Automorphic Forms Learning Seminar Notes

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These notes are essentially a summary of Goldfeld's *Automorphic Forms and L-functions for the Group* $GL(n, \mathbb{Z})$.

1 Maass forms and Whittaker functions for SL$(n, \mathbb{Z})$

- The spectral parameter approach follows Dorian's book. Supposedly useful in some analytic applications.
- The Langlands parameter approach follows the paper "A template method for Fourier coefficients of Langlands Eisenstein series" by Goldfeld, Miller, and Woodbury. Also known as Satake parameters. They come from automorphic representations at the Archimedean place.

1.1 Maass forms

- Recall that we are interested in SL$(n, \mathbb{Z})$ acting on
$$\mathfrak{h}^n = \mathrm{GL}(n, \mathbb{R})/(O(n, \mathbb{R}) \times \mathbb{R}^*).$$
- Recall that we can represent elements of
$$\mathfrak{h}^n = x \cdot y,$$
where x is an upper triangular matrix with 1s on the diagonal and $x_{i,j}$ off the diagonal, and y is a diagonal matrix with elements of the form $1, y_1, y_1 y_2, \ldots, y_1 y_2 \ldots y_{n-1}$.
- We will consider two parameterizations: the spectral parameters v and the Langlands parameters, denoted $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{C}^n$, with $\alpha_1 + \cdots + \alpha_n = 0$. (This is abuse of notation; we really refer to the set of the α_i.)
- Recall that we defined
$$b_{ij} = \begin{cases} ij & i+j \leq n \\ (n-i)(n-j) & i+j \geq n \end{cases}$$
The b_{ij} come from the inverse of the Cartan matrix for GL(n).
- We have the following relation between the two sets of parameters:
$$v_i = \frac{\alpha_i - \alpha_{i+1} + 1}{n},$$
and conversely
$$\alpha_i = \begin{cases} B_{n-1}(v) & i = 1 \\ B_{n-i}(v) - B_{n-i+1}(v) & 1 < i < n \\ -B_1(v) & i = n \end{cases},$$
where $B_j(s) = \sum_{i=1}^{n-1} b_{i,j}(v_i - 1/n)$.
- Example: For $n = 2$, $\alpha_1 = -\alpha_2 = v - \frac{1}{2}$. For $n = 3$, $\alpha_1 = 2v_1 + v_2 - 1$, $\alpha_2 = -v_1 + v_2$, and $\alpha_3 = -v_1 - 2v_2 + 1$.

- We have the center of the universal enveloping algebra of $\mathfrak{gl}(n,\mathbb{R})$ is $\mathfrak{D}^n$. For any $D \in \mathfrak{D}^n$ and $v = (v_1, \dots, v_{n-1})$, we know that
$$I_v(z) = \prod_{i=1}^{n-1}\prod_{j=1}^{n-1} y_i^{b_{i,j}v_j}$$
is an eigenfunction of every $D \in \mathfrak{D}^n$.
Alternatively, in terms of Langlands parameters, let $\rho_i = \frac{n+1}{2} - i$. We can express the power function as
$$I(z,\alpha) = \prod_{i=1}^{n}\left(\prod_{j=1}^{n-i} y_j\right)^{\alpha_i+\rho_i} = \prod_{i=1}^{n-i} y_i^{\sum_{j=1}^{n-i}(\alpha_j+\rho_j)}.$$
The eigenvalue is independent of the permutation of α. This will follow from the proof of the funcational equation for the Whittaker function.

- We express $DI_v(z) = \lambda_D I_v(z)$. Note that $\lambda_{D_1 \cdot D_2} = \lambda_{D_1} \cdot \lambda_{D_2}$, hence λ_D is a character of $\mathfrak{D}^n$, called the Harish-Chandra character.

- Maass form of type v for $\mathrm{SL}(n,\mathbb{Z})$: A function $\phi \in \mathcal{L}^2(\mathrm{SL}(n,\mathbb{Z})\backslash\mathfrak{h}^n)$ satsifying
  - $\phi(\gamma z) = \phi(z)$ for all $\gamma \in \mathrm{SL}(n,\mathbb{Z})$
  - $D\phi(z) = \lambda_D\phi(z)$ for all $D \in \mathfrak{D}^n$ for λ_D a Harish-Chandra character. (In particular, these are the same λ_D coming from $DI_v(z) = \lambda_D I_v(z)$, which is where the v condition is being used.)
  - $\int_{(\mathrm{SL}(n,\mathbb{Z})\cap U)\backslash U} \phi(uz)\,du = 0$ for all U that are matrices with diagonal matrices I_{r_i} on the diagonals and 0 below the diagonal.

  If the eigenvalues of ϕ agree with the eigenvalues of $I(\cdot,\alpha)$, then α are the Langlands parameters of ϕ.

- Remark: An alternative definition of a Maass (cusp) form replaces the $\mathcal{L}^2$ condition with a growth condition that
$$|\phi(xy)| \ll_N (y_1 \dots y_{n-1})^{-N}$$
for all $N > 0$; i.e. an exponential decay growth condition.

- For Laplace operator Δ, then if f is a Maass form, we have corresponding Laplace eigenvalue
$$\lambda_\Delta = \frac{n^3-n}{24} - \frac{\alpha_1^2 + \cdots + \alpha_n^2}{2}.$$
Generalized Ramanujan-Selberg conjecture: All Maass forms for $\mathrm{SL}(n,\mathbb{Z})$ (and congruence subgroups) are tempered, i.e. all α_i are purely imaginary. Compare this to the Ramanujan-Selberg conjecture for $n = 2$.

1.2 Whittaker functions associated to Maass forms

- Idea: Emulate the Fourier expansion in higher dimensions.
- Let $U_n(\mathbb{R})$ be the group of upper triangular $n \times n$ matrices.
- Let $m = (m_1, \dots, m_{n-1}) \in \mathbb{Z}^{n-1}$. We have a character $\psi_m : U_n(\mathbb{R}) \to \mathbb{C}^*$ sending
$$\psi_m(u) = e^{2\pi i(m_1u_{1,2} + m_2u_{2,3} + \cdots + m_{n-1}u_{n-1,n})}.$$
We have that $\psi_m(uv) = \psi_m(u)\psi_m(v)$.

- For Maass form ϕ, we want Fourier coefficients like

$$\widetilde{\phi}_m(z) = \int_0^1 \cdots \int_0^1 \phi(u \cdot z)\overline{\psi_m(u)} \prod_{1\le i<j\le n} \mathrm{d}u_{i,j}$$

  such that we can write

$$\phi = \sum_m \widetilde{\phi}_m(z).$$

  This is the analogue of $W(y)e^{2\pi imx}$ in the rank 2 case.

  Since $U_n(\mathbb{R})$ is non-Abelian, we need to be careful over which m we sum.

- Properties of the Fourier coefficients (which we will show later are Whittaker functions):
  - $\widetilde{\phi}_m(u \cdot z) = \psi_m(u)\widetilde{\phi}_m(z)$
  - $D\widetilde{\phi}_m = \lambda_D\widetilde{\phi}_m$ for all $D \in \mathfrak{D}^n$, where λ_D is a Harish-Chandra character
  - $\int_{\Sigma_{\frac{\sqrt{3}}{2},\frac{1}{2}}} |\widetilde{\phi}_m(z)|^2 \mathrm{d}^*z < \infty$.

- Proof of properties:
  - The substitution $u \mapsto u \cdot u'$, for $u' \in U_n(\mathbb{R})$, does not change the measure.
  - Follows by definition of a Maass form.
  - Follows from Cauchy-Schwarz, ϕ is automorphic, and that ϕ is L^2.

1.3 Fourier expansions on $SL(n, \mathbb{Z})$

- Every Maass form for $\mathrm{SL}(n, \mathbb{Z})$ has the Fourier expansion

$$\phi(z) = \sum_{\gamma\in U_{n-1}(\mathbb{Z})\backslash \mathrm{SL}(n-1,\mathbb{Z})} \sum_{m_1\neq 0} \sum_{m_2=1}^{\infty} \cdots \sum_{m_{n-1}=1}^{\infty} \widetilde{\phi}_{m_1,\ldots,m_{n-1}}\left(\begin{pmatrix}\gamma & \\ & 1\end{pmatrix} z\right).$$

  The sum is independent of the choice of representatives γ. Recall that

$$\widetilde{\phi}_{(m_1,\ldots,m_{n-1})}(z) = \int_0^1 \cdots \int_0^1 \phi(u \cdot z)e^{-2\pi i(m_1u_{1,2}+\cdots+m_{n-1}u_{n-1,n})}\mathrm{d}^*u$$

  where $\mathrm{d}^*u = \prod_{i=1}^n \mathrm{d}u_{i,i+1}$.

- Idea of proof: Inductively construct the Fourier expansion. Use standard Fourier expansion to get expansion with variables determined by the $n-1$ variables in the last column; i.e. let

$$v = \begin{pmatrix} 1 & & & & v_1 \\ & 1 & & & v_2 \\ & & \ddots & & \vdots \\ & & & 1 & v_{n-1} \\ & & & & 1 \end{pmatrix}$$

  and

$$\widehat{\phi}_m(z) = \int_0^1 \cdots \int_0^1 \phi(vz)e^{-2\pi\langle v,m\rangle}\mathrm{d}^*v.$$

  Since ϕ is periodic when multiplying by v, we get

$$\phi(z) = \sum_{m\in\mathbb{Z}^{n-1}} \widehat{\phi}_m(z).$$

  Rewrite this sum in terms of gcd of variables in last column and representatives of $\mathrm{SL}(n-1, \mathbb{Z})$ by $P(n-1, \mathbb{Z})$, where $P(n-1, \mathbb{Z})$ is matrices whose last row is e_{n-1}. Nothing corresponding to $m_{n-1} = 0$ because ϕ is a cuspform. Repeat inductively on all columns. m_1 also has negative coefficients because $\mathrm{SL}(1, \mathbb{Z})$ treats the orbits a and $-a$ separately.

1.4 Whittaker functions for $\mathbf{SL}(n,\mathbb{R})$

- A $\mathrm{SL}(n,\mathbb{Z})$ Whittaker function of type $v=(v_1,\dots,v_{n-1})\in\mathbb{C}^{n-1}$ associated to a character $\psi : U_n(\mathbb{R})\to\mathbb{C}$ is a smooth function $W:\mathfrak{h}^n\to\mathbb{C}$ such that
  - $W(uz)=\psi(u)W(z)$ for any $u\in U_n(\mathbb{R})$
  - $DW(z)=\lambda_D W(z)$ for any $D\in\mathfrak{D}^n$
  - $\int_{\Sigma_{\frac{\sqrt{3}}{2},\frac{1}{2}}}|W(z)|^2\mathrm{d}^*z<\infty$.
- In particular, note that the Fourier coefficient defined before

$$\widetilde{\phi}_m(z)=\int_0^1\dots\int_0^1\phi(u\cdot z)\overline{\psi}(u)\prod_{1\le i<j\le n}\mathrm{d}u_{i,j}$$

  is a Whittaker function.

1.5 Jacquet's Whittaker function

- Goal: Construct non-trivial (zero nowhere) Whittaker functions for rank n.
- Let $m=(m_1,\dots,m_{n-1})$ (corresponding to Fourier frequency) and $v=(v_1,v_2,\dots,v_{n-1})$ corresponding to the type of the Maass form. Alternatively, let $\alpha=(\alpha_1,\dots,\alpha_n)$ be the Langlands parameters.
- Notation: for the upper triangular matrix u, we denote

$$u_i=u_{n-i,n-i+1}.$$

- Let $\psi_m : U_n(\mathbb{R})\to\mathbb{C}$ to be the character

$$\psi_m(u)=e^{2\pi i(m_1u_1+\dots+m_{n-1}u_{n-1})}.$$

  Note that this has the reverse coefficients as expected in Section 5.3; i.e. the summation for Fourier expansion will be reversed.
- Let $z=xy$ and suppose all of the m_i are nonzero, we define the Jacquet Whittaker function $\mathfrak{h}^n\to\mathbb{C}$

$$W(z;v,\psi_m)=\int_{U_n(\mathbb{R})}I_v(wuz)\overline{\psi_m(u)}\mathrm{d}^*u,$$

  where

$$w=\begin{pmatrix} & & & 1\\ & & 1 & \\ & \iddots & & \\ 1 & & & \end{pmatrix}$$

  and the integral is integrated with respect to all $u_{i,j}$ from $-\infty$ to ∞.

  In terms of Langlands parameters, we have

$$W_\alpha(z)=\int_{U_n(\mathbb{R})}I(wuz,\alpha)\overline{\psi_m(u)}\mathrm{d}^*u.$$

  Note that this exactly matches the construction for $\mathbb{H}$.
- Remark: Dorian's book uses w_n ($-1^{\lfloor n/2\rfloor}$ in the top right corner), the long element of the Weyl group. This is equivalent because of the wedge product definition of $I_v(s)$, using that $e_jw_n=e_jw$ for all $j>1$. We will use the original definition to show the functional equation.

- If $\text{Re}(v_i) > 1/n$ for all i and $m_i \neq 0$ for all i, then:
  - W converges absolutely and uniformly on compact subsets of $\mathfrak{h}^n$
  - W has meromorphic continuation for all $v \in \mathbb{C}^{n-1}$
  - W is an $SL(n,\mathbb{Z})$-Whittaker function of type v and character ψ_m
  -
  $$W(z;v,\psi_m) = c_{v,m} W(Mz; v, \psi_{m_1/|m_1|,\ldots,m_{n-1}/|m_{n-1}|}) = c_{v,m}\psi_m(x) W(My; v, \psi_{1,\ldots,1}),$$
  where
  $$c_{v,m} = \prod_{i=1}^{n-1} |m_i|^{\left(\sum_{j=1}^{n-1} b_{i,j} v_j\right) - i(n-i)}$$
  and
  $$M = \begin{bmatrix} |m_1 m_2 \ldots m_{n-1}| & & & \\ & \ddots & & \\ & & |m_1| & \\ & & & 1 \end{bmatrix}.$$
- Remark: From the Fourier expansion, we only care about m where $m_1, \ldots, m_{n-2}$ are positive. By the above properties, it is sufficient to care only about $m = (1, \ldots, 1, \pm 1)$.
- Proof idea:
  - Proof that W is a Whittaker function:
    * $W(az) = \psi_m(a) W(z)$: Change of variable. Also proves part of second equation of fourth point.
    * $DW = \lambda_D W$: Use that $DI_v = \lambda_D I_v$.
    * Assume that integral converges absolutely and uniformly on compacts to an $\mathcal{L}^2$ function.
  - Proof of first equation of fourth point: Make the changes of variables in the integral from $wuMz$ to $wMuz$ to $wMw \cdot wuz$. This gives the correct constant $c_{v,m}$; see Broughan 2009, Theorem 6.1 for more details.
  - Proof of second equation of fourth point: Let δ_j be the identity matrix, except with $\varepsilon_j = m_j/|m_j|$ at the $n-j$th row. Replace u with $\delta_j \delta_j u$, then do a change of variable from $u \to \delta_j u$ and use that $\delta_{n-j} w = w \delta_j$, and since all the matrices in the integral are diagonal and $\delta_{n-j} \in O(n,\mathbb{R})$, δ_{n-j} can be ignored. Repeat for all j.
  - Proof of absolute convergence/meromorphic continuation for $n = 2$: Absolute convergence follows from computing the integral, which converges for $\text{Re}(v) > 1/2$. Meromorphic continuation: Follows from $K_v = K_{-v}$.

1.6 The exterior power of a vector space

- Let $\otimes^\ell(\mathbb{R}^n)$ be the space of ℓth tensor products of the vector space $\mathbb{R}^n$. Formally, we define
  $$\Lambda^\ell(\mathbb{R}^n) = \otimes^\ell(\mathbb{R}^n)/\mathfrak{a}_\ell$$
  where $\mathfrak{a}_\ell$ is the vector subspace generated by all elements $v_1 \otimes \cdots \otimes v_\ell$ where $v_i = v_j$ for some $i \neq j$.
- In other words, we have the set of $v_1 \wedge \cdots \wedge v_\ell$ with the rules $v \wedge v = 0$, $v \wedge w = -w \wedge v$, and $(a_1 v_1 + a_2 v_2) \wedge w = a_1 v_1 \wedge w + a_2 v_2 \wedge w$.
- On $\otimes^\ell(\mathbb{R}^n)$, we have the (canonical) inner product
  $$\langle v, w \rangle_{\otimes^\ell} = \prod_{i=1}^{\ell} \langle v_i, w_i \rangle.$$

- Let $e_1, \dots, e_n$ be the canonical basis for $\mathbb{R}^n$. Then letting
$$a = \sum_{1 \le i_1, \dots, i_\ell \le n} a_{i_1,\dots,i_\ell} e_{i_1} \wedge \cdots \wedge e_{i_\ell},$$
we define $\phi_\ell : \Lambda^\ell(\mathbb{R}^n) \to \otimes^\ell(\mathbb{R}^n)$ such that
$$\phi_\ell(a) = \frac{1}{\ell!} \sum_{1 \le i_1, \dots, i_\ell \le n} a_{i_1,\dots,i_\ell} \sum_{\sigma \in S_\ell} \mathrm{Sign}(\sigma) \cdot e_{i_{\sigma(1)}} \otimes \cdots \otimes e_{i_{\sigma(\ell)}}.$$
This is a well-defined injection, and hence $\Lambda^\ell(\mathbb{R}^n)$ can be viewed as a subspace of $\otimes^\ell(\mathbb{R}^n)$. We then define the inner product on Λ^ℓ to be
$$\langle v, w \rangle_{\Lambda^\ell} = \langle \phi_\ell(v), \phi_\ell(w) \rangle_{\otimes^\ell}.$$
- We define the action of $\mathrm{SL}(n, \mathbb{R})$ on $\Lambda^\ell(\mathbb{R}^n)$ via
$$v \circ g = (v_1 \cdot g) \wedge \cdots \wedge (v_\ell \cdot g)$$
and similarly for $\otimes^\ell$.
- For $k \in O(n, \mathbb{R})$, $\langle v, w \rangle_{\Lambda^\ell} = \langle v \circ k, w \circ k \rangle_{\Lambda^\ell}$, and $\|v\| = \sqrt{\langle v, v \rangle_{\Lambda^\ell}} = \|v \circ k\|$. Proof: Prove the same properties from $\otimes^\ell$, and then apply ϕ_ℓ.
- For any upper triangular matrix u,
$$(e_{n-\ell} \wedge \cdots \wedge e_n) \circ u = e_{n-\ell} \wedge \cdots \wedge e_n.$$
- Cauchy-Schwarz: $|\langle v, w \rangle_{\Lambda^\ell}|^2 \le \langle v, v \rangle_{\Lambda^\ell} \cdot \langle w, w \rangle_{\Lambda^\ell}$, and $\|v \wedge w\|_{\Lambda^\ell} \le \|v\|_{\Lambda^\ell} \|w\|_{\Lambda^\ell}$. Proof: Use that $\|v\|^2_{\Lambda^\ell} = \sum_{i_1,\dots,i_\ell} |a_{i_1,\dots,i_\ell}|^2$, and apply normal Cauchy-Schwarz.

1.7 Construction of the I_v function using wedge products

- We can write
$$I_v(z) = \left(\prod_{i=0}^{n-2} \|(e_{n-i} \wedge \cdots \wedge e_n) \circ z\|^{-nv_{n-i-1}} \right) \cdot |\det(z)|^{\sum_{i=1}^{n-1} i v_{n-i}},$$
and hence we can write W in terms of a wedge product.
- Check that operations inside are invariant under $SO(n, \mathbb{R})$ and $\mathbb{R}^*$, so the operation is well-defined. Moreover, use that x is upper triangular to get that $I_v^*(z) = I_v^*(y)$. Finish by doing the computation on y.
- Why is this helpful? Shows that we can choose to use w instead of w_n in the definition of the Jacquet-Whittaker function. Also can be used to explicitely compute the Whittaker function for the $\mathrm{SL}(n, 3)$ case:

$$W(y; v, \psi_m) = y_1^{v_1+2v_2} y_2^{2v_1+v_2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (y_1^2 y_2^2 + u_1^2 y_2^2 + (u_1 u_2 - u_3)^2)^{-3v_1/2} \cdot (y_1^2 y_2^2 + u_2^2 y_1^2 + u_3^2)^{-3v_2/2} e^{-2\pi i (m_1 u_1 + m_2 u_2)} \, du_1 \, du_2 \, du_3$$

1.8 Convergence of Jacquet's Whittaker function

- This section is incorrect; equation 5.8.2 is the wrong direction.
- Heuristic: The integral will be on the order of something like the product of
$$\int (1 + u_{j,j+1}^2 + \cdots + u_{j,n}^2)^{-\frac{n}{2} \sum_{i=1}^{n-j} \mathrm{Re} v_i} \, du$$
for all j, and the integral converges if $\frac{n}{2} \sum_{i=1}^{n-j} \mathrm{Re} v_i > \frac{n-j}{2}$, or if $\sum_{i=1}^{n-j} \mathrm{Re} v_i > \frac{n-j}{n}$ for all j. Hence the convergence is for $\mathrm{Re} v_j > \frac{1}{n}$.

1.9 Functional equations of Jacquet's Whittaker function

- Everything stated here will be in terms of Langlands parameters, but it is possible to translate everything in terms of spectral parameters. It just is really annoying.
- Multiplicity one: Due to Shalika, there is only one Whittaker function of type v and character ψ up to constant multiple (implied here is the growth condition).
- Because in the Fourier expansion, we care only when $m_1, \dots, m_{n-2} > 0$, and we can relate $W(z; v, \psi_m)$ to $W(Mz; v, \psi_{1,\dots,1,\pm 1})$ by a constant, it suffices only to consider the following normalized Whittaker functions:
$$W_\alpha^\pm(z) = \prod_{1 \le j < k \le n} \frac{\Gamma\left(\frac{1+\alpha_j-\alpha_k}{2}\right)}{\pi^{\frac{1+\alpha_j-\alpha_k}{2}}} \int_{U_n(\mathbb{R})} I(wug, \alpha)\overline{\psi_{1,\dots,1,\pm 1}(u)} \mathrm{d}^* u.$$
- Properties: It is an absolutely convergent integral for $\mathrm{Re}(\alpha_i - \alpha_{i+1}) > 0$ and has holomorphic continuation to all $\alpha \in \mathbb{C}^n$ with $\sum \alpha_i = 0$.
- Functional equation: For any permutation α' of α,
$$W_\alpha^\pm = W_{\alpha'}^\pm.$$
In other words, there is no abuse of notation regarding the Langlands parameters.
- Proof idea: It suffices to consider adjacent swaps of α_i. Let α' be the permutation of α swapping α_i and α_{i+1}. Consider
$$\sigma_i = \begin{pmatrix} I_{n-i-1} & & & \\ & 0 & 1 & \\ & 1 & 0 & \\ & & & I_{i-1} \end{pmatrix}.$$
Letting $w_i = \sigma_i^{-1} w$, every $u \in U_n(\mathbb{R})$ can be written in the form
$$u = (w_i^{-1} n_i w_i) n_i',$$
where $n_i \in N_i$ is the set of matrices with 1s on the diagonal, real number at position $(n-i, n-i+1)$, and zeros elsewhere, and $n_i' \in N_i'$ is the subgroup of $U_n(\mathbb{R})$ with a zero at the position $(i, i+1)$. Hence, we can write
$$W_\alpha^\pm(z) = \int_{U_n(\mathbb{R})} I_\alpha(wug)\overline{\psi}(u) \mathrm{d}^* u = \int_{N_i'} \left(\int_{N_i} I_\alpha(\sigma_i n_i (w_i n_i' z)) \overline{\psi}(n_i) \, \mathrm{d}n_i \right) \overline{\psi}(n_i') \, \mathrm{d}n_i' \, .$$
The inner integral is a Whittaker function over $\mathrm{SL}(2, \mathbb{R})$, whose function equation is independent of choice of $w_i n_i' z$, and we remark that
$$I_\alpha(\sigma_i n_i) = (u^2 + 1)^{\frac{1}{2}(\alpha_i - \alpha_{i+1})},$$
where u is the nonzero element of n_i. This thus resembles the Bessel function, and swapping α_i and α_{i+1} changes α to α' and changes by the requisite constant.

1.10 Degenerate Whittaker functions

- It is possible to construct Whittaker using other elements of the Weyl group (elements of $\mathrm{SL}(n, \mathbb{Z})$ with exactly one 1 or -1 in each row or column) instead of w_n, as the only key property used was that I_v was an eigenfunction.
- However, these Whittaker functions will not contain all of the $u_{i,j}$. You can define the Whittaker functions by integrating only over the variables that appear.