Automorphic Forms Learning Seminar Notes

Austin Lei

Fall 2024-Spring 2025

These notes are essentially a summary of Goldfeld's *Automorphic Forms and L-functions for the Group* $GL(n, \mathbb{Z})$.

1 The Godement-Jacquet L-function

1.1 Maass forms for SL$(n, \mathbb{Z})$

- Recall that we showed that a Maass form ϕ had a Fourier expansion

$$\phi(z) = \sum_{\gamma \in U_{n-1}(\mathbb{Z}) \backslash \mathrm{SL}(n-1,\mathbb{Z})} \sum_{m_1=1}^{\infty} \cdots \sum_{m_{n-2}=1}^{\infty} \sum_{m_{n-1} \neq 0} \widetilde{\phi}_{(m_1,\ldots,m_{n-1})} \left(\begin{pmatrix} \gamma & \\ & 1 \end{pmatrix} z \right),$$

  where

$$\widetilde{\phi}_{(m_1,\ldots,m_{n-1})}(z) = \int_0^1 \cdots \int_0^1 \phi(u \cdot z) \overline{\psi_m(u)} \mathrm{d}^* u,$$

  where here $\mathrm{d}^* u$ is an integral over all the u_{ij} in the unipotent matrix u.

- We know that $\widetilde{\phi}_{(m_1,\ldots,m_{n-1})}$ is a Whittaker function. By the multiplicity one theorem for SL$(n, \mathbb{Z})$ and identities about Whittaker functions, we can write the Fourier expansion in the form

$$\phi(z) = \sum_{\gamma \in U_{n-1}(\mathbb{Z}) \backslash \mathrm{SL}(n-1,\mathbb{Z})} \sum_{m_1=1}^{\infty} \cdots \sum_{m_{n-2}=1}^{\infty} \sum_{m_{n-1} \neq 0} \frac{A(m_1, \ldots, m_{n-1})}{\prod_{k=1}^{n-1} |m_k|^{k(n-k)/2}} W_\alpha \left(M \begin{pmatrix} \gamma & \\ & 1 \end{pmatrix} z, \psi_{1,\ldots,1,\frac{m_{n-1}}{|m_{n-1}|}} \right),$$

  where the $A(m_1, \ldots, m_{n-1}) \in \mathbb{C}$ are the Fourier coefficients. The choice of normalization comes from the constant in the identity

$$W_\alpha(z, \psi_m) = c_{\alpha,m} W_\alpha(Mz, \psi_{1,\ldots,1,\pm 1});$$

  $c_{\alpha,m}$ is of the form $c_\alpha \prod_{k=1}^{n-1} |m_k|^{-k(n-k)/2}$, where c_α depends only on α and n.

- Now, since

$$\frac{A(m_1, \ldots, m_{n-1})}{\prod_{k=1}^{n-1} |m_k|^{k(n-k)/2}} W_\alpha \left(My, \psi_{1,\ldots,1,\frac{m_{n-1}}{|m_{n-1}|}} \right) = \int_0^1 \cdots \int_0^1 \phi(z) \overline{\psi_m(x)} \mathrm{d}^* x,$$

  fixing some choice constant choice of y and using that ϕ is bounded, we conclude that the

$$\frac{A(m_1, \ldots, m_{n-1})}{\prod_{k=1}^{n-1} |m_k|^{k(n-k)/2}}$$

  are bounded.

1.2 Dual and symmetric Maass forms

- Symmetric Maass forms: Let δ be a matrix of the form

$$\begin{pmatrix} \delta_1 \ldots \delta_{n-1} & & & & \\ & \ddots & & & \\ & & \delta_1 \delta_2 & & \\ & & & \delta_1 & \\ & & & & 1 \end{pmatrix}$$

where each of the δ_i is ± 1. We define the operator

$$T_\delta \phi(z) = \phi(\delta z) = \phi(\delta z \delta),$$

which takes xy to $x'y$, where x' consists of $\delta_i x_i$ on the offdiagonal.

A Maass form is symmetric if $T_\delta \phi = \pm\phi$ for all T_δ, where the choice of $\pm$ depends on δ.

- By linear algebra, every Maass form is a linear combination of symmetric Maass forms, so it suffices to only consider aymmetric Maass forms.

- Let
$$\delta_0 = \begin{pmatrix} -1 & & & & \\ & 1 & & & \\ & & \ddots & & \\ & & & 1 & \\ & & & & 1 \end{pmatrix}.$$
We say that ϕ is even if $T_{\delta_0}\phi = \phi$, and odd if $T_{\delta_0}\phi = -\phi$.

- If n is odd, then all Maass forms are even. In particular, this means that
$$A(m_1, \ldots, m_{n-2}, m_{n-1}) = -A(m_1, \ldots, m_{n-2}, -m_{n-1}).$$
Proof idea: ϕ is invariant by $\mathrm{SL}(n, \mathbb{Z})$ and $-I_n$, which has determinant -1, so $T_\delta \phi = \phi$ for all δ. Then looking at T_{δ_0}, only the x_{n-1} variable changes to $-x_{n-1}$, so the corresponding coefficients that must match are $A(m_1, \ldots, m_{n-1})$ and $A(m_1, \ldots, -m_{n-1})$.

- If n is even, then for a symmetric Maass form, we have
$$A(m_1, \ldots, m_{n-2}, m_{n-1}) = \pm A(m_1, \ldots, m_{n-2}, -m_{n-1}),$$
depending on if ϕ is odd or even.

  Proof idea: The δ matrices are determined by $\mathrm{SL}(n, \mathbb{Z})$ and T_{δ_0}; hence ϕ is symmetric iff $T_{\delta_0}\phi = \pm\phi$. (Typo on page 266.)

  Looking at T_{δ_0}, only the x_{n-1} variable changes to $-x_{n-1}$, so the corresponding coefficients when looking at that must match are $A(m_1, \ldots, m_{n-1})$ and $A(m_1, \ldots, -m_{n-1})$, with a $\pm$ depending on the sign of $T_{\delta_0}\phi = \pm\phi$; i.e. when ϕ is even or odd.

- Using the T_δ, we can interpret $A(m_1, \ldots, m_{n-1})$ for symmetric Maass forms, even when the m_i are negative. Note in particular that letting δ be the matrix with the $\delta_i = \frac{m_i}{|m_i|}$, then we can take $A(m_1, \ldots, m_{n-1})$ to be the $(|m_1|, \ldots, |m_{n-1}|)$-th coefficient of $T_\delta \phi$, multiplied by $\det \delta$ if ϕ is odd and 1 if ϕ is even.

- Dual Maass form: (Typos in book.) For ϕ a Maass form of Langlands parameters α, then we define the dual Maass form
$$\widetilde{\phi}(z) = \phi((z^{-1})^T),$$
a Maass form of Langlands parameters $\alpha' = (-\alpha_n, -\alpha_{n-1}, \ldots, -\alpha_1)$. Moreover, if ϕ is symmetric, then the $(m_1, \ldots, m_{n-1})$th coefficient of $\widetilde{\phi}$ is
$$\pm A(m_{n-1}, \ldots, m_1),$$
where the sign is $(-1)^{n-1+\lfloor n/2 \rfloor}$ if ϕ is odd, and 1 if ϕ is even. Equivalently, the sign is -1 if $4 \mid n$ and ϕ is odd, and 1 otherwise.

- Proof idea: $\widetilde{\phi}(\gamma z) = \widetilde{\phi}(z)$ because $\mathrm{SL}(n,\mathbb{Z})$ is preserved by inverse transpose.

  By $\mathrm{SL}(n,\mathbb{Z})$ left invariance and $O(n,\mathbb{R})$ right invariance, we have that

$$\widetilde{\phi}(z) = \phi(w(z^{-1})^T w^{-1}),$$

  where

$$w = \begin{pmatrix} & & & (-1)^{\lfloor n/2 \rfloor} \\ & & 1 & \\ & \dots & & \\ 1 & & & \end{pmatrix}$$

  is the long element of the Weyl group for $\mathrm{SL}(n,\mathbb{Z})$. In particular, we note that $w(z^{-1})^T w^{-1}$ has a Iwasawa decomposition of the form $x'y'$, where y' consists of the elements $y_1 \dots y_{n-1}, y_2 \dots y_{n-1}, \dots, y_{n-1}, 1$, and x' consists of 1s on the diagonal, $-x_i$ on the offdiagonal for $2 \le i \le n-1$ and $(-1)^{\lfloor n/2 \rfloor + 1} x_1$. One can show by induction (on what the remaining x' terms look like) that

$$\int_{\mathrm{SL}(n,\mathbb{Z}) \cap U) \backslash U} \widetilde{\phi}(z) \,\mathrm{d}u = 0$$

  for all U from the cuspidal condition.

  In particular, note that the y' reverses the order of the y_i, which in Langlands parameters corresponds to reversing the order of the parameters and negating them. Thus $I_{\alpha'}(z) = I_\alpha(w(z^{-1})^T w^{-1})$, and thus $\widetilde{\phi}$ has Langlands parameters α'.

  Finally, the $(m_1, \dots, m_{n-1})$th coefficient of $\widetilde{\phi}$, by the work above, computed by looking at

$$\int_0^1 \cdots \int_0^1 \widetilde{\phi}(z) e^{-2\pi i (m_1 x_1 + \cdots + m_{n-1} x_{n-1})} \mathrm{d}^* x$$

  corresponds to $A(-m_{n-1}, -m_{n-2}, \dots, -m_2, (-1)^{\lfloor n/2 \rfloor + 1} m_1)$ of ϕ, giving the desired result.

1.3 Hecke operators for $\mathrm{SL}(n,\mathbb{Z})$

- Recall the Hecke theory discussed in Section 3.10. Let $G = \mathrm{GL}(n,\mathbb{R})$, $\Gamma = \mathrm{SL}(n,\mathbb{Z})$, and $X = \mathfrak{h}^n$. In this case, we have the that the matrices

$$\begin{pmatrix} m_0 \dots m_{n-1} & & & \\ & \ddots & & \\ & & m_0 m_1 & \\ & & & m_0 \end{pmatrix},$$

  where the m_i are all positive integers lies in the commensurator. Let Δ be the semigroup generated by all of these matrices. Since Δ is preserved under transpose, the Hecke ring generated by Δ is commutative.

- Define the set

$$S_n = \left\{ \begin{pmatrix} c_1 & c_{1,2} & \dots & c_{1,n} \\ & c_2 & \dots & c_{2,n} \\ & & \ddots & \vdots \\ & & & c_n \end{pmatrix} \mid c_\ell \ge 1, \prod_{\ell=1}^n c_\ell = N, 0 \le c_{i,\ell} < c_\ell \right\}.$$

  Then we have the double coset partition

$$\bigcup_{m_0^n m_1^{n-1} \dots m_{n-1} = N} \Gamma \begin{pmatrix} m_0 \dots m_{n-1} & & & \\ & \ddots & & \\ & & m_0 m_1 & \\ & & & m_0 \end{pmatrix} \Gamma = \bigcup_{\alpha \in S_N} \Gamma \alpha.$$

  Proof idea: Use Hermite and Smith normal form.

- We can use the S_N to define Hecke operators:

$$T_N f(z) = \frac{1}{N^{\frac{n-1}{2}}} \sum_{\substack{\prod_{\ell=1}^n c_\ell = N \\ 0 \le c_{i,\ell} < c_\ell}} f\left(\begin{pmatrix} c_1 & c_{1,2} & \cdots & c_{1,n} \\ & c_2 & \cdots & c_{2,n} \\ & & \ddots & \vdots \\ & & & c_n \end{pmatrix} \cdot z \right).$$

- With respect to the standard inner product, the Hecke operators are no longer self-adjoint. However, the adjoint of a Hecke operator is itself a Hecke operator, and commutes with the original Hecke operator, so the operator is normal. Mathematically, we have that T_N^* is associated to the double coset union

$$\bigcup_{m_0^n m_1^{n-1} \ldots m_{n-1} = N} \Gamma \begin{pmatrix} N \cdot m_0^{-1} & & & \\ & N \cdot (m_0 m_1)^{-1} & & \\ & & \ddots & \\ & & & N \cdot (m_0 \ldots m_{n-1})^{-1} \end{pmatrix} \Gamma.$$

Proof idea: Apply a change of variable in the integral for $\langle T_N f, g \rangle$.

- To form the full Hecke ring, we take ring of operators generated by the T_δ, the $\mathrm{GL}(n, \mathbb{R})$-invariant differential operators, and then T_n.

- As expected, we can use the Hecke operators to gain information about the Fourier coefficients of Hecke-eigen Maass forms.

- For a Heckeeigen Maass form ϕ, if $A(1, \ldots, 1) = 0$, then $\phi = 0$. Otherwise, choosing to normalize to let $A(1, \ldots, 1) = 1$, we have that

$$T_m \phi = A(m, 1, \ldots, 1)\phi,$$

with multiplicativity relations

$$A(m_1, \ldots, m_{n-1}) A(m_1', \ldots, m_{n-1}') = A(m_1 m_1', \ldots, m_{n-1} m_{n-1}')$$

if $\gcd(m_1 \cdots m_{n-1}, m_1' \cdots m_{n-1}') = 1$, and

$$A(m, 1, \ldots, 1) A(m_1, \ldots, m_{n-1}) = \sum_{\substack{\prod_{\ell=1}^n c_\ell = m \\ c_i | m_i}} A\left(\frac{m_1 c_n}{c_1}, \frac{m_2 c_1}{c_2}, \ldots \frac{m_{n-1} c_{n-2}}{c_{n-1}} \right).$$

- Proof idea: Manually compute what the $(m_1, \ldots, m_{n-1})$ coefficient of $T_m \phi$ looks like as a sum over the c matrices. Do a change of variable to swap from cx to $x'c$ and working through the relationship gives the equation

$$\lambda_m A(m_1, \ldots, m_{n-1}) = \sum_{\substack{\prod_{\ell=1}^n c_\ell = m \\ c_i | m_i}} A\left(\frac{m_1 c_n}{c_1}, \frac{m_2 c_1}{c_2}, \ldots \frac{m_{n-1} c_{n-2}}{c_{n-1}} \right).$$

First assume $A(1, \ldots, 1) \neq 0$, with normalization $A(1, \ldots, 1) = 1$. We directly get that $T_m \phi = A(m, 1, \ldots, 1)\phi$ and the relation

$$A(m, 1, \ldots, 1) A(m_1, \ldots, m_{n-1}) = \sum_{\substack{\prod_{\ell=1}^n c_\ell = m \\ c_i | m_i}} A\left(\frac{m_1 c_n}{c_1}, \frac{m_2 c_1}{c_2}, \ldots \frac{m_{n-1} c_{n-2}}{c_{n-1}} \right).$$

Now, to prove multiplicativity, we use the above relation. One can inductively show

$$A(p^{K_1} m_1, p^{K_2} m_2, \ldots, p^{K_{n-1}} m_{n-1}) = A(p^{K_1}, p^{K_2}, \ldots, p^{K_{n-1}}) A(m_1, m_2, \ldots, m_{n-1})$$

for $p \nmid m_i$ by applying the relation to

$$A(p^{K_0}, 1, \dots, 1) A(p^{K_1} m_1, p^{K_2} m_2, \dots, p^{K_{n-1}} m_{n-1}),$$

first proving it for p^{K_1} only, then p^{K_1} and p^{K_2}, etc. See Goldfeld's paper for more details.

Now, if $A(1, \dots, 1) = 0$, then all of the $A(m, 1, \dots, 1)$ are 0, and inductively using the prime power idea, one can show that all of the $A(m_1, \dots, m_{n-1})$ are 0.

1.4 The Godement-Jacquet L-function

- Given a Hecke-eigen Maass form f, we have the Godement-Jacquet L-function

$$L_f(s) = \sum_{m=1}^{\infty} A(m, 1, \dots, 1) m^{-s},$$

absolutely convergent for $\mathrm{Re}(s) > \frac{n+1}{2}$.

- We have an Euler product

$$L_f(s) = \prod_p \phi_p(s),$$

where

$$\phi_p(s) = \sum_{k=0}^{\infty} \frac{A(p^k, 1, \dots, 1)}{p^{ks}}.$$

By using the multiplicativity relations, one can show that

$$\phi_p(s) = \Big(1 - A(p, \dots, 1) p^{-s} + A(1, p, \dots, 1) p^{-2s} - \cdots + (-1)^{n-1} A(1, \dots, p) p^{-(n-1)s} + (-1)^n p^{-ns}\Big)^{-1}$$

- For GL(2), taking the Mellin transform of the Maass form along the y-axis gives the L-function, which we then can use to get the functional equation. We want to do something similar for GL(n), but the $\begin{pmatrix} \gamma & \\ & 1 \end{pmatrix}$ terms cause issues.

- Inductively, we can show that

$$\int_0^1 \cdots \int_0^1 f(\widehat{u} z) e^{-2\pi i (u_1 + \cdots + u_{n-2})} \mathrm{d}^* \widehat{u}$$
$$= \sum_{m \neq 0} \frac{A(1, \dots, 1, m)}{|m|^{\frac{n-1}{2}}} e^{2\pi i m x_{n-1}} e^{2\pi i (x_1 + \cdots + x_{n-2})} W_\alpha \left(\begin{pmatrix} |m| & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix} y, \psi_{1,\dots,1} \right),$$

where $\widehat{u}$ is an integral over a unipotent matrix, except for the u_n element (top left offdiagonal element). The induction is by column (right to left).

- Taking the Mellin transform of this formula gives the L function (of the dual modular form, possibly with a minus sign), along with the integral of a Whittaker function. For $n = 3$, we can compute explicitely everything in terms of Gamma functions and Bessel functions to compute the functional equation.

- For higher n, we will get the functional equation from the functional equation of the Eisenstein series.