lambda_{E(*,w)}(1-s),$$

which exactly matches the functional equation for an even Maass form of type w.

- Moreover, the Eisenstein series is an eigenfunction of all the Hecke operators, giving an explanation for the Euler product.

- Idea of proof: The S_n defined previously ($\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$) act as coset representatives of $\Gamma_1 \backslash \Gamma_n$. The Eisenstein series are summed over $\Gamma_\infty \backslash \Gamma_1$. Swap the sums and swap the order of coset representatives and the correct value for $T_n E(z,s)$ falls out.

3.15 Converse theorems for $\mathbf{SL}(2,\mathbb{Z})$

- Just like for holomorphic modular forms, satsifying functional equation + sufficient boundedness conditions gives modularity.

- Hecke-Maass converse theorem: Let $L(s) = \sum a_n n^{-s}$ be an L-function that converges absolutely for Re (s) sufficiently large, and suppose that the completed L-function

$$\Lambda^v(s) = \pi^{-s}\Gamma\left(\frac{s+\varepsilon-1/2+v}{2}\right)\Gamma\left(\frac{s+\varepsilon+1/2-v}{2}\right)L(s)$$

satsifies the functional equation

$$\Lambda^v(s) = (-1)^\varepsilon \Lambda^v(1-s),$$

where $\varepsilon = 0, 1$, and $\Lambda^v(s)$ is entire and bounded on vertical strips. Then

$$f = \sum_{n \neq 0} a_n \sqrt{2\pi y} K_{v-1/2}(2\pi |n| y) e^{2\pi i n x}$$

is an even/odd Maass form, where $a_{-n} = (-1)^\varepsilon a_n$.

- Idea of proof: Only need to check modularity. You get T for free, so only need to check S. Can show that it suffices to check $x = 0$, y if you can show that $f(iy) - f(i/y)$ satisfies some initial conditions involving differentials: $F(iy) = 0$, $\frac{\partial F}{\partial x}|_{x=0} = 0$ implies F is 0. (This is the replacement for analytic continuation in the holomorphic case.) This follows out of since Λ is the Mellin transform of f, expressing f as the Mellin inverse of f, then applying the functional equation + bounded on vertical strips + f rapidly decaying toward infinity to get the final answer.

- Caveat: No such L-function has been found for Maass forms on $\mathrm{SL}(2,\mathbb{Z})$. (Related to that there are no known constructions of $\mathrm{SL}(2,\mathbb{Z})$-Maass forms) Closest is the Hecke L-function - these turn out to be the functional equation of a Maass form of a congruence subgroup.

3.16 The Selberg spectral decomposition

- It turns out that we can decompose any $\mathcal{L}^2(\mathrm{SL}(2,\mathbb{Z})\backslash\mathfrak{h}^2)$ function into

$$\mathbb{C} \oplus \mathcal{L}^2_{cusp}(\mathrm{SL}(2,\mathbb{Z})\backslash\mathfrak{h}^2) \oplus \mathcal{L}^2_{cont}(\mathrm{SL}(2,\mathbb{Z})\backslash\mathfrak{h}^2).$$

  Here cusp refers to integrals at cusps is 0, and will be an integral of an Eisenstein series.

- We have $\eta_j(z)$, for $j \geq 1$, be an orthonormal basis of Maass forms that are all Hecke eigencuspforms. Moreover, let

$$\eta_0(z) = \sqrt{3/\pi}.$$

- We get the Selberg spectral decomposition

$$f(z) = \sum_{j=0}^{\infty} \langle f, \eta_j \rangle \, \eta_j(z) + \frac{1}{4\pi i} \int_{1/2-i\infty}^{1/2+i\infty} \langle f, E(*, s) \rangle \, E(z, s) \, \mathrm{d}s$$

  where the inner product is the Petersson inner product.

- Why a countable basis of Maass forms? It turns out that the Laplacian on cuspforms is a compact operator, so from spectral theory we get that the spectrum is countable. See Iwaniec-Kowalski 15.2.

- Can show that if f is of rapid decay such that

$$\langle f, E(*, \overline{s}) \rangle$$

  converges absolutely, and f is orthogonal to 1, then f decomposes into a cusp form plus the correct integral by showing that $\langle f, E(*, \overline{s}) \rangle$ is the Mellin transform of the constant term of f, and that the constant term of the integral is the inverse Mellin transform of $\langle f, E(*, \overline{s}) \rangle$.

- Spectral theory of automorphic forms important - will lead to Selberg trace formula, etc.