The Stieltjes–Fekete problem and degenerate orthogonal polynomials

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Algebraic Geometry, Mathematical Physics, and Solitons
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- The Stieltjes–Fekete problem and classical orthogonal polynomials
- Solution to the Stieltjes–Fekete problem and degenerate orthogonal polynomials
- Exactly solvable quartic anharmonic oscillator, Shapiro-Tater conjecture and Painlevé II equation

Based on "Exactly solvable anharmonic oscillator, degenerate orthogonal polynomials and Painlevé II", arXiv:2203.16889
and "The Stieltjes–Fekete problem and degenerate orthogonal polynomials", http://arxiv.org/abs/2206.06861 **Joint work with Marco Bertola and Eduardo Chavez-Heredia**

Stiltjes-Fekete problem

Find the configuration of points $(z_1, \dots, z_n) \in \mathbb{C}^n$ that we call weighted Fekete points, that provides the maximum of the weighted Fekete functional

$$\mathcal{F}(z_1, \dots, z_n) = \prod_{j=1}^{n} \prod_{\substack{k=1 \\ k \neq j}}^{n} \left| z_j - z_k \right| \mathrm{e}^{\frac{Q(z_j)+Q(z_k)}{2(n-1)}},$$

where $Q(z, \overline{z})$, $z \in \mathbb{C}$, is a real–valued *external potential*. Equivalently, the weighted Fekete points provide the minimum of the energy functional

$$\mathcal{E}(z_1, \dots, z_n) = -2 \sum_{1 \leqslant i < j \leqslant n}^{n} \log |z_j - z_k| - \sum_{j=1}^{n} Q(z_j).$$

Depending on the setup, one may require that the points belong to some assigned domain $\mathcal{D}$.

The critical points of the energy functional satisfy the equation

$$2 \sum_{k \neq j} \frac{1}{z_j - z_k} = -\partial Q(z_j), \quad j = 1, \dots, n,$$

where $\partial = \frac{1}{2}\left(\frac{\partial}{\partial x} - i \frac{\partial}{\partial y}\right)$, $z = x + iy$.

Classical cases

- $\mathcal{D} = \mathbb{R}$ and $Q(x) = -x^2$. The Fekete points are the zeros of the Hermite polynomials and are the global minimum of the energy

$$\mathcal{E}(x_1, \dots, x_n) = -2 \sum_{1 \leqslant i < j \leqslant n}^{n} \log |x_j - x_k| + \sum_{j=1}^{n} x_j^2.$$

The variational equations yield

$$\sum_{\substack{k=1 \\ k \neq j}}^{n} \frac{1}{x_j - x_k} = x_j, \quad j = 1, \dots, n.$$

- The domain $\mathcal{D} = (-1, 1)$ and the weight $Q(x) = (\alpha + 1)\log(1 - x) + (\beta + 1)\log(1 + x)$, $\alpha, \beta > -1$. The Fekete points are the zeros of the Jacobi polynomials and are the global minimum of the energy

$$\mathcal{E}(x_1, \dots, x_n) = -2 \sum_{1 \leqslant i < j \leqslant n}^{n} \log |x_j - x_k| - \sum_{j=1}^{n} \log(1 - x_j)^{\alpha+1} - \sum_{j=1}^{n} \log(1 + x_j)^{\beta+1}.$$

The variational equations yield

$$\sum_{\substack{k=1 \\ k \neq j}}^{n} \frac{1}{x_j - x_k} = \frac{\alpha + 1}{2(1 - x_j)} - \frac{\beta + 1}{2(1 + x_j)}, \quad j = 1, \dots, n.$$

- The domain $\mathcal{D} = (0, \infty)$ and the weight $Q(x) = -x + (\alpha + 1)\log x$, $\alpha > -1$. The Fekete points are the zeros of the Laguerre polynomials and are the global minimum of the energy

$$\mathcal{E}(x_1, \ldots, x_n) = -2 \sum_{1 \leqslant i < j \leqslant n}^{n} \log |x_j - x_k| + \sum_{j=1}^{n} [x_j - (\alpha + 1)\log x_j].$$

The variational equations yield

$$\sum_{\substack{k=1 \\ k \neq j}}^{n} \frac{1}{x_j - x_k} = \frac{1}{2} - \frac{\alpha + 1}{2x_j}, \quad j = 1, \ldots, n.$$

Remark. For the three above particular choices of $Q(x)$ the weighted Fekete points are the zeros of (classical) orthogonal polynomials and give the global minimum to the energy $\mathcal{E}(x_1, \ldots, x_n)$.

- Why the global minimum of the energy should be considered? Which other types of equilibria described above could be linked to the zeros of polynomials?
- What is the appropriate model for the complex zeros (when they exist)? Marcellán–Martínez-Finkelshtein–Martínez-González, J. Comp. Appl. Math (2007)

We consider a *holomorphic* version of the condition of criticality in the form

$$\sum_{k\neq j}\frac{1}{z_j-z_k}=\frac{A(z_j)}{2B(z_j)},\quad j=1,\dots,n, \tag{1}$$

where A, B are two relatively prime polynomials. The solutions of the above equation turn out to be the critical points for the energy

$$\mathscr{E}(z_1,\dots,z_n)=-2\sum_{1\leqslant i<j\leqslant n}^{n}\log|z_j-z_k|-\sum_{j=1}^{n}Q(z_j)$$

where $Q(z)=-\Re(\widehat{\theta}(z))$ with $\widehat{\theta}$ real–analytic, except for finite number of singularities and branch cuts, and $\widehat{\theta}(z)=\int^z\frac{A(z')}{B(z')}\,\mathrm{d}z'$. Equations (1) are also sometimes referred to as *Stieltjes–Bethe* equations because of their appearance in the Bethe-Ansatz for spin-chains. When

$$\frac{A(z)}{B(z)}=-\sum_{\ell=0}^{p}\frac{\nu_\ell}{z-a_\ell}$$

with $a_j\in\mathbb{C}$ all distinct and ν_j real, one obtains the Heine-Stieltjes electrostatic problem

$$\mathscr{E}(z_1,\dots,z_n)=-2\sum_{1\leqslant i<j\leqslant n}^{n}\log|z_j-z_k|-\sum_{j=1}^{n}\sum_{\ell=0}^{p}\nu_\ell\log|a_\ell-z_j|.$$

that was studied by T.J. Stiltjes (1885), H.E.Heine (1878), E.B.Van Vleck (1898), F.Klein (1894), G. Polya (1912), D. Dimitrov and W. Van Assche (2000), D.Dimitrov-B.Shapiro (2018), A.Varchenko (1995).

First main result

Given two relatively prime polynomials $A(z)$ and $B(z)$ with $\deg A \geqslant \deg B$, there is a one-to-one correspondence between the solutions of the Stiltjes-Bethe equations

$$\sum_{k \neq j} \frac{1}{z_j - z_k} = \frac{A(z_j)}{2B(z_j)}, \quad j = 1, \ldots, n,$$

and the zeros of *maximally degenerate* orthogonal polynomials of degree n for a *semiclassical moment functional* of type (A, B).

- Bethe ansatz equations is an ansatz method for finding the exact spectrum of integrable quantum many-body models in the form $E = \sum_i e(u_i)$ where u_i satisfy a system of algebraic or transcendental equations known as Bethe equations.
- Bethe ansatz equations comes out naturally when solving the following problem: "*when a linear equation with rational (trigonometric, elliptic) coefficients has rational (trigonometric elliptic) solution?*" For example

$$\psi_{n+1}(x) = \psi_n(x+1) - v_n \psi_n(x), \quad n \in \mathbb{Z}$$
$$v_n(x) = \frac{y_n(x) y_{n+1}(x+1)}{y_n(x+1) y_{n+1}(x)}$$

  where $y_n(x) = \prod_{i=1}^{k_n}(x - u_i^{(n)})$ has rational solution $\psi_n(x)$ with poles at the zeros of $y_n(x)$ if and only if

$$\frac{y_{n-1}(u_j^{(n)}+1) y_n(u_j^{(n)}-1) y_{n+1}(u_j^{(n)})}{y_{n-1}(u_j^{(n)}) y_n(u_j^{(n)}+1) y_{n+1}(u_j^{(n)}-1)} = -1$$

  that are the Bethe ansatz equation for the sl_N XXX quantum integrable model (Krichever, Lipan, Wigner Zabrodin 2016, Krichever-Varchenko 2021). Connection with discrete mKdV and Ruijsenaars-Schneider systems is obtained.)
- Connection of solutions of ODEs with rational (trigonometric, elliptic) coefficients and the theory of Calogero-Moser and Ruijsenaars-Schneider systems was pioneered by Krichever.

First main result

Given two relatively prime polynomials $A(z)$ and $B(z)$ with $\deg A \geqslant \deg B$, there is a one-to-one correspondence between the solutions of the Stiltjes-Bethe equations

$$\sum_{k \neq j} \frac{1}{z_j - z_k} = \frac{A(z_j)}{2B(z_j)}, \quad j = 1, \ldots, n,$$

and the zeros of *maximally degenerate* orthogonal polynomials of degree n for a *semiclassical moment functional* of type (A, B).

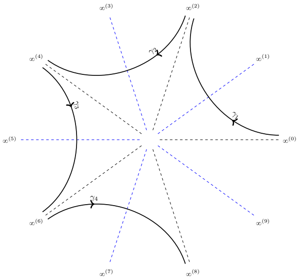
- A moment functional is a linear map on the space of polynomials $\mathcal{M} : \mathbb{C}[z] \to \mathbb{C}$.
- A moment functional $\mathcal{M} : \mathbb{C}[z] \to \mathbb{C}$ is *semiclassical of type* (A, B) where $A = A(z), B = B(z)$ are two relatively prime polynomials of degree a and b respectively if

$$\mathcal{M}[B(z)p'(z)] = \mathcal{M}[A(z)p(z)], \qquad \forall p(z) \in \mathbb{C}[z].$$

Studied by Maroni ['87], Ismail-Masson-Rahman ['91], Marcellán-Rocha ['98]: any such moment functional can be represented as:

$$\mathcal{M}[p] = \sum_{\ell=1}^{d} s_\ell \int_{\gamma_\ell} p(z) \mathrm{e}^{\theta(z)} \, \mathrm{d}z \ , \qquad \theta'(z) = -\frac{A(z) + B'(z)}{B(z)}$$

with $d = \max\{a, b-1\}$, s_ℓ are arbitrary complex parameters and γ_ℓ are contours that approach the poles of $\theta'(z)$ in $\overline{\mathbb{C}}$ along steepest descent directions ($\Re\theta \to -\infty$).



$\theta(z) = -z^5$, namely $B(z) = 1$, $A(z) = 5z^4$ and $d = 4$

$\infty^{(j)}$ are the asymptotic directions $\arg(z) = \frac{j\pi}{5}$ for $j = 0, \ldots, 9$

Contours

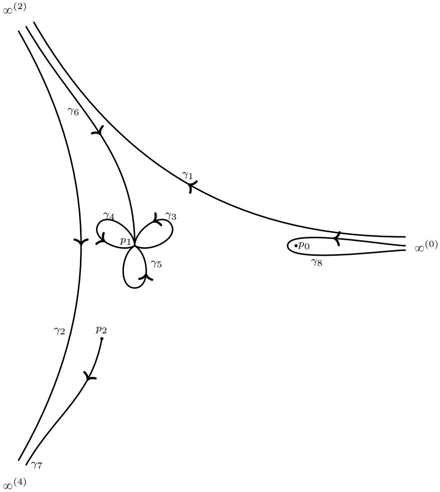


Figure: The contours in the case $B(z) = (z - p_0)(z - p_1)^4(z - p_2)$ and $A(z) = z^8$. $\theta'(z) = -\frac{A(z)+B'(z)}{B(z)}$, and $\text{Res}_{z=p_2}\theta'(z) > -1$ and $\text{Res}_{z=p_0}\theta'(z) < -1$,

Orthogonal polynomais

The polynomials $p_n(z) = z^n + \dots$, that are orthogonal with respect to the semiclassical moment functional $\mathcal{M}$ of type (A, B), are determined by the condition

$$\mathcal{M}\Big[p_n(z)z^k\Big] = \sum_{j=1}^{d} s_j \int_{\gamma_j} p_n(z)z^k \mathrm{e}^{\theta}\,\mathrm{d}z = 0, \qquad k = 0, \dots, n-1.$$

Let $\mu_k(\mathbf{s}) = \mathcal{M}[z^k]$, $\mathbf{s} = (s_1, \dots, s_d)$,

$$D_n(\mathbf{s}) := \det\Big[\mu_{j+k}(\mathbf{s})\Big]_{j,k=0}^{n-1}$$

Notice that $D_n(\mathbf{s})$ is a homogeneous polynomial of degree n in the paramaters $s_1, \dots, s_d$.
The monic orthogonal polynomial $p_n(z)$ of degree n is then given by

$$p_n(z) = \frac{1}{D_n(\mathbf{s})} \det \begin{bmatrix} \mu_0 & \mu_1 & \cdots & \mu_n \\ \mu_1 & \mu_2 & \cdots & \mu_{n+1} \\ \vdots & & & \\ \mu_{n-1} & \cdots & & \mu_{2n-1} \\ 1 & z & \dots & z^n \end{bmatrix}.$$

Degenerate orthogonality

Definition

The polynomial p_n is called ℓ–**degenerate orthogonal** if, in addition

$$\mathcal{M}\Big[p_n(z)z^{n+k}\Big] = 0, \quad k = 0, 1, \ldots, \ell - 1, \ \ell \geqslant 1.$$

Lemma

The orthogonal polynomial p_n is ℓ–degenerate if and only if

$$D_{n+1,k}(\mathbf{s}) := \det H_{n+1,k} = 0, \qquad k = 0, 1, \ldots, \ell - 1,$$

$$H_{n+1,k} := \left[\begin{array}{cccc} \mu_0 & \mu_1 & \cdots & \mu_n \\ \mu_1 & \mu_2 & \cdots & \mu_{n+1} \\ \vdots & & & \\ \mu_{n-1} & \cdots & & \mu_{2n-1} \\ \hline \mu_{n+k} & \mu_{n+k+1} & \cdots & \mu_{2n+k} \end{array}\right].$$

$D_{n+1,k}(\mathbf{s})$ is a homogeneous polynomials of degree $n+1$ is the parameters $s_1, \ldots s_d$.
Maximal degeneracy: $D_{n+1,k}(\mathbf{s}) = 0$ for $k = 0, \ldots d-2$ gives $d-1$ **homogeneous polynomial relations** on the "weights" $s_1, \ldots, s_d$,

Theorem

Let $\mathscr{Z} = \{z_1, \dots, z_n\}$ *be a critical configuration satisfying the Stilties-Bethe equations*

$$\sum_{k \neq j} \frac{1}{z_j - z_k} = \frac{A(z_j)}{2B(z_j)}, \quad j = 1, \dots, n,$$

where $A(z), B(z)$ *are relatively prime arbitrary polynomials with* $\deg A \geqslant \deg B$. *Then*

(1) *the polynomial* $p_n(z) = \prod_{j=1}^{n}(z - z_j)$ *is a maximally degenerate orthogonal polynomial for a semiclassical moment functional* $\mathcal{M}$ *of type* (A, B).

(2) *The polynomial* $p_n(z)$ *satisfies the ODE*

$$B(z)y''(z) - A(z)y'(z) + Q(z)y(z) = 0$$

where the polynomial $Q = B(\frac{p_n''}{p_n} + \theta' \frac{p_n'}{p_n}) + B' \frac{p_n'}{p_n}$ *has degree at most* $d - 1$ *with* $d = \max\{\deg A,\ \deg B - 1\}$.

Viceversa, if p_n *is a semiclassical, maximally degenerate orthogonal polynomial of degree* n *for a semiclassical moment functional* $\mathcal{M}$ *of type* (A, B) *then its zeroes satisfy the Stilties-Bethe equations and the ODE as above.*

Example

$\theta(z) = -z^4/2$, and look for the critical points of the energy

$$\mathcal{E}(z_1, \dots, z_n) = -2 \sum_{1 \leqslant i < j \leqslant n}^{n} \log|z_j - z_k| + \sum_{j=1}^{n} \frac{z_j^4}{2}.$$

on the **real axis.** Our theorem says that the corresponding polynomial $p_n(z) = \prod_{j=1}^{n}(z - z_j)$, with z_j solution of the Stieltjes–Bethe equation, is indeed an orthogonal polynomial, *not* for the orthogonality on the real axis but

$$\mathcal{M}[z^k] = \int_{\mathbb{R}} z^k \mathrm{e}^{-\frac{z^4}{2}}\,\mathrm{d}z + s \int_{i\mathbb{R}} z^k \mathrm{e}^{-\frac{z^4}{2}}\,\mathrm{d}z, \qquad s \simeq \begin{cases} 0.00001349595\, i & n = 10 \\ -3.79352745\, 10^{-6}\, i & n = 11. \end{cases}$$

Note that $p_n(z)$ is a 2-degenerate orthogonal, polynomial, namely

$$\mathcal{M}[z^k p_n(z)] = 0, \quad k = 0, \dots, n, n+1$$

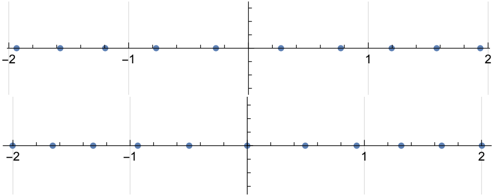


Figure: The numerically computed Fekete points with $n = 10, 11$ on the real axis for the Freud weight $\mathrm{e}^{-Z^4/2}$.

- Technical issue: count the number of solution of the Stieltjes–Bethe equations. This corresponds to count the number of solutions of

$$D_{n+1,k}(\mathbf{s}) = 0, \quad k = 0, \dots d-2$$

with $D_{n+1,k}(\mathbf{s})$ homogeneous polynomials of degree $n+1$ in $s_1, \dots, s_d$. However there are some degeneracies, for example the equations

$$D_{n+1,0}(\mathbf{s}) = 0 = D_{n,0}(\mathbf{s}),$$

implies that $p_n(z) \equiv 0$.

- Extend the analysis to the case $\deg A < \deg B$. In this case the contours of the semiclassical moment functional are Pochhammer contours.
- The function $F(z) = \sqrt{B(z)} P_n(z) \mathrm{e}^{\frac{1}{2}\theta(z)}$ solves the differential equation

$$F''(z) - W(z)F(z) = 0$$

where the potential $W(z)$ is a rational function with poles only at the zeros of $B(z)$ at most of twice the order. In the case $B = 1$, $A(z) = z^2 - t$, the potential $W(z)$ is a quartic polynomial. The condition that $P_n(z)$ is a degenerate orthogonal polynomial is equivalent to the condition that the spectrum of the quartic anharmonic oscillator is *exactly solvable.*

Exactly solvable anharmonic oscillator degenerate orthogonal polynomials and Painlevé II

Quartic Anharmonic Oscillators

By this we mean the spectrum of a Sturm–Liouville problem :

$$L_J(y(z)) := y''(z) - \left(z^4 + tz^2 + 2Jz\right) y(z) = \Lambda y(z) \tag{2}$$

$$y(z) \to 0 \text{ as } x \to \infty \text{ and } \arg(z) = \pm\pi/3, \tag{3}$$

Quasi–Exactly–Solvable spectrum

Bender–Boettcher ['98] showed that part of the spectrum (the "Exactly–Solvable") is explicit for $J = n + 1 \in \mathbb{N}$. The eigenfunctions are quasi-polynomials

$$y(z) = p_n(z) e^{\theta(z;t)} \quad \text{where } \theta(z;t) = \frac{z^3}{3} + \frac{tz}{2} \tag{4}$$

L_{n+1} maps the space of quasi-polynomials $\{p(z)e^{\theta}(z,t), \deg p \leqslant n\}$ to itself.

Eigenvalues

For $J = n + 1$ the rescaled eigenvalues $\lambda = \Lambda - \frac{t^2}{4}$ are obtained from the eigenvalue of the operator

$$\widehat{L_J} := \frac{\mathrm{d}^2}{\mathrm{d}z^2} + 2\left(z^2 + \frac{t}{2}\right)\frac{\mathrm{d}}{\mathrm{d}z} - 2(J-1)z \tag{5}$$

acting on the space of polynomials of degree up to n. The spectrum $\lambda = \lambda(t)$ is determined by $\det(\lambda\mathbf{1} - M_n(t)) = 0$ with $M_n(t)$ a $(n+1) \times (n+1)$ matrix

$$M_n(t) := \begin{bmatrix} 0 & -2n & & & & & \\ t & 0 & -2(n-1) & & & & \\ 2 & 2t & 0 & & & & \\ & 6 & 3t & 0 & \ddots & & \\ & & 12 & \ddots & \ddots & -4 & \\ & & & \ddots & & 0 & -2 \\ & & & & n(n-1) & nt & 0 \end{bmatrix} \tag{6}$$

The spectrum is real for $t < t_c^{(n)}$ (Bender–Boettcher) and for $t \in \mathbb{C}$ the spectrum is complex and can have repeated eigenvalues.

The discriminant locus

Shapiro-Tater $\simeq$'18 (formalized in '22)

What are the (complex) values of $t \in \mathbb{C}$ for which the spectrum is **not simple**?

$$D_n(t) := \mathrm{Disc}_\lambda \left(\det(\lambda \mathbf{1} - M_n(t))\right) = 0$$

The problem is non self-adjoint and the spectrum is complex (particularly so for $t \in \mathbb{C}$).

The numerics

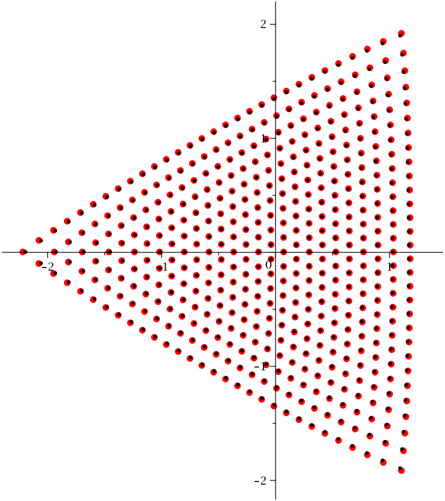


Figure: Scaled roots of the discriminant $D_n(n^{2/3}s)$ in black, for $n = 30$ and the rescaled roots of the Vorob'ev-Yablonsky polynomials $Y_n(n^{2/3}s)$ in red. This connection and particular scaling was conjectured by B.Shapiro–M.Tater '22.

n	$D_n(t)$
1	t
2	$t^3 + \frac{27}{8}$
3	$t^6 + \frac{35}{2}t^3 - \frac{243}{4}$
4	$t^{10} + \frac{215}{4}t^7 + \frac{89}{8}t^4 + \frac{4084101}{512}t$
5	$t^{15} + \frac{255}{2}t^{12} + \frac{76211}{32}t^9 + \frac{3730405}{64}t^6 - \frac{8700637815}{4096}t^3 - \frac{125005275}{32}$

n	$Y_n(t)$
1	t
2	$t^3 + 4$
3	$t^6 + 20t^3 - 80$
4	$t^{10} + 60t^7 + 11200t$
5	$t^{15} + 140t^{12} + 2800t^9 + 78400t^6 - 3136000t^3 - 6272000$

Table: The first five monic discriminant polynomials $D_n(t)$ and Vorob'ev–Yablonskii polynomials $Y_n(t)$.

Rational solutions of PII

$$\frac{\mathrm{d}^2 u(t)}{\mathrm{d}t^2} = 2u(t)^3 + tu(t) + \alpha, \tag{7}$$

Rational solution iff $\alpha = n \in \mathbb{Z}$;

$$u_n(t) = \frac{\mathrm{d}}{\mathrm{d}t} \log \frac{Y_{n-1}(t)}{Y_n(t)} \tag{8}$$

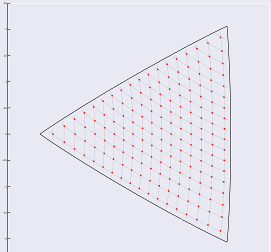
with Y_n the Vorob'ev–Yablonski polynomials of degree $n(n+1)/2$.

$$Y_{n+1}(t)Y_{n-1}(t) = tY_n^2(t) - 4\Big[Y_n''(t)Y_n(t) - \big(Y_n'(t)\big)^2\Big], \qquad n \geqslant 1,\ t \in \mathbb{C} \tag{VY}$$

with $Y_0(t) = 1, Y_1(t) = t$. Or otherwise

$$Y_n(t) = \left(-\frac{4}{3}\right)^{n(n+1)/6} \left(\prod_{k=1}^{n} (2k-1)!!\right) S_{(n,n-1,\dots,1)}\left(\left(-\frac{3}{4}\right)^{\frac{1}{3}} t, 0, 1, 0, 0, \dots\right).$$

The regularity of the pattern of zeroes of $Y_n(t)$ observed numerically by Clarkson ['03], and explained (asymptotically and analytically) by Buckingham–Miller ['14], Bertola-Bothner ['14];



Quartic Anharmonic Oscillators

$$y''(z) - \left(z^4 + tz^2 + 2Jz\right) y(x) = \Lambda y(z) \quad (9)$$

$$y(z) \to 0 \text{ as } z \to \infty \text{ and } \arg(z) = \pi, \pm\pi/3, \quad (10)$$

Quasi–Exactly–Solvable spectrum

If there are only two boundary conditions ⇒ Bender–Boettcher ['98]. Part of the spectrum (the "Exactly–Solvable") is explicit for $J \in \mathbb{N}$.

Exactly–Solvable spectrum

Three boundary conditions ⇒ $J \in \mathbb{N}$ and all the spectrum is explicit.

Proposition

The boundary value problem

$$y''(z) - (z^4 + tz^2 + 2Jz + \Lambda)y(z) = 0 \tag{11}$$

$$y(se^{k\pi i/3}) \to 0, \quad s \to +\infty, \ k = 1, 3, 5, \tag{12}$$

has solution if and only if $J = n + 1 \in \mathbb{N}$ and $y(z) = p_n(z)e^{\theta(z;t)}$, with $\theta(z;t) = \frac{z^3}{3} + \frac{tz}{2}$, with $p_n(z)$ a polynomial of degree n satisfying

$$\left(\frac{d^2}{dz^2} + 2\left(z^2 + \frac{t}{2}\right)\frac{d}{dz} - 2nz\right)p_n(z) = \lambda p_n(z), \qquad \lambda = \Lambda - \frac{t^2}{4}.$$

Structural results: Exactly Solvable spectrum II

Proposition

If $p_n(z)$ is a polynomial as above then

- p_n is a **degenerate orthogonal polynomial**

$$\left(\kappa \int_{\infty_1}^{\infty_3} + \widetilde{\kappa} \int_{\infty_5}^{\infty_3}\right) p_n(z) z^k \mathrm{e}^{2\theta(z;t)}\, \mathrm{d}z = 0 \quad k = 0, 1, \cdots, n-1, \mathbf{n}. \tag{13}$$

- The coefficients $\kappa, \widetilde{\kappa}$ are

$$\kappa = \int_{\infty_2}^{\infty_0} \frac{\mathrm{e}^{-2\theta(z;t)}\, \mathrm{d}z}{p_n^2(z)} \qquad \widetilde{\kappa} = \int_{\infty_0}^{\infty_4} \frac{\mathrm{e}^{-2\theta(z;t)}\, \mathrm{d}z}{p_n^2(z)}$$

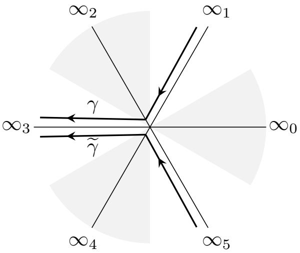


Figure: Directions at infinity ∞_k of argument $k\frac{i\pi}{3}$.

Structural results: degenerate spectrum

Proposition

If $p_n(z)$ is a polynomial as above then

- p_n is a **degenerate orthogonal polynomial**

$$\left(\kappa \int_{\infty_1}^{\infty_3} + \tilde{\kappa} \int_{\infty_5}^{\infty_3}\right) p_n(z) z^k \mathrm{e}^{2\theta(z;t)} \, \mathrm{d}z = 0 \quad k = 0, 1, \cdots, n-1, \mathbf{n}. \tag{14}$$

- The zeros of $p_n(z)$ satisfies the Fekete type relation

$$\theta'(z_j) = \sum_{k \neq j} \frac{1}{z_k - z_j}, \quad j = 1, \ldots, n.$$

- $t \in \mathbb{C}$ is such that the Exactly Solvable spectrum of (11)-(12) has a repeated eigenvalue iff the degenerate orthogonal polynomial $p_n(x)$ **additionally** satisfies

$$\int_{\infty_1}^{\infty_3} p_n^2(z) e^{2\theta(z;a)} \, \mathrm{d}z = 0, \quad \int_{\infty_3}^{\infty_5} p_n^2(z) e^{2\theta(z;a)} \, \mathrm{d}z = 0. \tag{15}$$

Second main result: the Shapiro-Tater conjecture

Proposition

The point t is a pole with residue -1 of the rational PII function $u(t)$ with parameter $\alpha = n$ (i.e. a zero of $Y_n(t)$) if and only if there is b such that the ODE (Its-Novokshenov)

$$f(z)'' - V_{JM}(z;t,b)f(z) = 0$$

$$V_{JM}(z;t,b) = z^4 + tz^2 + 2(n + \frac{1}{2})z + \left(\frac{7t^2}{36} + 10b\right)$$

manifests the Stokes' phenomenon indicated below [Buckingham–Miller '14]

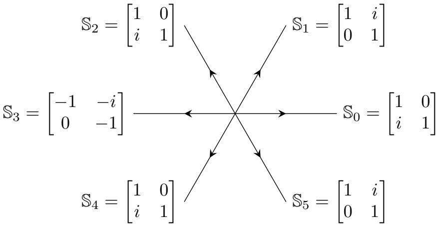


Figure: Stokes data for the Lax pair corresponding to rational solutions of Painlevé II

Proposition

The values t, Λ belong to the Exactly Solvable spectrum iff the solutions to

$$y''(z) - (z^4 + tz^2 + 2(n+1)z + \Lambda)y(z) = 0 \tag{16}$$

have the Stokes' phenomenon below. In addition the parameter t is in the discriminant locus if and only if

$$\int_{\infty_1}^{\infty_3} y^2(z)\,\mathrm{d}z = 0 = \int_{\infty_5}^{\infty_3} y^2(z)\,\mathrm{d}z$$

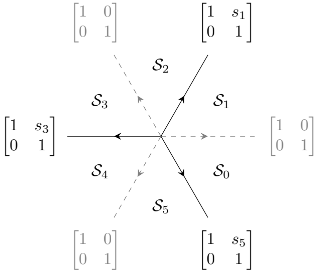


Figure: Stokes matrices and Stokes sectors for the Shapiro-Tater eigenvalue problem: the condition $s_j \in \mathbb{C}$ and $s_1 + s_3 + s_5 = 0$ holds.

Strategy

1. Scale t, Λ (and z) with $\hbar = (n+1)^{-1}$ or $\hbar = (n+\frac{1}{2})^{-1}$ to bring the equation to standard WKB form:

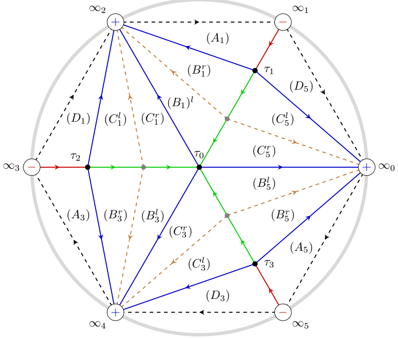
$$\hbar^2 y''(z) - Q(z; s, E) y(z) = 0, \qquad Q(z; s, E) = z^4 + sz^2 + 2z + E \tag{17}$$

$$s = \hbar^{\frac{2}{3}} t, \quad E = \hbar^{\frac{4}{3}} \Lambda \tag{18}$$

2. Use WKB to compute Stokes' data
3. Match Stokes' data with the one in the figure.
   1. For the VY case the Stokes' parameters are completely determined and this (implicitly) fixes the pair (s, E);
   2. For the ST case we need to **additionally** impose the degeneracy condition, which is equivalent to

$$\int_{\infty_1}^{\infty_3} y^2(z)\, \mathrm{d}z = 0 = \int_{\infty_5}^{\infty_3} y^2(z)\, \mathrm{d}z$$

These integrals must be estimated using the WKB approximation.



Stokes' complex of the ST problem
compatible with all the conditions

Figure: Labelled regions in the WKB Riemann-Hilbert problem.

Quantization conditions (leading order)

ST case

$$
\begin{aligned}
2(n+1)\int_{\tau_1}^{\tau_0}\sqrt{Q(z_+;s,E)}\,\mathrm{d}z &= \ln\left(\frac{-1}{1+\boldsymbol{\tau}(s,E)}\right) - 2i\pi(m_1+1)\\
2(n+1)\int_{\tau_2}^{\tau_0}\sqrt{Q(z_+;s,E)}\,\mathrm{d}z &= \ln\left(-1-\frac{1}{\boldsymbol{\tau}(s,E)}\right) - 2i\pi(m_2+1)\\
2(n+1)\int_{\tau_3}^{\tau_0}\sqrt{Q(z_+;s,E)}\,\mathrm{d}z &= \ln\left(\boldsymbol{\tau}(s,E)\right) - 2i\pi(m_3+1)\\
\boldsymbol{\tau}(s,E) = \frac{\displaystyle\int_{\tau_1}^{\tau_0}\frac{\mathrm{d}z}{\sqrt{Q(z_+;s,E)}}}{\displaystyle\int_{\tau_2}^{\tau_0}\frac{\mathrm{d}z}{\sqrt{Q(z_+;s,E)}}}&, \quad \Im(\boldsymbol{\tau}(s,E))>0\\
m_1+m_2+m_3 = n-1.
\end{aligned}
\tag{19}
$$

VY case

$$
\begin{aligned}
(2n+1)\int_{\tau_j}^{\tau_0}\sqrt{Q(z_+;s,E)}\,\mathrm{d}z &= -i\pi - 2i\pi k_j\\
k_1+k_2+k_3 &= n-1.
\end{aligned}
\tag{20}
$$

$$\omega := \int_{\tau_2}^{\tau_0} \frac{\mathrm{d}z}{\sqrt{Q(z_+; s, E)}}, \quad \omega' := \int_{\tau_1}^{\tau_0} \frac{\mathrm{d}z}{\sqrt{Q(z_+; s, E)}} \tag{21}$$

Theorem

Let (s_0, E_0) correspond to the first-order quantization conditions (19) or (20) in the bulk, namely, $m_j/n \simeq c_j \neq 0$. Then the neighbour points in the s–plane form a slowly modulated hexagonal lattice in the sense that the six closest neighbours of s_0 are

$$s_0 + 2\hbar \left(\omega \Delta m_1 - \omega' \Delta m_2\right) \tag{22}$$

where ω and ω' are the half periods of the holomorphic differentials in (21) and

$$\Delta m_j \in \{-1, 0, 1\}, \qquad |\Delta m_1 + \Delta m_2| \leqslant 1, \qquad |\Delta m_1| + |\Delta m_2| \geqslant 1.$$

Near the origin

If $(s, E) = \mathcal{O}(\hbar)$, the rescaled lattices of the zeroes of the VY Polynomials, and of the ST problem coincide within order $\mathcal{O}(\hbar^2) = \mathcal{O}(n^{-2})$ in a $\mathcal{O}(\hbar)$ neighbourhood of the origin in the s–plane.

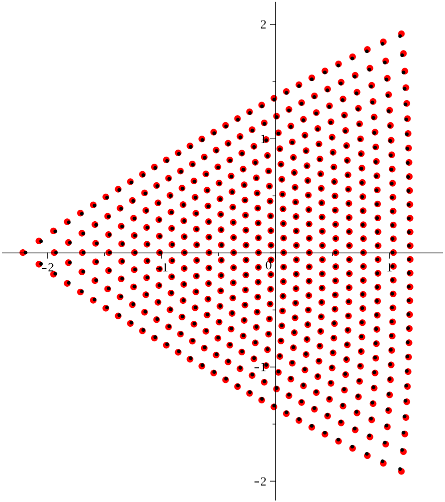


Figure: Scaled roots of the Vorob'ev-Yablonsky polynomials $Y_n(n^{2/3}s)$ in red, and roots of the discriminant $D_n(n^{2/3}s)$ in black, for $n = 30$.

HAPPY BIRTHDAY IGOR!