L-functions and motivic superpolynomials of plane curve singularities

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ABSTRACT: We will begin with a mini-review of the 4 main classical theories of zeta-functions and the passage to isolated singularities. Next, the motivic superpolynomials of plane curve singularities will be defined, conjecturally coinciding with the DAHA and Khovanov-Rozansky ones for algebraic knots. This is directly related to compactified Jacobians of plane curve singularities. Finally, the functional equation and Riemann hypothesis for motivic superpolynomials will be discussed, and some physics connections (LGSM, SCFT). See my "*2D Punctual Hilbert schemes*", JoA (2022), etc.

Igor contributed a lot to "Jacobians". So what happens if we switch to singularities, go to finite fields and consider their zeta-functions?

ZETA FUNCTIONS

Weil's conjectures
for smooth mnflds

CLASSICAL THEORIES

Riemann's zeta,
Dirichlet L-fncts

No connection
between these
theories

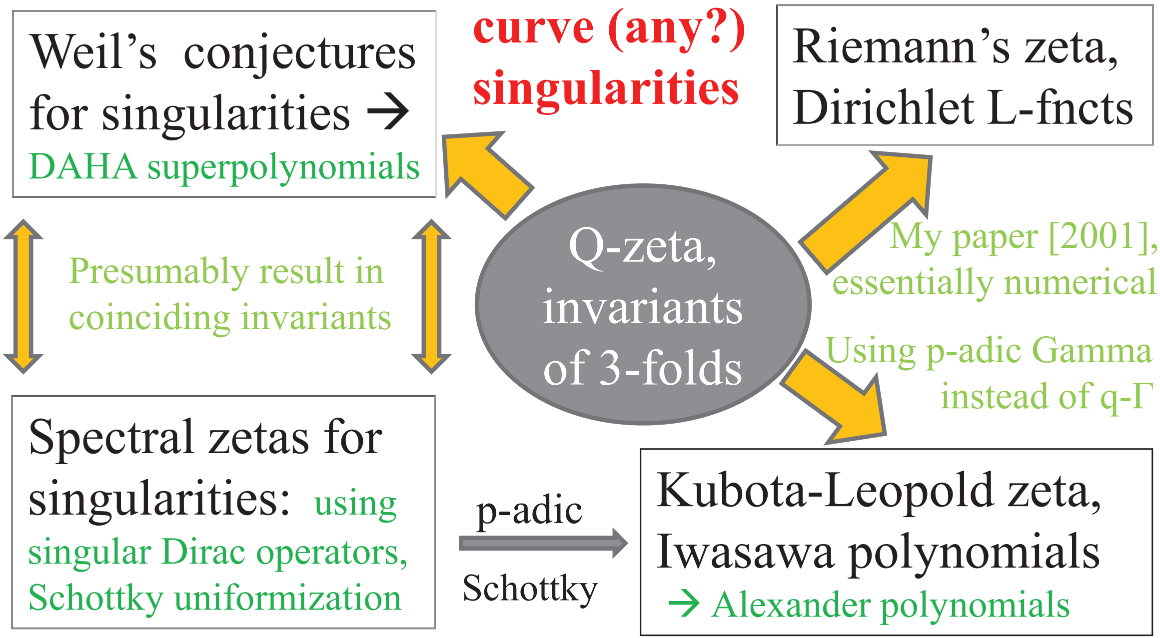
Spectral zetas,
Selberg's zeta

Kubota-Leopold ζ
Iwasawa polynms

For instance, Dirichlet L-functions have no counterparts among Weil's L-functions (and they have no ***q***) : two different universes. Also, ***zeta-equivalence*** of algebraic varieties over ***C*** (N. Katz) generally results only in the coincidence of their Hodge numbers.

FOCUS ON SINGULARITIES

These theories become connected for curve (any?) singularities



If they capture the topological (!) invariants of links or 3-folds, then 3 theories ( ) are ***a priori*** equivalent!

The key is to try to go to theory 4 via presumably 3-folds.

ZETAS AS TOPOLOGICAL INVARIANTS

First, we check that an isolated hypersurface singularity $0 \in \mathcal{X} \subset \mathbb{C}^n$ within its topological type, (say, the isotopy class of $\mathcal{X} \cap \mathbb{S}^{2n-1}$) can be defined over $\mathbb{Z}$ and generic (any?) $\mathbb{F}_q$, i.e. has good reductions at general prime p. Second, the compactified Jacobian $\bar{J}(\mathcal{X})$ of $\mathcal{X}$ (more generally, $Bun_G(\mathcal{X})$) is assumed of *strong polynomial count*: $|\bar{J}(\mathcal{X})(\mathbb{F}_q)|$ depend polynomially on q. Then the *flagged* $\zeta_{\mathcal{X}}(q,t,a)$ is a powerful topological invariant of $\mathcal{X}$. For instance, $\zeta_{\mathcal{C}}(q,t,a)$ for a plane curve singularity $\mathcal{C}$, readily provides the valuation semi-ring of $\mathcal{C}$, which determines the topological type of unibranch $\mathcal{C}$. Similarly, the singular and p-adic zetas are expected to capture the topological type of $\mathcal{C}$ too, so we expect some a priori equivalence of these 3 theories. The passage to the "Grand" ζ, L-functions is (hopefully) via toric-type surface singularities $\mathcal{X}$ (and Seifert 3-folds).

HILBERT SCHEMES AND KhR POLYNOMIALS

$\mathcal{C}=$ unibranch plane curve singularity; $\delta=$arithmetic genus.

For rational $C\subset\mathbb{C}P^2$ (Gopakumar-Vafa, Pandharipande-Thomas): $\sum_{n\geq 0} q^{n+1-\delta} e(C^{[n]}) = \sum_{0\leq i\leq\delta} n_C(i)\left(\frac{q}{(1-q)^2}\right)^{i+1-\delta}$, **for Euler numbers of** *Hilbert schemes* $C^{[n]}$; $n_C(i)\in\mathbb{Z}_+$ **(Göttsche, ..., Shende $\forall i$).**

ORS-Conjecture: **For** NESTED $\mathcal{C}^{[l\leq l+m]}$ **Hilbert schemes (pairs of ideals) and** $t \leftrightsquigarrow \mathfrak{w}=$*weight filtration* **(Serre, Deligne),**

$$\sum_{l,m\geq 0} q^{2l}\, a^{2m}\, t^{m^2}\, \mathfrak{w}(\mathcal{C}^{[l\leq l+m]}) \sim KhR^{\text{stab}}(Link(\mathcal{C})).$$

It adds t to Oblomkov-Shende conjecture, proved by Maulik.

ChD-Conjecture: **For any $\mathcal{C}$,** $\mathcal{H}_{DAHA}(\Box; a,q,t)=KhR^{\text{stab}}_{red}$, $\mathcal{H}_{DAHA}(\Box; q,t=1,a=0)=\sum_{i=0}^{\delta} q^i b_{2i}(\overline{J}(\mathcal{C}))$ **(incl.** $b_{2i+1}=0$**).** $\overline{J}(\mathcal{C})$ = **compactified Jacobian from Fundamental Lemma.**

MOTIVIC SUPERPOLYNOMIALS

Let $\mathcal{R} \subset \mathbb{C}[[z]]$ be the ring of a unibranch plane curve singularity. Flagged compactified Jacobian $\mathcal{F}$ is formed by *standard* **flags of $\mathcal{R}$-modules $M_0 \subset M_1 \subset \cdots \subset M_\ell \subset \mathcal{O} = \mathbb{C}[[z]]$ such that $(a)\, M_i \ni 1 + z(\cdot)$, $(b)\, \dim M_i/M_{i-1} = 1$ and $M_i = M_{i-1} \oplus \mathbb{C}\, z^{g_i}(1 + z(\cdot))$, (c) $(important)\, g_i < g_{i+1}$, $i \geq 1$.**

Conjecture (Ch, Philipp). Within its topological type, the singularity can be assumed over any $\mathbb{F} = \mathbb{F}_q$. Then $\mathcal{H}^{\square}_{DAHA} = \mathcal{H}^{mot}_{\mathcal{C}} \stackrel{\text{def}}{=\!=} \sum_{\{M_0 \subset \cdots \subset M_\ell\} \in \mathcal{F}(\mathbb{F})} t^{\dim(\mathcal{O}/M_\ell)} a^\ell$ for the DAHA superpolynomial corresponding to $\mathcal{C}$; $\mathcal{H}^{mot}_{\mathcal{C}}$ generalize p-adic orbital integrals (type A, nil-elliptic), which are for $t=1, a=0$.

This is checked very well, incl. many cases with "cells" in $\mathcal{F} = \cup_{\vec{\Delta}} \mathcal{F}_{\vec{\Delta}}$ that are *not* **$\mathbb{A}^N$; here $\vec{\Delta} \stackrel{\text{def}}{=\!=} \{\Delta_0 \subset \cdots \subset \Delta_\ell\}$ for modules $\Delta_i \stackrel{\text{def}}{=\!=}$ valuation$_z(M_i)$ over the** *semigroup* **$\Gamma \stackrel{\text{def}}{=\!=}$ valuation$_z(\mathcal{R})$.**

EXAMPLES OF PLAIN SINGULARITIES

The simplest one is for "trefoil" $T(3,2)$. The corresponding ring of singularity $\mathcal{R}=\mathbb{C}[\![z^2,z^3]\!]$ has the valuation semigroup $\Gamma=\mathbb{Z}_+\setminus\{1\}$. The latter remains unchanged over any(!) $\mathbb{F}_q$. The modules are $M_\lambda=(1+\lambda z)$ (called invertibles) of dim $\mathcal{O}/M=1$, and $M=\mathcal{O}$ (2 generators; dim$=0$). The standard 1-flags are $\{M_\lambda\subset\mathcal{O}\}$ (of dim 0). Thus $\mathcal{H}^{mot}=1$ (for $\mathcal{O}$) $+\,qt$ (invertibles) $+\,aq$ (for 1-flags).

The simplest non-torus one is the ring $\mathcal{R}=\mathbb{C}[\![z^4,z^6+z^7]\!]$, where $\Gamma=\mathbb{Z}_+\backslash$Gaps for Gaps$=\{1,2,3,5,7,9,11,15\}$, so $\delta=|$Gaps$|=8$. Here $(z^6+z^7)^2-(z^4)^3=2z^{13}+\ldots$ and $p=2$ is a place of bad reduction. This singularity can be also presented by $\mathcal{R}'=\mathbb{C}[\![z^4+z^5,z^6]\!]$, where the reduction is bad only at $p=3$. Thus it has no places of bad reduction; the same holds for any algebraic knots (higher dimensions?). Importantly, in contrast to torus knots, only 23 out of the 25 Γ-modules Δ come from some standard M (*the Piontkowski phenomenon*).

RIEMANN HYPOTHESIS

We substitute $q \to qt$: $\mathbf{H}(q,t,a) \stackrel{\textbf{def}}{=\!=} \mathcal{H}^{mot}_{\mathcal{C}}(qt,t,a)$, $\mathbf{H}(q,t,a) = \sum_{i=0}^{deg_a} \mathbf{H}_i(q,t)a^i$. Then $\mathbf{H}(q\to q, t\to 1/(qt), a) = q^{\cdot} t^{\cdot}\, \mathbf{H}(q,t,a)$ (superduality, proved for DAHA), and therefore if $t=\xi$ is a zero of $\mathbf{H}_i$, then so are $1/(q\xi)$ and (obviously) $\overline{\xi}$.

RH: For uncolored algebraic knots, all t-zeros ξ of $\mathbf{H}_i(q,t)$ satisfy $|\xi| = \sqrt{1/q}$ for $0 \le q \le \kappa$; $\boldsymbol{\kappa = 1/2}$ is sufficient for $\boldsymbol{i = 0}$. For algebraic *links* with p components, p–1 *non-RH* pairs are conjectured for $\mathbf{H}_{i=0}$; but $\mathbf{H}_{i=2}$ has 3 such pairs for $\{(y^3-x^2)(x^3-y^2)=0\}$. For torus knots, all $\mathbf{H}_i(q=1,t)$ are products of cyclotomic polynomials.

The connection goes through Galkin's ζ, Kapranov's motivic ζ, Göttsche-Shende conjectures, the ORS conjecture, the Cherednik-Philipp conjecture (related to Gorsky's combinatorial approach).

FLAGGED GALKIN-STÖHR ZETA

A flag of $\mathcal{R}$-*ideals* $\overrightarrow{M} = \{M_0 \subset M_1 \cdots \subset M_\ell\}$ **is called** *standardizable* **if** $\{z^{-m} M_i\}$ **becomes standard (as above) for** $m = min(\textbf{valuation}_z(M_\ell))$**. We set:** $\mathcal{Z}(q,t,a) \stackrel{\textbf{def}}{=\!=} \sum_{\vec{M}\subset\mathcal{R}} a^\ell t^{dim_{\mathbb{F}}(\mathcal{R}/M_\ell)}$, $\mathcal{L}(q,t,a) \stackrel{\textbf{def}}{=\!=} (1-t)\mathcal{Z}(q,t,a)$, **where the summation is over** *standardizable* **flags of ideals** $\overrightarrow{M}$**.**

Conjecture. $\mathbf{H}(\boldsymbol{q},\boldsymbol{t},\boldsymbol{a}) = \boldsymbol{\mathcal{L}}(\boldsymbol{q},\boldsymbol{t},\boldsymbol{a})$**, and** $\mathbf{H}(q,t,a = -\frac{1}{q}) = \mathcal{L}_{\textbf{prncpl}}(q,t,a=0)$ **(summation over principle** $\mathcal{M} \subset \mathcal{R}$**), where the latter "=" possibly holds for any Gorenstein** $\mathcal{R}$**.**

Conjecture is checked for many knots, including involved cases when $\mathcal{R} = \mathbb{F}[[z^6, z^8 + z^9]]$ **for** $\ell = 0, 1$ **and** $\mathcal{R} = \mathbb{F}[[z^6, z^9 + z^{10}]]$ **for** $\ell = 0$**.**

WHAT IS "MOTIVIC LGSM"?

Vafa-Warner's paper "Catastrophes..." (1989) promoted study of Landau-Ginzburg Sigma Models as directly as possible in terms of superpotentials $W(x,y)$ considered as equations of isolated singularities. They can be $u^2+W(x,y)$ or $uv+W(x,y)$; called local Gorenstein K_3, CY. E.g. for $W=x^r-y^s$ under gcd$(r,s)=1$, the Milnor number $\mu=(r-1)(s-1)$ is the number of chiral operators (coinciding here with the Witten index), $c_{eff}=6\big((\frac{1}{2}-\frac{1}{r})+(\frac{1}{2}-\frac{1}{s})\big)$ and so on: $\int\prod(\lambda^{1/2}dx_i)\exp[i\lambda W(x_i)]\sim O(\lambda^{c/6})$ as $\lambda\gg 0$.

DAHA superpolynomials are certain partition functions by construction; so $\mathcal{H}_{DAHA}=\mathcal{H}^{mot}$ implies that $\mathcal{H}^{mot}$ can be interpreted as partition functions in $LGSM$. In particular, physics S-duality in M-theory and the DAHA superduality become the functional equation for $W(x,y)=0$, where $t=q^{-s}$, $q=|\mathbb{F}|$.

Happy Birthday Igor!