Theta divisors and Chern-Dold character in complex cobordisms

Alexander Veselov
(j/w with Victor M. Buchstaber)

Algebraic Geometry, Mathematical Physics and Solitons
Celebrating the work of Igor Krichever, Columbia University, October 7-9, 2022

S.P. Novikov's School

Novikov's School in Ascona 2015: Buchstaber, AV, Novikov, IK, Dubrovin

Work of Igor Krichever in algebraic topology

Bordisms of groups acting freely on spheres. Uspehi Mat. Nauk 26 (1971), no. 6(162), 245-246.
Actions of finite cyclic groups on quasicomplex manifolds. Mat. Sb. (N.S.) 90(132) (1973), 306-319, 327.
Formal groups and the Atiyah-Hirzebruch formula. Izv. Akad. Nauk SSSR Ser. Mat. 38 (1974), 1289-1304.
Formulae for the fixed points of an action of the group Z_p. (jointly with S.M. Gusein-Zade) Uspehi Mat. Nauk 28 (1973), no. 1(169), 237-238.
The invariance of characteristic classes for manifolds of the homotopy type of CP^n. Uspehi Mat. Nauk 28 (1973), no. 5(173), 245-246.
Formal groups and the Atiyah-Hirzebruch formula. Izv. Akad. Nauk SSSR Ser. Mat. 38 (1974), 1289-1304.
Equivariant Hirzebruch genera. The Atiyah-Hirzebruch formula. Uspehi Mat. Nauk 30 (1975), no. 1(181), 243-244.
Obstructions to the existence of S^1-actions. Bordisms of branched coverings. Izv. Akad. Nauk SSSR Ser. Mat. 40 (1976), no. 4, 828-844, 950.
Generalized elliptic genera and Baker-Akhiezer functions. Math. Notes 47 (1990), no. 1-2, 132-142

Complex coborodisms

Let M^m be a smooth closed real manifold. By *stable complex structure* (or, simply *U-structure*) on M^m we mean an isomorphism of real vector bundles $TM^m \oplus (2N - m)_{\mathbb{R}} \cong r\xi$, where TM^m is the tangent bundle of M^m, $(2N - m)_{\mathbb{R}}$ is trivial real $(2N - m)$-dimensional bundle over M^m, ξ is a complex vector bundle over M^m and $r\xi$ is its real form. A manifold M^m with a chosen U-structure is called *U-manifold*.

Complex coborodisms

Let M^m be a smooth closed real manifold. By *stable complex structure* (or, simply *U-structure*) on M^m we mean an isomorphism of real vector bundles $TM^m \oplus (2N-m)_{\mathbb{R}} \cong r\xi$, where TM^m is the tangent bundle of M^m, $(2N-m)_{\mathbb{R}}$ is trivial real $(2N-m)$-dimensional bundle over M^m, ξ is a complex vector bundle over M^m and $r\xi$ is its real form. A manifold M^m with a chosen U-structure is called *U-manifold*.

Two closed smooth real m-dimensional manifolds M_1 and M_2 are called *bordant* if there exists a real $(m+1)$-dimensional U-manifold W such that the boundary ∂W is a disjoint union of M_1 and M_2 and the restriction of the stable tangent bundle TW to M_i is stably equivalent to TM_i, $i = 1, 2$.

Let M^m be a smooth closed real manifold. By *stable complex structure* (or, simply *U-structure*) on M^m we mean an isomorphism of real vector bundles $TM^m \oplus (2N-m)_{\mathbb{R}} \cong r\xi$, where TM^m is the tangent bundle of M^m, $(2N-m)_{\mathbb{R}}$ is trivial real $(2N-m)$-dimensional bundle over M^m, ξ is a complex vector bundle over M^m and $r\xi$ is its real form. A manifold M^m with a chosen U-structure is called *U-manifold*.

Two closed smooth real m-dimensional manifolds M_1 and M_2 are called *bordant* if there exists a real $(m+1)$-dimensional U-manifold W such that the boundary ∂W is a disjoint union of M_1 and M_2 and the restriction of the stable tangent bundle TW to M_i is stably equivalent to TM_i, $i=1,2$.

Similarly, two closed smooth real m-dimensional manifolds M_1 and M_2 are called *cobordant* if there exists a real $(m+1)$-dimensional U-manifold W such that the boundary ∂W is a disjoint union of M_1^m and M_2^m and the restriction of the stable normal bundle νW to M_i coincides with νM_i, $i=1,2$.

Ring structure

The sum of bordism classes of two closed U-manifolds M_1^m and M_2^m is defined as

$$[M_1^m] + [M_2^m] = [M_1^m \cup M_2^m],$$

where $M_1^m \cup M_2^m$ is the disjoint union of M_1^m and M_2^m. The product of the bordism classes of $M_1^{m_1}$ and $M_2^{m_2}$ is defined by

$$[M_1^{m_1}][M_2^{m_2}] = [M_1^{m_1} \times M_2^{m_2}].$$

This defines the commutative graded ring

$$\Omega^U = \sum_{m \geq 0} \Omega_m^U,$$

where Ω_m^U is the group of bordism classes of m-dimensional U-manifolds.

Ring structure

The sum of bordism classes of two closed U-manifolds M_1^m and M_2^m is defined as

$$[M_1^m] + [M_2^m] = [M_1^m \cup M_2^m],$$

where $M_1^m \cup M_2^m$ is the disjoint union of M_1^m and M_2^m. The product of the bordism classes of $M_1^{m_1}$ and $M_2^{m_2}$ is defined by

$$[M_1^{m_1}][M_2^{m_2}] = [M_1^{m_1} \times M_2^{m_2}].$$

This defines the commutative graded ring

$$\Omega^U = \sum_{m \geq 0} \Omega_m^U,$$

where Ω_m^U is the group of bordism classes of m-dimensional U-manifolds.

Similarly, we have the graded ring $\Omega_U = \sum_{m \geq 0} \Omega_U^{-m}$, where Ω_U^{-m} is the group of cobordism classes of m-dimensional U-manifolds.

Ring structure

The sum of bordism classes of two closed U-manifolds M_1^m and M_2^m is defined as

$$[M_1^m] + [M_2^m] = [M_1^m \cup M_2^m],$$

where $M_1^m \cup M_2^m$ is the disjoint union of M_1^m and M_2^m. The product of the bordism classes of $M_1^{m_1}$ and $M_2^{m_2}$ is defined by

$$[M_1^{m_1}][M_2^{m_2}] = [M_1^{m_1} \times M_2^{m_2}].$$

This defines the commutative graded ring

$$\Omega^U = \sum_{m \geq 0} \Omega_m^U,$$

where Ω_m^U is the group of bordism classes of m-dimensional U-manifolds.

Similarly, we have the graded ring $\Omega_U = \sum_{m \geq 0} \Omega_U^{-m}$, where Ω_U^{-m} is the group of cobordism classes of m-dimensional U-manifolds.

From the correspondence between stable complex structures in tangent and normal bundles it follows that the groups Ω_m^U and Ω_U^{-m} and the rings Ω_U and Ω^U are isomorphic.

Let $\lambda = (i_1, \dots, i_k)$, $i_1 \geq \dots \geq i_k$ be a partition of $n = i_1 + \dots + i_k$. Using the splitting principle one can define the Chern classes $c_\lambda(TM) \in H^{2n}(M, \mathbb{Z})$ of a U-manifold M corresponding to the monomial symmetric functions $m_\lambda(t) = t_1^{i_1} \dots t_k^{i_k} + \dots$.

Let $\lambda = (i_1, \dots, i_k)$, $i_1 \geq \dots \geq i_k$ be a partition of $n = i_1 + \dots + i_k$. Using the splitting principle one can define the Chern classes $c_\lambda(TM) \in H^{2n}(M, \mathbb{Z})$ of a U-manifold M corresponding to the monomial symmetric functions $m_\lambda(t) = t_1^{i_1} \dots t_k^{i_k} + \dots$.

The *Chern number* $c_\lambda(M^{2n})$, $|\lambda| = n$ of U-manifold M^{2n} is defined as

$$c_\lambda(M^{2n}) := (c_\lambda(TM^{2n}), \langle M^{2n} \rangle).$$

We have $p(n)$ Chern numbers $c_\lambda(M^{2n})$, which depend only on the bordism class of M^{2n}.

Chern numbers

Let $\lambda = (i_1, \dots, i_k)$, $i_1 \geq \dots \geq i_k$ be a partition of $n = i_1 + \dots + i_k$. Using the splitting principle one can define the Chern classes $c_\lambda(TM) \in H^{2n}(M, \mathbb{Z})$ of a U-manifold M corresponding to the monomial symmetric functions $m_\lambda(t) = t_1^{i_1} \dots t_k^{i_k} + \dots$.

The *Chern number* $c_\lambda(M^{2n})$, $|\lambda| = n$ of U-manifold M^{2n} is defined as

$$c_\lambda(M^{2n}) := (c_\lambda(TM^{2n}), \langle M^{2n} \rangle).$$

We have $p(n)$ Chern numbers $c_\lambda(M^{2n})$, which depend only on the bordism class of M^{2n}.

It will be more convenient for us to use the Chern numbers $c_\lambda^\nu(M^{2n})$ defined using the stable normal bundle νM^{2n}:

$$c_\lambda^\nu(M^{2n}) := (c_\lambda(\nu M^{2n}), \langle M^{2n} \rangle).$$

They can be expressed through the usual Chern numbers $c_\lambda(M^{2n})$ and contain the same information about U-manifold M^{2n}.

The following fundamental result is due to Milnor and Novikov.

Theorem (Milnor, Novikov, 1960)

The graded complex bordism ring Ω^U is isomorphic to the graded polynomial ring $\mathbb{Z}[y_1, y_2, \dots, y_n, \dots]$ of infinitely many variables with $\deg y_n = 2n$.

Two closed $2n$-dimensional U-manifolds M_1 and M_2 are U-bordant if and only if all the corresponding Chern numbers are the same.

The following fundamental result is due to Milnor and Novikov.

Theorem (Milnor, Novikov, 1960)

The graded complex bordism ring Ω^U is isomorphic to the graded polynomial ring $\mathbb{Z}[y_1, y_2, \dots, y_n, \dots]$ of infinitely many variables with $\deg y_n = 2n$.

Two closed $2n$-dimensional U-manifolds M_1 and M_2 are U-bordant if and only if all the corresponding Chern numbers are the same.

The same is true for the complex cobordism ring Ω_U with the degrees of the generators $-2n$.

Buchstaber 1970: **The Chern-Dold character in complex cobordisms**

$$ch_U : U^*(X) \to H^*(X, \Omega_U \otimes \mathbb{Q}),$$

is uniquely defined by its action

$$ch_U : u \to \beta(z), \quad \beta(z) := z + \sum_{n=1}^{\infty} [\mathcal{B}^{2n}] \frac{z^{n+1}}{(n+1)!},$$

where $u \in U^2(\mathbb{C}P^\infty)$ and $z \in H^2(\mathbb{C}P^\infty)$ are the first Chern classes of the universal line bundle over $\mathbb{C}P^\infty$ in the complex cobordisms and cohomology theory respectively, and $\mathcal{B}^{2n}$ are some implicitly characterised U-manifolds.

Buchstaber 1970: **The Chern-Dold character in complex cobordisms**

$$ch_U : U^*(X) \to H^*(X, \Omega_U \otimes \mathbb{Q}),$$

is uniquely defined by its action

$$ch_U : u \to \beta(z), \quad \beta(z) := z + \sum_{n=1}^{\infty} [\mathcal{B}^{2n}] \frac{z^{n+1}}{(n+1)!},$$

where $u \in U^2(\mathbb{C}P^\infty)$ and $z \in H^2(\mathbb{C}P^\infty)$ are the first Chern classes of the universal line bundle over $\mathbb{C}P^\infty$ in the complex cobordisms and cohomology theory respectively, and $\mathcal{B}^{2n}$ are some implicitly characterised U-manifolds.

The series $\beta(z)$ is the exponential of the commutative formal group

$$F(u, v) = u + v + \sum_{i,j} a_{i,j} u^i v^j$$

of the geometric complex cobordisms introduced by **Novikov**:

$$\beta(z + w) = F(\beta(z), \beta(w)).$$

Quillen identified this group with Lazard's universal one-dimensional commutative formal group.

Main result

The inverse of this series is the logarithm of this formal group, which was found explicitly by **Mischenko**:

$$\beta^{-1}(u) = u + \sum_{n=1}^{\infty} [\mathbb{C}P^n] \frac{u^{n+1}}{n+1}.$$

The question about explicit algebraic representatives of $[\mathcal{B}^{2n}]$ in the exponential was open since 1970.

Main result

The inverse of this series is the logarithm of this formal group, which was found explicitly by **Mischenko**:

$$\beta^{-1}(u) = u + \sum_{n=1}^{\infty} [\mathbb{C}P^n] \frac{u^{n+1}}{n+1}.$$

The question about explicit algebraic representatives of $[\mathcal{B}^{2n}]$ in the exponential was open since 1970.

Buchstaber, APV 2020:
As a representative of $[\mathcal{B}^{2n}]$ one can take a smooth theta divisor Θ^n of a general principally polarised abelian variety A^{n+1}:

$$\beta(z) = z + \sum_{n=1}^{\infty} [\Theta^n] \frac{z^{n+1}}{(n+1)!}.$$

Let $A^{n+1} = \mathbb{C}^{n+1}/\Gamma$ be a principally polarised abelian variety (ppav). It has a canonical line bundle L with one-dimensional space of sections generated by the classical Riemann θ-function

$$\theta(z, \tau) = \sum_{m \in \mathbb{Z}^{n+1}} \exp[\pi i(m, \tau m) + 2\pi i(m, z)],\ z \in \mathbb{C}^{n+1}, \quad \tau^t = \tau, Im\,\tau > 0.$$

Theta divisors

Let $A^{n+1} = \mathbb{C}^{n+1}/\Gamma$ be a principally polarised abelian variety (ppav). It has a canonical line bundle L with one-dimensional space of sections generated by the classical Riemann θ-function

$$\theta(z,\tau) = \sum_{m\in\mathbb{Z}^{n+1}} \exp[\pi i(m,\tau m) + 2\pi i(m,z)],\ z \in \mathbb{C}^{n+1}, \quad \tau^t = \tau, Im\,\tau > 0.$$

Andreotti and Mayer 1967: For generic ppav the theta divisor $\Theta^n \subset A^{n+1}$

given by $\theta(z,\tau) = 0$ is smooth irreducible algebraic variety of general type. The cobordism class $[\Theta^n]$ does not depend on the choice of such abelian variety.

Let $A^{n+1} = \mathbb{C}^{n+1}/\Gamma$ be a principally polarised abelian variety (ppav). It has a canonical line bundle L with one-dimensional space of sections generated by the classical Riemann θ-function

$$\theta(z,\tau) = \sum_{m\in\mathbb{Z}^{n+1}} \exp[\pi i(m,\tau m) + 2\pi i(m,z)],\ z\in\mathbb{C}^{n+1}, \quad \tau^t = \tau, Im\,\tau > 0.$$

Andreotti and Mayer 1967: For generic ppav the theta divisor $\Theta^n \subset A^{n+1}$

given by $\theta(z,\tau) = 0$ is smooth irreducible algebraic variety of general type. The cobordism class $[\Theta^n]$ does not depend on the choice of such abelian variety.

In particular, the theta divisor $\Theta^1 \cong \mathcal{C} \subset A^2 = J(\mathcal{C})$ is a smooth genus 2 curve, $\Theta^2 \cong S^2(\mathcal{C}) \subset A^3 = J(\mathcal{C})$ for a smooth non-hyperelliptic curve $\mathcal{C}$ of genus 3.

Theta divisors

Let $A^{n+1} = \mathbb{C}^{n+1}/\Gamma$ be a principally polarised abelian variety (ppav). It has a canonical line bundle L with one-dimensional space of sections generated by the classical Riemann θ-function

$$\theta(z,\tau) = \sum_{m \in \mathbb{Z}^{n+1}} \exp[\pi i(m,\tau m) + 2\pi i(m,z)],\ z \in \mathbb{C}^{n+1}, \quad \tau^t = \tau, Im\,\tau > 0.$$

Andreotti and Mayer 1967: For generic ppav the theta divisor $\Theta^n \subset A^{n+1}$

given by $\theta(z,\tau) = 0$ is smooth irreducible algebraic variety of general type. The cobordism class $[\Theta^n]$ does not depend on the choice of such abelian variety.

In particular, the theta divisor $\Theta^1 \cong \mathcal{C} \subset A^2 = J(\mathcal{C})$ is a smooth genus 2 curve, $\Theta^2 \cong S^2(\mathcal{C}) \subset A^3 = J(\mathcal{C})$ for a smooth non-hyperelliptic curve $\mathcal{C}$ of genus 3.

For $n \geq 3$ the general ppav is not Jacobian and the characterisation of the Jacobi varieties among all ppav is the famous **Riemann-Schottky problem** (see the recent survey by **Igor Krichever** and his plenary talk at ICM-2022).

Key lemma

The Chern numbers of the theta divisor Θ^n are

$$c_\lambda^\nu(\Theta^n) = 0$$

for any partition λ of n different from the one-part partition $\lambda = (n)$, and

$$c_{(n)}^\nu(\Theta^n) = (n+1)!.$$

Key lemma

The Chern numbers of the theta divisor Θ^n are

$$c^{\nu}_{\lambda}(\Theta^n) = 0$$

for any partition λ of n different from the one-part partition $\lambda = (n)$, and

$$c^{\nu}_{(n)}(\Theta^n) = (n+1)!.$$

Indeed, since the tangent bundle of abelian variety is trivial, the normal bundle $\nu\Theta^n$ is stably equivalent to the line bundle $\mathcal{L} = i^*(L)$, where $i : \Theta^n \to A^{n+1}$ is natural embedding and L is the principal polarisation line bundle on A^{n+1}.

Key lemma

The Chern numbers of the theta divisor Θ^n are

$$c_\lambda^\nu(\Theta^n) = 0$$

for any partition λ of n different from the one-part partition $\lambda = (n)$, and

$$c_{(n)}^\nu(\Theta^n) = (n+1)!.$$

Indeed, since the tangent bundle of abelian variety is trivial, the normal bundle $\nu\Theta^n$ is stably equivalent to the line bundle $\mathcal{L} = i^*(L)$, where $i : \Theta^n \to A^{n+1}$ is natural embedding and L is the principal polarisation line bundle on A^{n+1}.
The last equality follows from the well-known fact that

$$D^g = g! \in H^{2g}(A^g, \mathbb{Z}) = \mathbb{Z}$$

where $D \in H^2(A^g, \mathbb{Z})$ is the Poincare dual cohomology class of the theta divisor $\Theta \subset A^g$ of any principally polarised abelian variety.

Todd class in complex cobordisms

Buchstaber 1970: **The Todd class** $Td_U(\xi)$ of complex vector bundle ξ over CW-complex X with values in $H^*(X, \Omega_U \otimes \mathbb{Q})$ is uniquely defined by the following properties:

- For every two vector bundles ξ_1 and ξ_2 over X

$$Td_U(\xi_1 \oplus \xi_2) = Td_U(\xi_1) Td_U(\xi_2),$$

- For any U-manifold M^{2n}

$$(Td_U(TM^{2n}), \langle M^{2n} \rangle) = [M^{2n}].$$

Todd class in complex cobordisms

Buchstaber 1970: **The Todd class** $Td_U(\xi)$ of complex vector bundle ξ over CW-complex X with values in $H^*(X, \Omega_U \otimes \mathbb{Q})$ is uniquely defined by the following properties:

- For every two vector bundles ξ_1 and ξ_2 over X

$$Td_U(\xi_1 \oplus \xi_2) = Td_U(\xi_1) Td_U(\xi_2),$$

- For any U-manifold M^{2n}

$$(Td_U(TM^{2n}), \langle M^{2n} \rangle) = [M^{2n}].$$

Buchstaber, APV 2020:
The Todd class can be expressed in terms of theta divisors by the formula

$$Td_U(\xi) = \sum_{\lambda} c_\lambda(-\xi) \frac{[\Theta^\lambda]}{(\lambda+1)!},$$

where the sum is over all partitions $\lambda = (i_1, \dots, i_k)$ and $\Theta^\lambda = \Theta^{i_1} \times \cdots \times \Theta^{i_k}$.
In particular, for any U-manifold M we have

$$[M^{2n}] = \sum_{\lambda : |\lambda| = n} c_\lambda^\nu(M^{2n}) \frac{[\Theta^\lambda]}{(\lambda+1)!}.$$

Hirzebruch genus of theta divisors

The Hirzebruch genus in complex cobordisms is a homomorphism $\Phi : \Omega_U \to \mathcal{A}$, where $\mathcal{A}$ is some algebra over $\mathbb{Q}$, determined by its characteristic power series

$$Q(z) = 1 + \sum_{n=1}^{\infty} a_n z^n, \quad a_n \in \mathcal{A}.$$

Hirzebruch genus of theta divisors

The Hirzebruch genus in complex cobordisms is a homomorphism $\Phi : \Omega_U \to \mathcal{A}$, where $\mathcal{A}$ is some algebra over $\mathbb{Q}$, determined by its characteristic power series

$$Q(z) = 1 + \sum_{n=1}^{\infty} a_n z^n, \quad a_n \in \mathcal{A}.$$

Buchstaber, APV 2020: The exponential generating function of Hirzebruch genus of theta divisors $\Phi(\Theta^n)$ is

$$z + \sum_{n=1}^{\infty} \Phi(\Theta^n) \frac{z^{n+1}}{(n+1)!} = \frac{z}{Q(z)}.$$

The Hirzebruch genus in complex cobordisms is a homomorphism $\Phi : \Omega_U \to \mathcal{A}$, where $\mathcal{A}$ is some algebra over $\mathbb{Q}$, determined by its characteristic power series

$$Q(z) = 1 + \sum_{n=1}^{\infty} a_n z^n, \quad a_n \in \mathcal{A}.$$

Buchstaber, APV 2020: The exponential generating function of Hirzebruch genus of theta divisors $\Phi(\Theta^n)$ is

$$z + \sum_{n=1}^{\infty} \Phi(\Theta^n) \frac{z^{n+1}}{(n+1)!} = \frac{z}{Q(z)}.$$

In particular, for the Todd genus we have $Q(z) = \frac{z}{1-e^{-z}}$, so

$$z + \sum_{n=1}^{\infty} Td(\Theta^n) \frac{z^{n+1}}{(n+1)!} = 1 - e^{-z} = z + \sum_{n\in\mathbb{N}} (-1)^n \frac{z^{n+1}}{(n+1)!},$$

so that the Todd genus of the theta divisors is $Td(\Theta^n) = (-1)^n$.
As a corollary we have the Todd genus for any U-manifold M^{2n} as

$$Td(M^{2n}) = \sum_{\lambda : |\lambda| = n} c_\lambda^\nu(M^{2n}) \frac{(-1)^n}{(\lambda+1)!}.$$

Euler characteristic and signature

For the Euler characteristic $Q(z) = 1 + z$, so we have

$$z + \sum_{n=1}^{\infty} \chi(\Theta^n) \frac{z^{n+1}}{(n+1)!} = \frac{z}{1+z},$$

so $\chi(\Theta^n) = (-1)^n (n+1)!$.

Euler characteristic and signature

For the Euler characteristic $Q(z) = 1 + z$, so we have

$$z + \sum_{n=1}^{\infty} \chi(\Theta^n) \frac{z^{n+1}}{(n+1)!} = \frac{z}{1+z},$$

so $\chi(\Theta^n) = (-1)^n (n+1)!$.

The signature corresponds to Hirzebruch L-genus with $Q(z) = \frac{z}{\tanh z}$. Since

$$\frac{z}{Q(z)} = \tanh z = \sum_{k=0}^{\infty} 2^{2k+2}(2^{2k+2} - 1) B_{2k+2} \frac{z^{2k+1}}{(2k+2)!}$$

we have the signature

$$\tau(\Theta^n) = \frac{2^{n+2}(2^{n+2} - 1)}{n+2} B_{n+2},$$

where B_n are the classical **Bernoulli numbers**:

$$B_0 = 1,\ B_1 = -\frac{1}{2},\ B_2 = \frac{1}{6},\ B_4 = -\frac{1}{30},\ B_6 = \frac{1}{42},\ B_8 = -\frac{1}{30},\ B_{10} = \frac{5}{66}, \dots$$

Cohomology of theta divisors

The Betti numbers of the theta divisors Θ^n can be computed using the Lefschetz hyperplane theorem (**Izadi-Wang 2015**). Indeed, the embedding $i : \Theta^n \to A^{n+1}$ induces the isomorphisms

$$i_* : H_k(\Theta^n, \mathbb{Z}) \to H_k(A^{n+1}, \mathbb{Z}), \quad i_* : \pi_k(\Theta^n) \to \pi_k(A^{n+1})$$

for $k < n$, while for $k = n$ these homomorphisms are surjections.

Cohomology of theta divisors

The Betti numbers of the theta divisors Θ^n can be computed using the Lefschetz hyperplane theorem (**Izadi-Wang 2015**). Indeed, the embedding $i : \Theta^n \to A^{n+1}$ induces the isomorphisms

$$i_* : H_k(\Theta^n, \mathbb{Z}) \to H_k(A^{n+1}, \mathbb{Z}), \quad i_* : \pi_k(\Theta^n) \to \pi_k(A^{n+1})$$

for $k < n$, while for $k = n$ these homomorphisms are surjections.

Using Poincare duality we obtain all Betti numbers of Θ^n as

$$b_k(\Theta^n) = b_k(A^{n+1}) = \binom{2n+2}{k} = b_{2n-k}(\Theta^n), \; k < n,$$

except the middle one b_n, which can be found using the formula for the Euler characteristic:

$$b_n(\Theta^n) = (n+1)! + \frac{n}{n+2}\binom{2n+2}{n+1} = (n+1)! + nC_{n+1},$$

where $C_n = \frac{1}{n+1}\binom{2n}{n}$ is the n-th Catalan number.

The Betti numbers of the theta divisors Θ^n can be computed using the Lefschetz hyperplane theorem (**Izadi-Wang 2015**). Indeed, the embedding $i : \Theta^n \to A^{n+1}$ induces the isomorphisms

$$i_* : H_k(\Theta^n, \mathbb{Z}) \to H_k(A^{n+1}, \mathbb{Z}), \quad i_* : \pi_k(\Theta^n) \to \pi_k(A^{n+1})$$

for $k < n$, while for $k = n$ these homomorphisms are surjections.

Using Poincare duality we obtain all Betti numbers of Θ^n as

$$b_k(\Theta^n) = b_k(A^{n+1}) = \binom{2n+2}{k} = b_{2n-k}(\Theta^n), \ k < n,$$

except the middle one b_n, which can be found using the formula for the Euler characteristic:

$$b_n(\Theta^n) = (n+1)! + \frac{n}{n+2}\binom{2n+2}{n+1} = (n+1)! + nC_{n+1},$$

where $C_n = \frac{1}{n+1}\binom{2n}{n}$ is the n-th Catalan number.

However, the multiplication structure is yet to be understood. The appearance of both Bernoulli and Catalan numbers looks quite intriguing.

Milnor-Hirzebruch problem

There is a famous **Milnor-Hirzebruch problem**, which in algebraic version can be formulated as follows:

Which sets of $p(n)$ integers c_λ, $\lambda \in \mathcal{P}_n$ can be realised as the Chern numbers $c_\lambda(M^n)$ of some smooth irreducible complex algebraic variety M^n?

In this version it still remains largely open, although some arithmetic restrictions are known since the work of Milnor and Hirzebruch.

There is a famous **Milnor-Hirzebruch problem**, which in algebraic version can be formulated as follows:

Which sets of $p(n)$ integers c_λ, $\lambda \in \mathcal{P}_n$ can be realised as the Chern numbers $c_\lambda(M^n)$ of some smooth irreducible complex algebraic variety M^n?

In this version it still remains largely open, although some arithmetic restrictions are known since the work of Milnor and Hirzebruch.

In particular, we have the following Hirzebruch congruences for the usual Chern numbers of any almost complex manifold:

$$n = 1 : c_1 \equiv 0 \mod 2, \ n = 2 : c_2 + c_1^2 \equiv 0 \mod 12,$$

$$n = 3 : c_1 c_2 \equiv 0 \mod 24, \quad c_3 \equiv c_1^3 \equiv 0 \mod 2.$$

$$n = 4 : -c_4 + c_1 c_3 + 3c_2^2 + 4c_1^2 c_2 - c_1^4 \equiv 0 \mod 720, \ c_1^2 c_2 + 2c_1^4 \equiv 0 \mod 12,$$

$$-2c_4 + c_1 c_3 \equiv 0 \mod 4.$$

There is a famous **Milnor-Hirzebruch problem**, which in algebraic version can be formulated as follows:

Which sets of $p(n)$ integers c_λ, $\lambda \in \mathcal{P}_n$ can be realised as the Chern numbers $c_\lambda(M^n)$ of some smooth irreducible complex algebraic variety M^n?

In this version it still remains largely open, although some arithmetic restrictions are known since the work of Milnor and Hirzebruch.

In particular, we have the following Hirzebruch congruences for the usual Chern numbers of any almost complex manifold:

$$n = 1 : c_1 \equiv 0 \mod 2, \; n = 2 : c_2 + c_1^2 \equiv 0 \mod 12,$$

$$n = 3 : c_1 c_2 \equiv 0 \mod 24, \quad c_3 \equiv c_1^3 \equiv 0 \mod 2.$$

$$n = 4 : -c_4 + c_1 c_3 + 3c_2^2 + 4c_1^2 c_2 - c_1^4 \equiv 0 \mod 720, \; c_1^2 c_2 + 2c_1^4 \equiv 0 \mod 12,$$

$$-2c_4 + c_1 c_3 \equiv 0 \mod 4.$$

One can check that the first Hirzebruch congruence in each case is sharp for the theta divisors, which means that it cannot be improved in the algebraic setting. We hope that our results may lead to some progress in this direction.

References

F. Hirzebruch *Komplexe Mannigfaltigkeiten.* In: Proc. Intern. Congress of Math. 1958. Cambridge University Press, Cambridge, 1960, 119-136.

J. Milnor *On the cobordism ring Ω_* and complex analogue.* Part I., Amer. J. Math., **82:3** (1960), 505–521.

S.P. Novikov *On certain problems in topology of manifolds, connected with the theory of Thom spaces.* Dokl. AN SSSR **132:5** (1960), 1031-1034.

S.P. Novikov *Methods of algebraic topology from the viewpoint of cobordism theory.* Izv. Akad. Nauk SSSR, Ser. Mat. **31:4** (1967), 827–913.

V.M. Buchstaber *Chern-Dold character in cobordisms, I.* Math. Sbornik **83** (1970), 575-95.

V.M. Buchstaber, A.P. Veselov *Chern-Dold character in complex cobordisms and theta divisors.* arXiv 2007.05782 (2020).

Igor and Mika