Automorphic Forms Learning Seminar Notes

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These notes are essentially a summary of Goldfeld's *Automorphic Forms and L-functions for the Group $GL(n, \mathbb{Z})$*.

Contents

1 Discrete Group Actions

1.1 Action of a Topological Space

- Left group action of G on X: continuous if $x \to g \circ x$ is continuous for all g. We denote the set of orbits $G\backslash X$ (right cosets).
- $\Gamma \subseteq G$ is discrete if for any compact K, there exists finitely many $\gamma \in \Gamma$ such that $\gamma K \cap K \neq \emptyset$.
- SL$(2,\mathbb{Z})$ is a discrete subgroup of SL$(2,\mathbb{R})$.
- $\Gamma_\infty = \left\{ \begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix} | m \in \mathbb{Z} \right\}$.

- Proof that $\mathrm{SL}(2,\mathbb{Z})$ is discrete: Finitely many $\gamma \in \Gamma_\infty \backslash \mathrm{SL}(2,\mathbb{Z})$ such that a rectangle intersects itself after translation by Γ. Multiplication by something in Γ_∞ corresponds to translation, and only finitely many possible translate can hit the same rectangle.
- Standard action of $\mathrm{SL}(2,\mathbb{Z})$ on $\mathbb{H}$, and fundamental domain.

1.2 Iwasawa Decomposition

- Iwasawa decompostion for $\mathrm{GL}(2,\mathbb{R})$: We can express
$$g = zkd,$$
where z is upper triangular with 1 in the lower right corner, k is orthogonal, and d is diagonal with the same entry along the diagonal. k and d are unique up to multiplication by $\pm I$, and z is unique.
- Generalized upper half-plane: $\mathfrak{h}^n$ is the set of all matrices in $\mathrm{GL}(n,\mathbb{R})$ of the form xy, where x is upper triangular with 1s on the diagonal, and
$$y = \begin{pmatrix} y_1y_2\cdots y_{n-1} & & & & \\ & y_1y_2\cdots y_{n-2} & & & \\ & & \ddots & & \\ & & & y_1 & \\ & & & & 1 \end{pmatrix}.$$
- $\mathfrak{h}^3$ does not have complex structure, compared to $\mathfrak{h}^2$. This is what makes $\mathrm{GL}(n)$ automorphic forms different.
- Iwasawa decomposition for $\mathrm{GL}(n)$: We have that
$$\mathrm{GL}(n,\mathbb{R}) = \mathfrak{h}^n O(n,R) Z_n,$$
where Z_n is the center of $\mathrm{GL}(n,\mathbb{R})$, i.e. diagonal matrices with everything the same along the diagonal. Letting $g = zkd$ be the decomposition, k and d are unique up to multiplication by $\pm I$, and z is unique. Hence
$$\mathfrak{h}^n \cong \mathrm{GL}(n,\mathbb{R})/(O(n,\mathbb{R})\mathbb{R}^*),$$
defining an action of $\mathrm{GL}(n,\mathbb{R})$ (and $\mathrm{GL}(n,\mathbb{Z})$) on $\mathfrak{h}^n$.
- Proof of decomposition: explicit computation involving factoring gg^T in terms of upper and lower triangular matrices.

1.3 Siegel Sets

- Siegel set: $\Sigma_{a,b} \subseteq \mathfrak{h}^n$ is the set of $z = x \cdot y \in \mathfrak{h}^n$ such that $|x_{i,j}| \leq b$ and $y_i > a$.
- $\Gamma^n = \mathrm{GL}(n,\mathbb{Z})$ acts discretely on $\mathfrak{h}^n$. In particular, for any $z \in \mathfrak{h}^n$, there are only finitely many $g \in \Gamma^n$ such that $gz \in \Sigma_{\sqrt{3}/2,1/2}$. In fact, we can write
$$\mathrm{GL}(n,\mathbb{R}) = \bigcup_{g\in\Gamma^n} g\Sigma_{\sqrt{3}/2,1/2}.$$
Hence $\Sigma_{\sqrt{3}/2,1/2}$ serves as a "good approximation" for a fundamental domain for $\mathfrak{h}^n$.
- Proof idea: Reduce to $\mathrm{SL}(n,\mathbb{R})$ and $\mathrm{SL}(n,\mathbb{Z})$. Show that if $\phi(\gamma z)$ is minimized for $\gamma \in \mathrm{SL}(n,\mathbb{Z})$, where ϕ is the norm of the last row (which exists because $\mathrm{SL}(n,\mathbb{Z})$ is a lattice) then $\gamma z \in \Sigma^*_{\sqrt{3}/2,1/2}$ (determinant 1 version). This proves the cover of $\mathrm{GL}(n,\mathbb{R})$ by elements of $\Sigma_{\sqrt{3}/2,1/2}$.

- Proof of discreteness of action: We show that there are finitely many $\gamma \in \mathrm{SL}(n,\mathbb{Z})$ such that $\gamma z \in \Sigma^*_{\sqrt{3}/2,1/2}$. (This is good enough because $\mathrm{GL}(n,\mathbb{Z})/\mathrm{SL}(n,\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}$.) Define $\phi_i(z) = \|e_i\gamma z\|$, well-defined on $\mathrm{SL}(n,\mathbb{R})/\mathrm{SO}(n,\mathbb{R})$. Letting $z = xy$ and explicitely computing $\phi_i(z)$ shows that $\gamma z \in \Sigma^*_{\sqrt{3}/2,1/2} \implies \phi_i(z)$ bounded, hence since $e_i\gamma z$ has lattice structure, there are only finitely many ith rows of γ so that γz lies in $\Sigma^*_{\sqrt{3}/2,1/2}$, and hence finitely many γ.

1.4 Haar Measure

- Topological group: A topological space G such that G is a group, and

$$(g,h) \mapsto g \cdot h^{-1}$$

  is continuous in both variables; i.e. multiplication and inversion is continuous.

- Locally compact: every point has compact neighborhood
- Hausdorff: distinct elements can be separated by opens
- In particular, $\mathrm{GL}(n,\mathbb{R})$ is a locally compact Hausdorff topological group, coming from the subspace topology of $\mathrm{GL}(n,\mathbb{R}) \subseteq \mathrm{Mat}(n,\mathbb{R}) = \mathfrak{gl}(n,\mathbb{R})$.
- (left) Haar measure: For locally compact Hausdorff topological group, we want a positive Borel measure μ on G, left invariant on the action by G, i.e. $\mu(gE) = \mu(E)$. Same for right. If left invariant measure means right invariant measure on G, G is unimodular.

  Can define differential one form, such that that we have integrals for compactly supported $f : G \to \mathbb{C}$

$$\int_G f(g)\,\mathrm{d}\mu(g)\,,$$

  and

$$\int_E \mathrm{d}\mu(g) = \mu(E).$$

  This $\mathrm{d}\mu(g)$ is the Haar measure.

- Key theorem: For any locally compact Hausdorff topological group, there exists a unique left Haar measure on G, up to positive real multiples. Proof of uniquness: Fubini.
- Haar measure on $\mathrm{GL}(n,\mathbb{R})$: For $g = (g_{i,j})_{i,j} \in \mathrm{GL}(n,\mathbb{R})$, the unique left-right invariant measure on $\mathrm{GL}(n,\mathbb{R})$ is

$$\mathrm{d}\mu(g) = \frac{\prod_{1\le i,j\le n} \mathrm{d}g_{i,j}}{\det(g)^n}.$$

  Proof: Decompose $\mathrm{GL}(n,\mathbb{R})$ into Z_n (center of $\mathrm{GL}(n,\mathbb{R})$) and elements that are 1 on the diagonal and $x_{r,s}$ and (r,s), and do casework.

1.5 Invariant measure on coset spaces

- Let G be a locally compact Hausdorff topological group, and H a compact subgroup of G, with corresponding Haar measure μ and ν, respectively. Then there exists a unique (up to scalar multiple) quotient measure $\widetilde{\mu}$ on G/H such that

$$\int_G f(g)\,\mathrm{d}\mu(g) = \int_{G/H} \left(\int_H f(gh)\,\mathrm{d}\nu(h) \right) \mathrm{d}\widetilde{\mu}(gH)\,.$$

- $\mathfrak{h}^n$ and $\mathrm{GL}(n,\mathbb{R})$: The measure left invariant $\mathrm{GL}(n,\mathbb{R})$ measure on $\mathfrak{h}^n$ can be expressed as

  $$\mathrm{d}^*z = \mathrm{d}^*x\mathrm{d}^*y,$$

  with

  $$\mathrm{d}^*x = \prod_{1\le i<j\le n} \mathrm{d}x_{i,j}$$

  and

  $$\mathrm{d}^*y = \prod_{1\le i\le k} {y_k}^{-k(n-k)-1}\,\mathrm{d}y_k\,.$$

  Proof: Check invariance under diagonal matrices, upper triangular matrices with 1s on diagonal, and transpositions.

1.6 Volume of $\mathbf{SL}(n,\mathbb{Z})\backslash\mathbf{SL}(n,\mathbb{R})/\mathbf{SO}(n,\mathbb{R})$

- Note that $\mathrm{SL}(n,\mathbb{R})/\mathrm{SO}(n,\mathbb{R}) \cong \mathrm{GL}(n,\mathbb{R})/(\mathrm{O}(n,\mathbb{R})\cdot\mathbb{R}^*)$.
- The volume of $\mathrm{SL}(n,\mathbb{Z})\backslash\mathrm{SL}(n,\mathbb{R})/\mathrm{SO}(n,\mathbb{R})$ can be explicitly computed to be

  $$n2^{\lfloor n/2\rfloor}\prod_{\ell=2}^{n}\frac{\zeta(\ell)}{\mathrm{Vol}(S^{\ell-1})}.$$

  (Note that this is different from the book.)
- Proof idea: Induction.
- Base case ($n=2$): Can directly integrate using the fundamental domain. Or use the technique from the general case.
- General case: Define a test function f, then create a periodic function

  $$F(z) = \sum_{m\in\mathbb{Z}^n} f(m\cdot z).$$

  Split sum in casework by last row: take out common factor ℓ, then treat as coset of $\mathbb{P}_n\backslash\mathfrak{h}^n$, where P_n is anything with e_n as the last row. Integrate over a fundamental domain $\Gamma_n\backslash\mathfrak{h}^n$. Now break up $\ell e_n\cdot z$ via the Iwasawa decomposition into three components; one that is integrating over $\mathrm{SL}(n-1,\mathbb{Z})$, one over $(\mathbb{R}/\mathbb{Z})^{n-1}$ (corresponding to $x_{j,n}$), and one integrating over $(0,\infty)$; corresponding to $t = \left(\prod_{i=1}^{n-1} y_i^{n-i}\right)^{-1/n}$. Applying spherical integration techniques, this can be computed in terms of $\widehat{f}(0)$. Now, applying Poisson summation, replace f by $\widehat{f}$, and get the same formula but with $\widehat{f}$ and f switched. Choosing an appropriate f, this gives the desired result.

2 Invariant differential operators

- The periodic functions $e^{2\pi inx}$ on $\mathcal{L}^2(\mathbb{Z}\backslash\mathbb{R})$ are precisely the eigenfunctions for the Laplacian operator $\frac{\mathrm{d}^2}{\mathrm{d}x^2}$, with eigenvalue $-4\pi^2n^2$. This directly leads to Fourier theory.
- Thus, we are motivated to consider differential operators invariant under the discrete group, and their eigenvalues/functions.

2.1 Lie algebra

- Associative algebra: Associative algebra A over field K is a vector space over K with an associative product closed in A satisfying the distributive law.
- Lie algebra: Vector space over K with bilinear map $[\cdot,\cdot] : L \times L \to L$ such that
  - $[a, \beta b + \gamma c] = \beta[a,b] + \gamma[a,c]$
  - $[a,a] = 0$
  - $[a,b] = -[b,a]$
  - $[a,[b,c]] + [b,[c,a]] + [c,[a,b]] = 0.$
- Given an associative algebra A, the associated Lie algebra $\mathrm{Lie}(A)$ is A equipped with the bracket
$$[a,b] = ab - ba.$$
- Universal enveloping algebra: For any Lie algebra $\mathfrak{L}$ over K, consider the tensor algebra
$$T(\mathfrak{L}) = \oplus_{k=0}^{\infty} \otimes^k \mathfrak{L},$$
where the tensor product is taken over K. Let $I(\mathfrak{L})$ be the two-sided ideal generated by $X \otimes Y - Y \otimes X - [X,Y]$. Then the universal enveloping algebra is precisely
$$U(\mathfrak{L}) = T(\mathfrak{L})/I(\mathfrak{L}),$$
an associative algebra with the product $X \circ Y = X \otimes Y \pmod{I(\mathfrak{L})}$. In particular, by definition,
$$\mathfrak{L} \subseteq \mathrm{Lie}(U(\mathfrak{L}))$$
with the inclusion respecting the bracket.

2.2 Universal enveloping algebra of $\mathfrak{gl}(n,\mathbb{R})$

- $\mathfrak{gl}(n,\mathbb{R})$: Precisely $\mathrm{Mat}(n,\mathbb{R})$, with Lie bracket
$$[\alpha,\beta] = \alpha\beta - \beta\alpha.$$
- We have the (left invariant) differential operators D_α, for $\alpha \in \mathfrak{gl}(n,\mathbb{R})$, acting on the set of smooth functions $\mathrm{GL}(n,\mathbb{R}) \to \mathbb{C}$, via
$$D_\alpha F(g) = \frac{\partial}{\partial t} F(g\exp(t\alpha))|_{t=0} = \frac{\partial}{\partial t} F(g + tg\alpha)|_{t=0}.$$
Denote $\mathcal{D}^n$ to be the associative algebra generated by the D_α, where the multiplication is composition.
- Some properties of the differential operators:
  - $D_\alpha(FG) = D_\alpha F \cdot G + F \cdot D_\alpha G$
  - $D_\alpha(F(G(g))) = (D_\alpha F)(G(g)) D_\alpha(G)(g)$
  - $D_{\alpha+\beta} = D_\alpha + D_\beta$
  - $D_\alpha \circ D_\beta - D_\beta \circ D_\alpha = D_{[\alpha,\beta]}$.
  - $D_\alpha \circ D_\beta = D_\beta \circ D_\alpha \implies D_{\alpha\beta} = D_{\alpha\beta}$.

  In particular, $\mathcal{D}^n$ can be realized as the universal enveloping algebra of $\mathfrak{gl}(n,\mathbb{R})$. Letting $[D,D'] = D \circ D' - D' \circ D$ be the bracket for the induced Lie algebra (from the universal enveloping algebra), we have that $[D_\alpha, D_\beta \circ D] = [D_\alpha, D_\beta] \circ D + D_\beta \circ [D_\alpha, D]$.

  Proof of properties: Direct calculation using multivariate chain rule.
- If $f : \mathrm{GL}(n,\mathbb{R}) \to \mathbb{C}$ is left-invariant by $\mathrm{GL}(n,\mathbb{Z})$ and right-invariant by Z_n, then for all $D \in \mathcal{D}^n$, Df is also left-invariant by $\mathrm{GL}(n,\mathbb{Z})$ and right-invariant by Z_n.

2.3 The center of the universal enveloping algebra of $\mathfrak{gl}(n, \mathbb{R})$

- Denote $\mathfrak{D}^n$ to be the center of the universal enveloping algebra $\mathcal{D}^n$.
- If $D \in \mathfrak{D}^n$, and f is a smooth function

$$f : \mathrm{GL}(n, \mathbb{Z})\backslash\mathrm{GL}(n, \mathbb{R})/(O(n, \mathbb{R})Z_n) \to \mathbb{C},$$

  then Df is also well defined over $\mathrm{GL}(n, \mathbb{Z})\backslash\mathrm{GL}(n, \mathbb{R})/(O(n, \mathbb{R})Z_n)$; i.e. it is left-invariant by $\mathrm{GL}(n, \mathbb{Z})$ and right invariant by $O(n, \mathbb{R})Z_n$.
- Proof idea: Use the fact that $SO(n, \mathbb{R})$ is generated by exp of skew-symmetric matrices, and that exp commutes well with the definition of D. Then use that D lies in the center, and f is right-invariant by $O(n, \mathbb{R})$ (including δ_1!).
- Casimir operators: Let $E_{i,j}$ be the matrix with 1 at i, j and 0 elsewhere, and let $D_{i,j} = D_{E_{i,j}}$. Then for each $m \geq 2$, we have the Casimir operator

$$\sum_{i_1=1}^{n} \sum_{i_2=1}^{n} \cdots \sum_{i_m=1}^{n} D_{i_1,i_2} \circ D_{i_2,i_3} \circ \cdots \circ D_{i_m,i_1},$$

  which lies in $\mathfrak{D}^n$.
- For $\mathfrak{gl}(n, \mathbb{R})$: The center is a rank n algebra. Any element in the center can be expressed as a polynomial in $\mathbb{R}$ in the Casimir operators defined before and D_{I_n}. Moreover, D_{I_n} annihilates any function invariant under Z_n.

2.4 Eigenfunctions of invariant differential operators

- We want a smooth function $f : \mathfrak{h}^n \to \mathbb{C}$ that is an eigenfunction for all $D \in \mathfrak{D}^n$; i.e. we want

$$Df(z) = \lambda_D f(z)$$

  for all D in the center of the universal enveloping algebra and $z \in \mathfrak{h}^n$.
- Power function: We have the I_s function, a generalization of the imaginary part function raised to the power s;

$$I_s(z) = \prod_{i=1}^{n-1} \prod_{j=1}^{n-1} y_i^{b_{i,j} s_j},$$

  where

$$b_{i,j} = \begin{cases} ij & i + j \leq n \\ (n-i)(n-j) & i + j > n \end{cases}$$

- On $\mathrm{GL}(2, \mathbb{R})$, this is just y^s. We have that $\Delta = y^2(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2})$ generates $\mathfrak{D}^2$ for functions over $\mathfrak{h}^2$ (we can ignore D_{I_2}, as all functions are right-invariant by the center). In particular, note that

$$\Delta I_s(z) = s(s-1)I_s(z).$$

- The proposition in the book proving this is incorrect. See Dorian's classes' notes.
- In particular, $I_s(z)$ is such an eigenfunction for all $D \in \mathfrak{D}^n$.
- Theme: Any function in just ys is an eigenfunction.

3 Automorphic forms and L-functions for $\mathbf{SL}(2,\mathbb{Z})$

- Key idea: Automorphicity is equivalent to existence of functional equation for certain L-functions - this is the idea of converse theorems.
- Hecke operators: Simultaneous eigenfunction of all Hecke operators corresponds to Euler product for L-function.

3.1 Eisenstein series

- Hyperbolic Laplacian:
$$\Delta = -y^2\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right),$$
invariant under the action by $\mathrm{GL}(2,\mathbb{R})^+$.
- y^s is an eigenfunction of this operator, with eigenvalue $s(1-s)$.
- Automorphic function: Smooth function $\mathrm{SL}(2,\mathbb{Z})\backslash\mathfrak{h}\to\mathbb{C}$.
- To construct automorphic function, we average over the group to get the Eisenstein series:
$$E(z,s)=\sum_{\gamma\in\Gamma_\infty\backslash\mathrm{SL}(2,\mathbb{Z})}\frac{I_s(\gamma z)}{2}=\frac{1}{2}\sum_{\substack{c,d\in\mathbb{Z}\\ \gcd(c,d)=1}}\frac{y^s}{|cz+d|^{2s}}$$
- $E(z,s)$ converges absolutely and uniformly on compact subset for $z\in\mathfrak{h}^2$ and $\mathrm{Re}(s)>1$.
- Real analytic in z and complex analytic in s.
- More properties:
  - $|E(z,s)-y^s|\le c(\varepsilon)y^{-\varepsilon}$ for $\sigma\ge 1+\varepsilon>1$.
  - $E(\gamma z,s)=E(z,s)$ for $\gamma\in\mathrm{SL}(2,\mathbb{Z})$.
  - $\Delta E(z,s)=s(1-s)E(z,s)$.
- Bessel function:
$$K_s(y)=\frac{1}{2}\int_0^\infty e^{-\frac{1}{2}y(u+1/u)}u^s\frac{\mathrm{d}u}{u}.$$
In particular, $K_s(y)=K_{-s}(y)$.
- Fourier coefficients of Eisenstein series: We have that
$$E(z,s)=y^s+\phi(s)y^{1-s}+\frac{2\pi^s\sqrt{y}}{\Gamma(s)\zeta(2s)}\sum_{n\neq 0}\sigma_{1-2s}(n)|n|^{s-1/2}K_{s-1/2}(2\pi|n|y)e^{2\pi inx},$$
where
$$\phi(s)=\sqrt{\pi}\frac{\Gamma(s-1/2)}{\Gamma(s)}\frac{\zeta(2s-1)}{\zeta(2s)}.$$
- Idea of proof: Integral calculation. A little bit of Ramanujan sums. Some identities from Gamma integrals involve rewriting the Gamma integral then performing a change of variable.
- Properties of ϕ:
  - $\phi(s)\phi(1-s)=1$,
  - $E(z,s)=\phi(s)E(z,1-s)$.

- We have the modified function $E^*(z,s) = \pi^{-s}\Gamma(s)\zeta(2s)E(z,s)$. It is meromorphic, with simple poles at $s = 0, 1$. It has functional equation

$$E^*(z,s) = E^*(z,1-s)$$

(which follows by examining the Fourier coefficients) and has

$$\mathrm{Res}_{s=1} E(z,s) = \frac{3}{\pi}$$

for all $z \in \mathfrak{h}^2$.

- Why do we care? Useful in the Rankin-Selberg method (discussed previously) to get functional equations for L-functions. Will also arise the in the Selberg spectral decomposition of $\mathcal{L}^2(\mathrm{SL}(2,\mathbb{Z})\backslash\mathfrak{h}^2)$ functions.

3.2 Hyperbolic Fourier expansion of Eisenstein series

- Idea: We can use a hyperbolic Fourier expansion of the Eisenstein series to recover the functional equation for the Hecke L-function associated to $\mathbb{Q}(\sqrt{D})$, where this is a real quadratic field (also for D of specific form).

- Let $\rho = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$. be a hyperbolic element; i.e. $\gamma > 0$ and $|\alpha+\delta| > 2$. This has two fixed points

$$\omega = \frac{\alpha - \delta + \sqrt{D}}{2\gamma}$$

and

$$\omega' = \frac{\alpha - \delta - \sqrt{D}}{2\gamma},$$

where $D = (\alpha+\delta)^2 - 4$.

- Let

$$\kappa = \begin{pmatrix} 1 & -\omega \\ 1 & -\omega' \end{pmatrix}.$$

Then $\kappa\rho\kappa^{-1}$ (the diagonalization) is equivalent to

$$\begin{pmatrix} \varepsilon & 0 \\ 0 & \varepsilon^{-1} \end{pmatrix},$$

where

$$\varepsilon = (\alpha + \delta - \sqrt{D})/2.$$

so that $\varepsilon + \varepsilon^{-1} = \alpha + \delta$. Moreover, it is a unit in $\mathbb{Q}(\sqrt{D})$. Since $\mathbb{Q}(\sqrt{D})$ is a quadratic extension of $\mathbb{Q}$, the ring of integers is rank 1, and we suppose that ε is a fundamental unit of the group of units.

- In particular, we have that

$$E(\kappa^{-1}z, s) = E(\kappa^{-1}(\varepsilon^2 z), s).$$

Consider this series as a function of v, where $z = iv$. We get a Fourier expansion

$$\zeta(2s)E(\kappa^{-1}(iv), s) = \sum_{n\in\mathbb{Z}} b_n(s) v^{\frac{\pi i n}{\log\varepsilon}},$$

with

$$b_n(s) = \frac{1}{2\log\varepsilon}\int_1^{\varepsilon^2} \zeta(2s)E(\kappa^{-1}(iv), s) v^{\frac{\pi i n}{\log\varepsilon}} \frac{\mathrm{d}v}{v}$$

- After some tedious calculation, you get that

$$b_n(s) = \frac{(\omega-\omega')^s}{4\log\varepsilon}\sum_{\beta\neq 0} N(\beta)^{-s}\left|\frac{\beta}{\beta'}\right|^{\frac{-\pi i n}{\log\varepsilon}}\int_{|\beta'/\beta|}^{\varepsilon^2|\beta'/\beta|}\left(\frac{v}{v^2+1}\right)^s v^{-\pi i n/\log\varepsilon}\frac{\mathrm{d}v}{v}$$

Note that there are typos in the book: extra factor of $1/2$, and inside term is v and not v^2.

- The $\beta = c\omega + d$ lie in an (fractional) ideal $\mathfrak{b}$ such that $N(\mathfrak{b}) = \frac{1}{\gamma}$, so using the definition of two principal ideals being equal (using that ε is a fundamental unit) and an integral calculation similar to Bump Proposition 1.9.1, we have that

$$b_n(s) = \frac{\Gamma\left(\frac{s-\frac{\pi i n}{\log\varepsilon}}{2}\right)\Gamma\left(\frac{s+\frac{\pi i n}{\log\varepsilon}}{2}\right)}{\Gamma(s)}\frac{(N(\mathfrak{b})\sqrt{D})^s}{8\log\varepsilon}\sum_{\mathfrak{b}|(\beta)\neq 0}\left|\frac{\beta}{\beta'}\right|^{-\pi i n/\log\varepsilon} N(\beta)^{-s}.$$

(TODO: This 8 should maybe be a 2? Double check the gamma integral. Maybe check The Spectrum of Hyperbolic Surfaces, Bergeron)

- We have the Hecke grossencharakter

$$\psi((\beta)) = \left|\frac{\beta}{\beta'}\right|^{-\pi i n/\log\varepsilon}$$

and Hecke L-function

$$L_{\mathfrak{b}}(s,\psi^n) = \sum_{\mathfrak{b}|(\beta)\neq 0}\psi^n((\beta))N(\beta)^{-s}.$$

- Hence the expansion for the Eisenstein series invovles the Hecke L-function:

$$E^*(\kappa^{-1}(iv), s) = \frac{(N(\mathfrak{b})\sqrt{D})^s}{8\pi^s\log\varepsilon}\sum_{n\in\mathbb{Z}}\Gamma\left(\frac{s-\frac{\pi i n}{\log\varepsilon}}{2}\right)\Gamma\left(\frac{s+\frac{\pi i n}{\log\varepsilon}}{2}\right)L_{\mathfrak{b}}(s,\psi^n)v^{\pi i n/\log\varepsilon}.$$

- The functional equation for the Eisenstein series hence gives the functional equation for the Hecke L-function: $L_{\mathfrak{b}}(s,\psi^n)$ has meromorphic continuation except a simple pole at $s=1$, and letting

$$\Lambda^n_{\mathfrak{b}}(s) = \frac{(N(\mathfrak{b})\sqrt{D})^s}{\pi^s}\Gamma\left(\frac{s-\frac{\pi i n}{\log\varepsilon}}{2}\right)\Gamma\left(\frac{s+\frac{\pi i n}{\log\varepsilon}}{2}\right)L_{\mathfrak{b}}(s,\psi^n),$$

we have $\Lambda^n_{\mathfrak{b}}(s) = \Lambda^n_{\mathfrak{b}}(1-s)$.

3.3 Maass forms

- We have a Hilbert space of $\mathcal{L}^2(\mathrm{SL}(2,\mathbb{Z})\backslash\mathfrak{h}^2)$ with the inner product given by the Petersson inner product:

$$\int_{\mathrm{SL}(2,\mathbb{Z})\backslash\mathfrak{h}^2} f(z)\overline{g(z)}\frac{\mathrm{d}x\,\mathrm{d}y}{y^2}.$$

- We define a Maass form of type v to be a function $f\in\mathcal{L}^2(\mathrm{SL}(2,\mathbb{Z})\backslash\mathfrak{h}^2)$ such that
  - $f(\gamma z) = f(z)$ for $\gamma\in\mathrm{SL}(2,\mathbb{Z})$.
  - $\Delta f = v(1-v)f$
  - $\int_0^1 f(z)\,\mathrm{d}x = 0$.

  (In other sources, the last condition is for a Maass cuspform.)

- Δ is a self-adjoint operator. Hence $v(1-v)$ is real and nonnegative. Proof idea: Use that
$$\int_{\Gamma\backslash\mathfrak{h}^2}\left(\frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y^2}\right)\overline{f}=\int_{\Gamma\backslash\mathfrak{h}^2}\left|\frac{\partial f}{\partial x}\right|^2+\left|\frac{\partial f}{\partial y}\right|^2$$
for f a Maass form. This follows from Green's theorem.

- Maass form of types 0 or 1 must be constant. Follows from properties of harmonic functions. (Why is f bounded as the imaginary part of $z\to\infty$? Answer: Follows from Fourier expansion; we show later that each of the Fourier expansions has rapid decay, so the function is hence bounded. See: `https://math.stackexchange.com/questions/4980702/a-question-on-properties-of-mass-forms`)

3.4 Whittaker expansions and multiplicity one for $\mathbf{GL}(2,\mathbb{R})$

- For a Maass form f, using the transformation property gives a Fourier expansion
$$f(z)=\sum_{m\in\mathbb{Z}}A_m(y)e^{2\pi imx},$$
and $A_m(y)e^{2\pi imx}$ satsifies the two properties
  - $\Delta W_m(z)=v(1-v)W_m(z)$
  - $W_m\left(\begin{pmatrix}1 & u\\ 0 & 1\end{pmatrix}z\right)=W_m(z)e^{2\pi imu}$

  This motivates the following definition.

- A Whittaker function of type v with additive character $\psi: R\to S^1$ is a smooth nonzero function $W:\mathfrak{h}^2\to\mathbb{C}$ such that
  - $\Delta W(z)=v(1-v)W(z)$
  - $W\left(\begin{pmatrix}1 & u\\ 0 & 1\end{pmatrix}z\right)=W_m(z)\psi(u)$

- On $\mathrm{GL}(2,\mathbb{R})$, we can construct these functions explicitely. We can check that
$$W(z,v,\psi_m)=\sqrt{2}\frac{(\pi|m|)^{v-1/2}}{\Gamma(v)}\sqrt{2\pi y}K_{v-1/2}(2\pi|m|y)e^{2\pi imx}.$$
where
$$K_v(y)=\frac{1}{2}\int_0^\infty e^{-1/2y(u+1/u)}u^v\frac{\mathrm{d}u}{u}.$$

- Multiplicity 1: For $\mathrm{SL}(2,\mathbb{Z})$-Whittaker functions of type not 0 or 1 with rapid decay at infinity, it must be a constant multiple of the W computed before. In particular, if $\psi=1$, then $a=0$. Moreover, if ψ is non-trivial, we can assume W has polynomial growth.

- Proof follows from differential equation theory. In the nontrivial case, there are two solutions; one has rapid decay ($K_v(y)$) and one has rapid growth, and it is precisely the function defined before.

3.5 Fourier-Whittaker expansions on $\mathbf{GL}(2,\mathbb{R})$

- Corollary of Multiplicity One theorem: Every nonconstant Maass form of type v (i.e. type not 0 or 1) has Whittaker expansion of the form
$$f(z)=\sum_{n\neq 0}a_n\sqrt{2\pi y}K_{v-1/2}(2\pi|n|y)e^{2\pi inx}.$$

- Proof: The integral condition requires that the $e^{2\pi i0}$ coefficient is 0. Maass forms being $\mathcal{L}^2$ implies that it has polynomial growth. Hence all of the Whittaker functions corresponding to each Fourier coefficient (corresponding not to 0) must be at worst polynomial growth. By multiplicity one, this forces every Fourier coefficient to be of the form above.

3.6 Ramunujan-Petersson Conjecture

- For holomorphic modular cuspforms of weight k, we have the Ramunujan-Petersson conjecture
$$|a_n| = O(n^{(k-1)/2}d(n))$$
where $d(n)$ is the number of divisors of n.
- Idea: Non-constant Maass forms like holomorphic modular forms of weight 0.
- This leads to Ramanujan-Petersson conjecture for Maass forms:
$$|a_n| = O(d(n)),$$
with constant only dependent on the Petersson norm of f.
- What we can show: If f has Petersson norm 1 and is of type v, we have that
$$|a_n| = O_v(\sqrt{|n|}).$$
Proof idea: Integrate from $x \in [0,1]$, $y \in [Y, \infty)$ of $|f(z)|^2$, then isolate the a_n. Then use a change of variable $y \mapsto Yy$ to get a $\frac{1}{Y}$ factor times the area of $|f|^2$ over the fundamental domain.

3.7 Selberg eigenvalue conjecture

- We know that for a non-constant Maass form f, $\Delta f = v(1-v)f$, where $\lambda = v(1-v)$ is real and positive. How small can λ be?
- Maass form for a congruence subgroup Γ: Smooth on $\mathfrak{h}^2$, automorphic on Γ, lies in $\mathcal{L}^2(\Gamma \backslash \mathfrak{h}^2)$, constant terms of Fourier expansions at cusps vanish, and $\Delta f = v(1-v)f$.
- Selberg conjecture: If f is a Maass form of type v for a congruence subgroup Γ, then $v(1-v) \geq 1/4$; i.e. $\mathrm{Re}(v) = 1/2$.
- For Maass forms on $\mathrm{SL}(2,\mathbb{Z})$, can prove (according to M-F Vigneras) that $v(1-v) \geq \frac{3\pi^2}{2}$.
- Proof idea: Use again that
$$\int_{\Gamma \backslash \mathfrak{h}^2} \left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} \right) \overline{f} = \int_{\Gamma \backslash \mathfrak{h}^2} \left| \frac{\partial f}{\partial x} \right|^2 + \left| \frac{\partial f}{\partial y} \right|^2$$
for f a Maass form. This follows from Green's theorem.

3.8 Finite dimensionality of the eigenspaces

- Let $\mathfrak{S}_\lambda$ be the space of Maass forms of eigenvalue $\Lambda = v(1-v)$ under Δ. This space is finite dimensional.
- Idea of proof: If $a_n = 0$ for $n \leq n_0$, for n_0 sufficiently large, then f itself must be 0. Get bound using that $a_n = O(\sqrt{|n|})$ and use that $K_v(y) \asymp \frac{e^{-y}}{\sqrt{y}}$.

3.9 Even and odd Maass forms

- T_{-1}: Operator such that
$$T_{-1}f(x+iy) = f(-x+iy).$$
Notation is written to match Hecke operators later. This sends Maass forms of type v to Maass forms of type v.
- In particular, the eigenvalues of T_{-1} must be ± 1, since $T_{-1}^2 = I$.

- If $T_{-1}f = f$, then f is even. If $T_{-1}f = -f$, then f is odd.
- If f is even, then $a_n = a_{-n}$. If f is odd, then $a_n = -a_{-n}$. Proof: Clear from Fourier inversion after making substitution $x \mapsto -x$.
- Any Maass form of type v can be expressed as the sum of an even and odd Maass form:
$$f = \frac{1}{2}\left(f + T_{-1}f\right) + \frac{1}{2}\left(f - T_{-1}f\right);$$
the left is an even Maass form and the right is an odd Maass form.

3.10 Hecke operators

We will show this in more generality, then later apply to the case of $\Gamma = \mathrm{SL}(2,\mathbb{Z})$ and $X = \mathfrak{h}^2$.

- G is a group acting continuously on topological space X, Γ is a discrete subgroup of G, $\Gamma\backslash X$ as left Γ-invariant measure $\mathrm{d}x$.
- We have the commensurator of Γ
$$C_G(\Gamma) = \left\{g \in G | (g^{-1}\Gamma g) \cap \Gamma \text{ has finite index in both } \Gamma \text{ and } g^{-1}\Gamma g\right\}.$$
- For any $g \in C_G(\Gamma)$, we have the decomposition
$$\Gamma = \cup_{i=1}^{d}\left((g^{-1}\Gamma g) \cap \Gamma\right)\delta_i,$$
giving double coset decomposition
$$\Gamma g \Gamma = \cup_{i=1}^{d} \Gamma g \delta_i$$
for some representatives $\delta_i \in \Gamma$, where $d = [\Gamma : (g^{-1}\Gamma g) \cap \Gamma]$.
- We define the Hecke operator
$$T_g : \mathcal{L}^2(\Gamma\backslash X) \to \mathcal{L}^2(\Gamma\backslash X)$$
by
$$T_g(f(x)) = \sum_{i=1}^{d} f(g\delta_i x).$$
- This is well-defined; the choice of δ_i is preserved because f is invariant under left-multiplication under Γ, and $T_g(f(\gamma x)) = T_g(f(x))$ for $\gamma \in \Gamma$ because $\delta_i\gamma = \gamma_i'\delta_{\sigma(i)}$ for $\gamma_i' \in g^{-1}\Gamma g \cap \Gamma$, and so
$$g\delta_i\gamma = g\gamma_i'\delta_{\sigma(i)} \in \Gamma g \delta_{\sigma_i},$$
and then we invoke left invariance of Γ.
- We get the Hecke ring by considering formal sums
$$\sum_k m_k T_{g_k}.$$
- For multiplication, we consider the multiplication of the double cosets:
$$(\Gamma g \Gamma)(\Gamma h \Gamma) = \cup_j \Gamma g \Gamma \beta_j = \cup_{i,j} \Gamma \alpha_i \beta_j = \cup_{\Gamma w \subseteq \Gamma g \Gamma h \Gamma} \Gamma w = \cup_{\Gamma w \Gamma \subseteq \Gamma g \Gamma h \Gamma} \Gamma w \Gamma.$$
- Then
$$T_g T_h = \sum_{\Gamma w \Gamma \subseteq \Gamma g \Gamma h \Gamma} m(g,h,w) T_w,$$
where $m(g,h,w)$ is the number of i,j such that $\Gamma\alpha_i\beta_j = \Gamma w$. This product ends up being associative.

- Let Δ be a semigroup such that $\Gamma \subseteq \Gamma \subseteq C_G(\Gamma)$. The Hecke ring $\mathcal{R}_{\Gamma,\Delta}$ is the set of all formal sums
$$\sum_k c_k T_{g_k}$$
with $c_k \in \mathbb{Z}$ and $g_k \in \Delta$.
- Antiautomorphism: $g \mapsto g^*$ such that $(gh)^* = h^* g^*$. For example, transpose of matrix, which is what we care about.
- Commutativity of Hecke ring: If there exists antiautomorphism $g \mapsto g^*$ of $C_G(\Gamma)$ such that $\Gamma^* = \Gamma$ and $(\Gamma g \Gamma)^* = \Gamma g \Gamma$ for all $g \in \Delta$, then $\mathcal{R}_{\Gamma,\Delta}$ is a commutative ring.
- Proof: Idea: Use the antiautomorphism to show that left and right coset decompositions are basically the same. Then use antiautomorphism to show that products should come out to the same thing.

3.11 Hermite and Smith normal forms

- Hermite normal form: Every matrix $A \in \mathrm{GL}(n,\mathbb{Z})^+$ is left-equivalent under $\mathrm{SL}(n,\mathbb{Z})$ to a matrix B, i.e. $B = \gamma A$, with $\gamma \in \mathrm{SL}(n,\mathbb{Z})$, of the form
$$\begin{pmatrix} d_1 & \alpha_{2,1} & \dots & \alpha_{n,1} \\ 0 & d_2 & \dots & \alpha_{n,2} \\ 0 & 0 & \ddots & \vdots \\ 0 & 0 & \dots & d_n \end{pmatrix}$$
where the d_i are positive integers and $0 \le \alpha_{k,j} < d_k$.
- Idea of proof: You can get this form by performing row operations that preserve the determinant, which is equivalent to left-multiplication by $\gamma \in \mathrm{SL}(n,\mathbb{Z})$.
- Smith normal form: Every matrix $A \in \mathrm{GL}(n,\mathbb{Z})^+$ is left-right equivalent under $\mathrm{SL}(n,\mathbb{Z})$ to a matrix D; i.e. $D = \gamma_1 A \gamma_2$, where D is a diagonal matrix, with d_n in the top left and d_1 in the bottom right, such that $d_i \mid d_{i+1}$, and the $d_i > 0$.
- Idea of proof: Same idea, but now with both row and column operations. Uniqueness: GCD of all $k \times k$ components determines d_k.

3.12 Hecke operators for $\mathcal{L}^2(\mathbf{SL}(2,\mathbb{Z})\backslash\mathfrak{h}^2)$

- In this case, we have $G = \mathrm{GL}(2,\mathbb{R})$, $\Gamma = \mathrm{SL}(2,\mathbb{Z})$, and $X = \mathfrak{h}^2$.
- The matrix
$$\begin{pmatrix} n_0 n_1 & 0 \\ 0 & n_0 \end{pmatrix}$$
for integers $n_0, n_1 \ge 1$ lies in $C_G(\Gamma)$, we can let Δ be the semigroup generated by Γ and these matrices.
- For this Δ, we have the antiautomorphism of transposition. We have that
$$\Gamma^t = \Gamma$$
and
$$(\Gamma g \Gamma)^t = \Gamma g \Gamma$$
for $g \in \Delta$, as g is generated by diagonal matrices and elements of Γ, so the Hecke ring $\mathcal{R}_{\Gamma,\Delta}$ is commutative.

- Let
$$S_n = \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} | ad = n, 0 \le b < d \right\}$$
then
$$\cup_{m_0^2 m_1 = n} \Gamma \begin{pmatrix} m_0 m_1 & 0 \\ 0 & m_0 \end{pmatrix} \Gamma = \cup_{\alpha \in S_n} \Gamma \alpha$$
is a disjoint decomposition.
- Proof idea: Basically follows from Hermite/Smith normal forms.
- Thus, we use the double coset disjoint union on the right to define the Hecke operator
$$T_n f(z) = \frac{1}{n} \sum_{\substack{ad=n \\ 0 \le b < d}} f\left(\frac{az+b}{d}\right),$$
where $1/\sqrt{n}$ is a normalization factor to help with formulas later.
- The Hecke operators are self-adjoint wrt the Petersson inner product:
$$\langle T_n f, g \rangle = \langle f, T_n g \rangle .$$
Proof idea: Use that diagonal matrices are invariatn under transposition, and that
$$S \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} \right)^T S^{-1} = \frac{1}{ad} \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$
and acting on z this is the same as $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$.
- In particular, one can check that the Hecke operators, T_{-1}, and Δ all commute.
- Hence, we can simulatenously diagonalize with respect to all of the operators, giving Maass Hecke-eigenforms. These must be either even or odd.
- In particular, letting
$$f(z) = \sum_{n \neq 0} a_n \sqrt{2\pi y} K_{v-1/2}(2\pi |n| y) e^{2\pi i n x}$$
be the Fourier-Whittaker decomposition, we have that for a Maass eigenform of type v $a(1) = 0 \implies f = 0$. If f is nonzero and we normalize such that $a(1) = 1$, then we have the following properties:
  - $T_n f = a_n f$
  - $a_m a_n = a_{mn}$, $\gcd(m, n) = 1$
  - $a_m a_n = \sum_{d|(m,n)} a_{mn/d^2}$
  - $a_{p^{r+1}} = a_p a_{p^r} - a_{p^{r-1}}$

  for all primes p and $r \ge 1$.
- Proof idea: Direct computation using the definition of the Hecke operators.

3.13 L-functions associated to Maass forms

- Let f be a Maass Hecke eigenform of type v that is also an eigenfunction for T_{-1}. We have the L-function associated to f
$$L_f(s) = \sum_{n=1}^{\infty} a_n n^{-s}.$$
Since we know that $a_n = O(\sqrt{n})$, this series is absolutely convergent for $\mathrm{Re}(s) > 3/2$.

- Since the a_n are multiplicative, we have the Euler product
$$L_f(s) = \prod_p \left(\sum_{\ell=0}^{\infty} \frac{a_{p^\ell}}{p^{\ell s}} \right).$$

- Using the previous formulas for the a_{p^r} gives that
$$L_f(s) = \prod_p (1 - a_p p^{-s} + p^{-2s})^{-1} = \prod_p (1 - \alpha_p p^{-s})^{-1}(1 - \beta_p p^{-s})^{-1},$$
where $\alpha_p \beta_p = 1$ and $\alpha_p + \beta_p = a_p$.

- We have the following holomorphic continuation and functional equation for $L_f(s)$: Let $\varepsilon = 0, 1$ be such that $T_{-1} f = (-1)^\varepsilon f$. Then the completed L-function is
$$\Lambda_f(s) = \pi^{-s} \Gamma\left(\frac{s + \varepsilon - 1/2 + v}{2}\right) \Gamma\left(\frac{s + \varepsilon + 1/2 - v}{2}\right) L_f(s),$$
and we have the functional equation
$$\Lambda_f(s) = (-1)^\varepsilon \Lambda_f(1 - s).$$

- Proof: Consider $x = 0$, and consider as function of y for $y > 0$. Take the Mellin transform of the function. The two gamma factors arise out of the Mellin transform of Bessel functions. If f is even, use that $a_n = a_{-n}$. If f is odd, instead take the Mellin transform of $\frac{\partial}{\partial x} f$.

3.14 L-functions associated to Eisenstein series

- Recall that for $\mathrm{Re}(w) > 1$, we had the Eisenstein series
$$E(z, w) = \frac{1}{2} \sum_{\substack{c,d \in \mathbb{Z} \\ \gcd(c,d)=1}} \frac{y^w}{|cz + d|^{2w}}$$
with Fourier-Whittaker expansion
$$E(z, w) = y^w + \phi(w) y^{1-w} + \frac{2^{1/2} \pi^{w-1/2}}{\Gamma(w)\zeta(2w)} \sum_{n \neq 0} \sigma_{1-2w}(n) |n|^{w-1} \sqrt{2\pi|n|y} K_{w-1/2}(2\pi|n|y) e^{2\pi i n x}.$$
Note that the $\sigma_{1-2w}(n)|n|^{w-1/2}$ are analogous to the a_n for Maass forms.

- Hence we define the L-function associated to $E(z, w)$ by
$$L_{E(*,w)}(s) = \sum_{n=1}^{\infty} \sigma_{1-2w}(n) n^{w-1/2-s}.$$

- It turns out that
$$L_{E(*,w)}(s) = \zeta(s+w-1/2)\zeta(s-w+1/2),$$
so letting (completing in the natural way for each zeta)
$$\Lambda_{E(*,w)}(s) = \pi^{-s}\Gamma\left(\frac{s+w-1/2}{2}\right)\Gamma\left(\frac{s-w+1/2}{2}\right)L_{E(*,w)}(s),$$
we get the functional equation
$$\Lambda_{E(*,w)}(s) = \Lambda_{E(*,w)}(1-s),$$
which exactly matches the functional equation for an even Maass form of type w.
- Moreover, the Eisenstein series is an eigenfunction of all the Hecke operators, giving an explanation for the Euler product.
- Idea of proof: The S_n defined previously ($\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$) act as coset representatives of $\Gamma_1\backslash\Gamma_n$. The Eisenstein series are summed over $\Gamma_\infty\backslash\Gamma_1$. Swap the sums and swap the order of coset representatives and the correct value for $T_nE(z,s)$ falls out.

3.15 Converse theorems for SL$(2,\mathbb{Z})$

- Just like for holomorphic modular forms, satsifying functional equation + sufficient boundedness conditions gives modularity.
- Hecke-Maass converse theorem: Let $L(s) = \sum a_n n^{-s}$ be an L-function that converges absolutely for $\mathrm{Re}(s)$ sufficiently large, and suppose that the completed L-function
$$\Lambda^v(s) = \pi^{-s}\Gamma\left(\frac{s+\varepsilon-1/2+v}{2}\right)\Gamma\left(\frac{s+\varepsilon+1/2-v}{2}\right)L(s)$$
satsifies the functional equation
$$\Lambda^v(s) = (-1)^\varepsilon\Lambda^v(1-s),$$
where $\varepsilon = 0, 1$, and $\Lambda^v(s)$ is entire and bounded on vertical strips. Then
$$f = \sum_{n\neq 0} a_n\sqrt{2\pi y}K_{v-1/2}(2\pi|n|y)e^{2\pi inx}$$
is an even/odd Maass form, where $a_{-n} = (-1)^\varepsilon a_n$.
- Idea of proof: Only need to check modularity. You get T for free, so only need to check S. Can show that it suffices to check $x = 0$, y if you can show that $f(iy) - f(i/y)$ satisfies some initial conditions involving differentials: $F(iy) = 0$, $\frac{\partial F}{\partial x}|_{x=0} = 0$ implies F is 0. (This is the replacement for analytic continuation in the holomorphic case.) This follows out of since Λ is the Mellin transform of f, expressing f as the Mellin inverse of f, then applying the functional equation + bounded on vertical strips + f rapidly decaying toward infinity to get the final answer.
- Caveat: No such L-function has been found for Maass forms on SL$(2,\mathbb{Z})$. (Related to that there are no known constructions of SL$(2,\mathbb{Z})$-Maass forms) Closest is the Hecke L-function - these turn out to be the functional equation of a Maass form of a congruence subgroup.

3.16 The Selberg spectral decomposition

- It turns out that we can decompose any $\mathcal{L}^2(\mathrm{SL}(2,\mathbb{Z})\backslash\mathfrak{h}^2)$ function into
$$\mathbb{C}\oplus\mathcal{L}^2_{cusp}(\mathrm{SL}(2,\mathbb{Z})\backslash\mathfrak{h}^2)\oplus\mathcal{L}^2_{cont}(\mathrm{SL}(2,\mathbb{Z})\backslash\mathfrak{h}^2).$$
Here cusp refers to integrals at cusps is 0, and will be an integral of an Eisenstein series.

- We have $\eta_j(z)$, for $j \geq 1$, be an orthonormal basis of Maass forms that are all Hecke eigencuspforms. Moreover, let
$$\eta_0(z) = \sqrt{3/\pi}.$$
- We get the Selberg spectral decomposition
$$f(z) = \sum_{j=0}^{\infty} \langle f, \eta_j \rangle \, \eta_j(z) + \frac{1}{4\pi i} \int_{1/2-i\infty}^{1/2+i\infty} \langle f, E(*, s) \rangle \, E(z, s) \, \mathrm{d}s$$
where the inner product is the Petersson inner product.
- Why a countable basis of Maass forms? It turns out that the Laplacian on cuspforms is a compact operator, so from spectral theory we get that the spectrum is countable. See Iwaniec-Kowalski 15.2.
- Can show that if f is of rapid decay such that
$$\langle f, E(*, \overline{s}) \rangle$$
converges absolutely, and f is orthogonal to 1, then f decomposes into a cusp form plus the correct integral by showing that $\langle f, E(*, \overline{s}) \rangle$ is the Mellin transform of the constant term of f, and that the constant term of the integral is the inverse Mellin transform of $\langle f, E(*, \overline{s}) \rangle$.
- Spectral theory of automorphic forms important - will lead to Selberg trace formula, etc.

4 Existence of Maass forms

- We defined Maass forms. We have some examples for Maass forms for congruence subgroups, but whatabout for other groups?
- Phillips-Sarnak (1985) - conjectured that for most non-congruence subgroups, Maass forms do not exist
- For full modular group: Selberg trace formula to get Weyl's law on number of Maass forms of type v, with $|v| \leq x$. Has been extended to all $\mathrm{SL}(n, \mathbb{Z})$.
- Lindenstrauss-Venkatesh - another proof using Hecke operators. Extends better to higher rank than trace formula.
- We essentially will be discussing this proof.

4.1 The infinitude of odd Maass forms for $\mathbf{SL}(2, \mathbb{Z})$

- Recall that we have the hyperbolic Laplacian
$$\Delta = -y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right).$$
Maass forms of type v are $\mathcal{L}^2$ eigenfunctions of Δ with eigenvalue $v(1-v)$ such that f is invariant under the action by $\mathrm{SL}(2, \mathbb{Z})$, and the cusp is 0 $\int_0^1 f(x+iy) \, \mathrm{d}x = 0$.
- We have the Fourier expansion for Maass forms
$$f(z) = \sum_{n \neq 0} a_n \sqrt{2\pi y} K_{v-1/2}(2\pi |n| y) e^{2\pi i n x}.$$
- Recall that we have odd and even Maass forms, defined by the sign of being an eigenfunction for
$$T_{-1} f(x+iy) = f(-x+iy)$$
For odd Maass forms, $a_n = -a_{-n}$, and for even Maass forms $a_n = a_{-n}$.

- It is simple to show that there are an infinite number of odd Maass forms for $\mathrm{SL}(2,\mathbb{Z})$; define the operator
$$J : \mathcal{L}^2(\mathrm{SL}(2,\mathbb{Z})\backslash\mathfrak{h}) \to \mathcal{L}^2(\mathrm{SL}(2,\mathbb{Z})\backslash\mathfrak{h})$$
via
$$Jf = f - T_{-1}f.$$
Note that for any f,
$$\int_0^1 Jf\,\mathrm{d}x = \int_0^1 f(x+iy)\,\mathrm{d}x - \int_0^1 f(-x+iy)\,\mathrm{d}x = \int_0^1 f(x+iy)\,\mathrm{d}x + \int_0^{-1} f(x+iy)\,\mathrm{d}x = 0,$$
so the image of J is cuspidal. Moreover $T_{-1}(Jf) = -(Jf)$. Hence sending any eigenfunction of Δ gives an odd Maass form. We have control of the a_n in the Fourier expansion of a Maass form, so we can get a nontrivial image.

- Hence it suffices to show that even Maass forms are also infinite dimensional.

4.2 Integral operators

Note to self: Present 4.3 first.

- Hyperbolic distance: $d(z,z') = d(\alpha z, \alpha z')$ for all $\alpha \in \mathrm{SL}(2,\mathbb{R})$. Defined by functions
$$u(z,z') = \frac{|z-z'|^2}{4\,\mathrm{Im}\,z\,\mathrm{Im}\,z'} = \sinh^2(d(z,z')/2).$$
In particular, the hyperbolic distance between i and iy_0 is $\log(y_0)$.

- Abel transform: For $f(x)$ on $[0,\infty)$, the Abel transform is defined by
$$F(x) = \int_{-\infty}^{\infty} f(x+\xi^2/2)\,\mathrm{d}\xi = \sqrt{2}\int_x^{\infty} \frac{f(v)}{\sqrt{v-x}}\,\mathrm{d}v\,.$$
Inverse Abel transformation: we have that
$$f(x) = -\frac{1}{2\pi}\int_{-\infty}^{\infty} F'(x+\eta^2/2)\,\mathrm{d}\eta\,.$$
Proof: switch to polar coordinates, differentiate under the integral sign.

- (Inverse) Selberg/Harish-Chandra transform: Variant of Abel transform. For g an even smooth function of compact support on $\mathbb{R}$, we have Fourier transform
$$h(t) = \int_{-\infty}^{\infty} g(x)e^{itx}\,\mathrm{d}x$$
and transformations
$$q(v) = \frac{1}{2}g(2\log\big(\sqrt{v+1}+\sqrt{v}\big))$$
$$g(r) = 2q((\sinh r/2)^2)$$
and
$$k(u) = -\frac{1}{\pi}\int_u^{\infty}(v-u)^{-1/2}\,\mathrm{d}q(v)$$
$$q(v) = \int_v^{\infty} k(u)(u-v)^{-1/2}\,\mathrm{d}u$$
Then the transformation from $k(u)$ to $h(t)$ is the Selberg transform.
k is compactly supported an continuous; if g supported in $[-M,M]$, then $k(u)$ is 0 for $u > \sinh^2(M/2)$.

- Notationally, let $u := u(z, w)$ for $z, w \in \mathfrak{h}^2$.
- Point pair invariant: Given $g : \mathbb{R} \to \mathbb{C}$ an even smooth function of compact support, the point pair invariant $K : \mathfrak{h}^2 \times \mathfrak{h}^2 \to \mathbb{C}$ is defined to be

$$K(z, w) = k(u(z, w)) = k(u).$$

  This is continuous, and is supported for $d(z, w) \leq R$, where R depends only on the support of g. Moreover, $K(\alpha z, \alpha w) = K(z, w)$ for all $\alpha \in \mathrm{SL}(2, \mathbb{R})$.
- Integral operator: For $f \in \mathcal{L}^1(\mathrm{SL}(2, \mathbb{Z})\backslash\mathfrak{h}^2)$, we have

$$(K * f)(z) = \int_{\mathfrak{h}^2} K(z, w) f(w) \mathrm{d}^* w = \int_0^\infty \int_{-\infty}^\infty K(z, \mu + i\nu) f(\mu + i\nu) \frac{\mathrm{d}\mu \, \mathrm{d}\nu}{\nu^2}.$$

  $f \to K * f$ is a self-adjoint continuous endomorphism of $\mathcal{L}^2(\mathrm{SL}(2, \mathbb{Z})\backslash\mathfrak{h}^2)$.
- Lemma:

$$\int_{-\infty}^\infty K(i, t + ie^x) \, \mathrm{d}t = e^{x/2} g(x).$$

  Proof: Plug into the Selberg/Harish-Chandra transform. This is the main computational tool to computing anything involving these point pair invariants: we transform $z \to i$ and then do the integral.
- Intuition: For f an eigenfunction of Δ with eigenvalue $1/4 + r^2$, $K * f = h(r) f$. Can check by computing $K * y^{1/2+ir}$.
- Goal: We will want to choose kernels K corresponding to g that are δ-functions. We will approximate the δ function by a sequence of $g^{(j)}$ with support shrinking to 0 but area under the integral is 1 as $j \to \infty$.
- In particular, note that

$$\int_{\mathfrak{h}^2} K^{(j)}(i, w) \mathrm{d}^* w = \int_{-\infty}^\infty e^{-x/2} g^{(j)}(x) \, \mathrm{d}x \to 1$$

  under the substitution $\nu = e^x$. Moreover, once can show that for any z,

$$\int_{\mathfrak{h}^2} K^{(j)}(i, w) \mathrm{d}^* w = \int_{\mathfrak{h}^2} K^{(j)}(z, w) \mathrm{d}^* w.$$

  Then for any choice of continuous $f \in \mathcal{L}^1(\mathrm{SL}(2, \mathbb{Z})\backslash\mathfrak{h}^2)$, we get that

$$K^{(j)} * f(z) = \int_{h^2} K^{(j)}(z, w) f(z) \mathrm{d}^* w + \int_{h^2} K^{(j)}(z, w)(f(w) - f(z)) \mathrm{d}^* w \to f(z)$$

  (pointwise, not uniformly in z), using that the support of $K^{(j)}$ goes to 0 as $g^{(j)} \to \delta$.
- Thus, $K^{(j)} * f(z) \to f(z)$.

4.3 The endomorphism ♡

- Goal: Define an endomorphism $\heartsuit : \mathcal{L}^2(\mathrm{SL}(2, \mathbb{Z})\backslash\mathfrak{h}^2) \to \mathcal{L}^2(\mathrm{SL}(2, \mathbb{Z})\backslash\mathfrak{h}^2)$ whose image is purely cuspidal. In other words, $\heartsuit f$ will be cuspidal even when f is not. We will use the arithmetic structure of $\mathrm{SL}(2, \mathbb{Z})$.
- We will use the Eisenstein series to get intuition for what we want out of ♡.
- Recall that we had the Hecke operators

$$T_p f(z) = \frac{1}{\sqrt{p}} \left(f(pz) + \sum_{k=0}^{p-1} f\left(\frac{z + k}{p}\right) \right).$$

We also had Eisenstein series $E(z,1/2+ir)$, defined by

$$E(z,s)=\frac{1}{2}\sum_{(c,d)\neq(0,0)}\frac{y^s}{|cz+d|^{2s}}$$

- Easy to compute that
$$T_pE(z,1/2+ir)=(p^{ir}+p^{-ir})E(z,1/2+ir).$$
- Formally, define
$$\heartsuit=T_p-p^{\sqrt{1/4-\Delta}}-p^{-\sqrt{1/4-\Delta}}.$$
Then $\heartsuit E(z,1/2+ir)=0$, and same with the constant function.
- Suppose that $\heartsuit$ can be made rigorous, and that it is self-adjoint on the space of even square integrable automorphic functions. Then since $\heartsuit$ kills the continuous spectrum and the constant functions, the image of $\heartsuit$ must be cuspidal. Thus, we need to find an f such that the image under $\heartsuit$ is nonzero, and we are done.
- How to make $\heartsuit$ rigorous - wave equation or convolution operators.

4.4 How to interpret $\heartsuit$ an explicit operator with purely cuspidal image

- Intuitively, suppose that g_p is the sum of δ functions at $\log p$ and $-\log p$. Then $h_p(r)=p^{ir}+p^{-ir}$, and for an eigenfunction of Δ with spectral parameter r,
$$K_p*f=(p^{ir}+p^{-ir})f,$$
which is what we want our formal $p^{\sqrt{\Delta-1/4}}+p^{-\sqrt{\Delta-1/4}}$ to behave like.
- To make this more rigorous, we will take sequences of even compactly supported smooth functions approaching a delta function.
- Let g_0 be an even smooth function of compact support on $\mathbb{R}$. Define
$$g_p(x)=g_0(x+\log p)+g_0(x-\log p)$$
(emulating the g that we want). Define k_p via the Selberg transform, and the point pair invariant $K_p(z,w)$.
- Thinking of the g_0 as a delta function, the corresponding h_0 is the constant function 1, so K_0*f is like the constant function. Hence we should expect $K_p*f-T_p(K_0*f)$ to annihilate the continuous spectrum.
- Proof: Explictely compute the constant terms of both operators, using the definition of the Hecke operator. You take an integral from $[0,1]$ in x and over all $\mathfrak{h}^2$, and at one point you need to unfold the integral to integrate over all x, while restricting the domain in the integral over $\mathfrak{h}^2$.

4.5 There exist infinitely many even cusp forms for SL$(2,\mathbb{Z})$

- Now that we have this self-adjoint operator $\heartsuit$ that sends even $\mathcal{L}^2$ functions to cuspidal even functions, we need to show that this image is nontrivial. We will do this by leveraging the different behavior of the operators as one approaches the cusp at infinity.
- Consider $\mathfrak{G}(T)=\left\{z\in\mathfrak{h}^2:0\leq x<1,y>T\right\}$. Hence the space of smooth compactly supported functions on $\mathfrak{G}(T)$ is a subset of the smooth compactly supported functions on $\mathrm{SL}(2,\mathbb{Z})\backslash\mathfrak{h}^2$, and similarly for $\mathcal{L}^2$ functions.
- Consider R sufficiently large such that $k_0(z,w)$ and $k_p(z,w)$ are supported in $d(z,w)\leq R$. Let $Y\geq pe^R$. Then $\heartsuit$ maps $C^\infty(\mathfrak{G}(Y))$ into $C^\infty(\mathfrak{G}(1))\subseteq C_c^\infty(\mathrm{SL}(2,\mathbb{Z})\backslash\mathfrak{h}^2)$.

- Moreover,
$$\begin{aligned} a_{n,K_p*f}(y) &= \int_{x\in\mathbb{Z}\backslash\mathbb{R}} e^{-2\pi inx} \int_{w\in\mathfrak{h}^2} K_p(x+iy,w)f(w)\mathrm{d}^*w\,\mathrm{d}x \\ &= \int_{x\in\mathbb{Z}\backslash\mathbb{R}} e^{-2\pi inx} \int_{w\in\mathfrak{h}^2} K_p(iy,w-x)f(w)\mathrm{d}^*w\,\mathrm{d}x \\ &= \int_{w\in\mathfrak{h}^2} K_p(iy,w) \int_{x\in\mathbb{Z}\backslash\mathbb{R}} e^{-2\pi inx} f(w+x)\,\mathrm{d}x\,\mathrm{d}^*w, \end{aligned}$$
so if $a_{n,f}(y)$ (the nth coefficient) is identically 0, so is $a_{n,K_p*f}(y)$.

- Fix an integer N, not divisible by p. Consider $f\in C_c^\infty(\mathfrak{G}(Y))$ that is also even, such that $a_{n,f}$ vanishes identically for all $n\neq\pm N$. Then a_{n,K_p*f} also vanishes identically for all $n\neq\pm N$.

- Similarly,
$$T_p(K_0*f)(z) = \frac{1}{\sqrt{p}}\left((K_0*f)(pz)+\sum_{k=0}^{p-1}(K_0*f)\left(\frac{z+k}{p}\right)\right)$$
so by a similar argument (using that $p\nmid N$) $a_{n,T_p(K_0*f)}$ vanishes identically for all $n\neq\pm pN$.

- Moreover, note that the K_0*f is nonzero as we take g_0 to approach the delta function, as $K_0*f\to f$ pointwise, which is not f. Hence $\heartsuit f$ is nonzero and even. We have free reign over the choice of N, so we can generate infinitely many even cusp forms.

4.6 A weak Weyl law

- We wish to make the number of cusp forms quantitative.

- Lemma: Order the spectrum of a non-negative self-adjoint operator A on H $\lambda_1\leq\lambda_2\leq\dots$. Suppose $V\subseteq H$ is finite such that $\|Av\|\leq\Lambda\|v\|$ whenever $v\in V$. Then $\#\{\lambda_i\leq\Lambda\}\geq\dim(V)$.

- One can show that the number of even cusp forms with eigenvalue $\leq\Lambda$ is at least $c\Lambda$ for some constant c, by specifying the right choice of functions to pass through the heart function.

4.7 Interpretation via wave equation and the role of finite propogation speed

- This section is more or less included in the next section.

4.8 Intepretation via wave equation: higher rank case

- For more details, see Lindenstrauss and Venkatesh (2005) - Existence and Weyl's law for spherical cusp forms.

- One can intepret $\heartsuit$ via the automorphic wave eqaution
$$u_{tt} = -\Delta u + u/4$$
where t is time and position is $x+iy$ in the hyperbolic plane

- You can define an operator U_t taking $f(x+iy)$ to $2u(x+iy,t)$, where u is the solution such that $u|_{t=0}=f$ and $u_t|_{t=0}=0$. This operator is well-defined and self-adjoint, and formally can be written by
$$U_t = e^{t\sqrt{\Delta-1/4}} + e^{-t\sqrt{\Delta-1/4}},$$
hence this can be interpreted as the T_p operator from before.

- The value of this operator only depends on $d(z,w) \le t$, representing the finite propogation speed of waves in the hyperbolic plane. Hence the operator $p^{\sqrt{\Delta - 1/4}} + p^{-\sqrt{\Delta-1/4}}$ corresponds to propagating a wave for time $\log p$.
- The $\heartsuit$ operator can be generalized in higher rank by doing something similar - using the number of parameters for Eisenstein series to find an operator that kills the Eisenstein series, then construct an function that is not killed by this operator.

4.9 Extra: Selberg trace formula and Weyl's law for rank 2

This section is roughly based on Iwaniec's *Spectral Methods for Automorphic Forms* and Marklof's *Selberg trace formula: A Introduction*, as extra info for a talk.

- Idea: consider trace of kernel $K(z,w)$ associated to a specific test function h satsifying nice enough conditions.
- One side, consider the spectral decomposition of Δ to get information relating to the eigenvalues.
- Other side is geometric side. Related to computing integrals by considering conjugacy classes of Γ and considering the parabolic, hyperbolic, and elliptic motions.
- Discrete spectrum:
$$N_\Gamma(T) = \#\{j : |t_j| \le T\}$$
- Continuous spectrum:
$$M_\Gamma(T) = \frac{1}{4\pi}\int_{-T}^{T} \frac{-\phi'}{\phi}(1/2+it)\,\mathrm{d}t\,,$$
where $\phi(s) = \pi^{1/2}\frac{\Gamma(s-1/2)}{\Gamma(s)}\frac{\zeta(2s-1)}{\zeta(2s)}$ (coming from the Fourier expansions of the Eisenstein series). Some computation with complex analysis gives $M_\Gamma(T)$ as the number of poles of $\phi(s)$ on the left of the critical line of height less than T up to an error term $O(T)$.
- Selberg trace formula for $h(t) = e^{-\delta t^2}$
$$\sum_j e^{-\delta t_j^2} + \frac{1}{4\pi}\int_{-\infty}^{\infty} \frac{-\phi'}{\phi}(1/2+it)e^{-\delta t^2}\,\mathrm{d}t = \frac{|F|}{4\pi\delta} + \frac{h\log\delta}{4\sqrt{\pi\delta}} - \frac{\gamma h}{4\sqrt{\pi\delta}} + O(1).$$
where h is the rank of some matrix and $|F|$ is the area of a fundamental domain.
- Tauberian theorem implies that
$$N_\Gamma(T) + M_\Gamma(T) \sim \frac{|F|}{4\pi}T^2$$
- $M_\Gamma(T)$ can be shown to be $O(T\log T)$, giving the final asymptotic
$$N_\Gamma(T) \sim \frac{|F|}{4\pi}T^2 + O(T\log T).$$
Compare this with the previous bound.
- Remark: This approach allows to compute an error term for the Weyl law, which is not available in the Lindenstrauss-Venkatesh approach.

5 Maass forms and Whittaker functions for SL$(n,\mathbb{Z})$

- The spectral parameter approach follows Dorian's book. Supposedly useful in some analytic applications.
- The Langlands parameter approach follows the paper "A template method for Fourier coefficients of Langlands Eisenstein series" by Goldfeld, Miller, and Woodbury. Also known as Satake parameters. They come from automorphic representations at the Archimedean place.

5.1 Maass forms

- Recall that we are interested in $\mathrm{SL}(n,\mathbb{Z})$ acting on
$$\mathfrak{h}^n = \mathrm{GL}(n,\mathbb{R})/(O(n,\mathbb{R}) \times \mathbb{R}^*).$$
- Recall that we can represent elements of
$$\mathfrak{h}^n = x \cdot y,$$
where x is an upper triangular matrix with 1s on the diagonal and $x_{i,j}$ off the diagonal, and y is a diagonal matrix with elements of the form $1, y_1, y_1y_2, \ldots, y_1y_2\cdots y_{n-1}$.
- We will consider two parameterizations: the spectral parameters v and the Langlands parameters, denoted $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{C}^n$, with $\alpha_1 + \cdots + \alpha_n = 0$. (This is abuse of notation; we really refer to the set of the α_i.)
- Recall that we defined
$$b_{ij} = \begin{cases} ij & i+j \leq n \\ (n-i)(n-j) & i+j \geq n \end{cases}$$
The b_{ij} come from the inverse of the Cartan matrix for $\mathrm{GL}(n)$.
- We have the following relation between the two sets of parameters:
$$v_i = \frac{\alpha_i - \alpha_{i+1} + 1}{n},$$
and conversely
$$\alpha_i = \begin{cases} B_{n-1}(v) & i = 1 \\ B_{n-i}(v) - B_{n-i+1}(v) & 1 < i < n \\ -B_1(v) & i = n \end{cases},$$
where $B_j(s) = \sum_{i=1}^{n-1} b_{i,j}(v_i - 1/n)$.
- Example: For $n = 2$, $\alpha_1 = -\alpha_2 = v - \frac{1}{2}$. For $n = 3$, $\alpha_1 = 2v_1 + v_2 - 1$, $\alpha_2 = -v_1 + v_2$, and $\alpha_3 = -v_1 - 2v_2 + 1$.
- We have the center of the universal enveloping algebra of $\mathfrak{gl}(n,\mathbb{R})$ is $\mathfrak{D}^n$. For any $D \in \mathfrak{D}^n$ and $v = (v_1, \ldots, v_{n-1})$, we know that
$$I_v(z) = \prod_{i=1}^{n-1}\prod_{j=1}^{n-1} y_i^{b_{i,j}v_j}$$
is an eigenfunction of every $D \in \mathfrak{D}^n$.

  Alternatively, in terms of Langlands parameters, let $\rho_i = \frac{n+1}{2} - i$. We can express the power function as
$$I(z,\alpha) = \prod_{i=1}^{n}\left(\prod_{j=1}^{n-i} y_j\right)^{\alpha_i+\rho_i} = \prod_{i=1}^{n-i} y_i^{\sum_{j=1}^{n-i}(\alpha_j+\rho_j)}.$$
The eigenvalue is independent of the permutation of α. This will follow from the proof of the functional equation for the Whittaker function.
- We express $DI_v(z) = \lambda_D I_v(z)$. Note that $\lambda_{D_1\cdot D_2} = \lambda_{D_1} \cdot \lambda_{D_2}$, hence λ_D is a character of $\mathfrak{D}^n$, called the Harish-Chandra character.
- Maass form of type v for $\mathrm{SL}(n,\mathbb{Z})$: A function $\phi \in \mathcal{L}^2(\mathrm{SL}(n,\mathbb{Z})\backslash\mathfrak{h}^n)$ satsifying
  - $\phi(\gamma z) = \phi(z)$ for all $\gamma \in \mathrm{SL}(n,\mathbb{Z})$

  - $D\phi(z) = \lambda_D \phi(z)$ for all $D \in \mathfrak{D}^n$ for λ_D a Harish-Chandra character. (In particular, these are the same λ_D coming from $DI_v(z) = \lambda_D I_v(z)$, which is where the v condition is being used.)
  - $\int_{(\mathrm{SL}(n,\mathbb{Z})\cap U)\backslash U} \phi(uz)\,\mathrm{d}u = 0$ for all U that are matrices with diagonal matrices I_{r_i} on the diagonals and 0 below the diagonal.

  If the eigenvalues of ϕ agree with the eigenvalues of $I(\cdot, \alpha)$, then α are the Langlands parameters of ϕ.

- Remark: An alternative definition of a Maass (cusp) form replaces the $\mathcal{L}^2$ condition with a growth condition that
$$|\phi(xy)| \ll_N (y_1 \dots y_{n-1})^{-N}$$
for all $N > 0$; i.e. an exponential decay growth condition.

- For Laplace operator Δ, then if f is a Maass form, we have corresponding Laplace eigenvalue
$$\lambda_\Delta = \frac{n^3 - n}{24} - \frac{\alpha_1^2 + \cdots + \alpha_n^2}{2}.$$
Generalized Ramanujan-Selberg conjecture: All Maass forms for $\mathrm{SL}(n, \mathbb{Z})$ (and congruence subgroups) are tempered, i.e. all α_i are purely imaginary. Compare this to the Ramanujan-Selberg conjecture for $n = 2$.

5.2 Whittaker functions associated to Maass forms

- Idea: Emulate the Fourier expansion in higher dimensions.
- Let $U_n(\mathbb{R})$ be the group of upper triangular $n \times n$ matrices.
- Let $m = (m_1, \dots, m_{n-1}) \in \mathbb{Z}^{n-1}$. We have a character $\psi_m : U_n(\mathbb{R}) \to \mathbb{C}^*$ sending
$$\psi_m(u) = e^{2\pi i (m_1 u_{1,2} + m_2 u_{2,3} + \cdots + m_{n-1} u_{n-1,n})}.$$
We have that $\psi_m(uv) = \psi_m(u)\psi_m(v)$.
- For Maass form ϕ, we want Fourier coefficients like
$$\widetilde{\phi}_m(z) = \int_0^1 \cdots \int_0^1 \phi(u \cdot z)\overline{\psi_m(u)} \prod_{1 \le i < j \le n} \mathrm{d}u_{i,j}$$
such that we can write
$$\phi = \sum_m \widetilde{\phi}_m(z).$$
This is the analogue of $W(y)e^{2\pi i m x}$ in the rank 2 case.

  Since $U_n(\mathbb{R})$ is non-Abelian, we need to be careful over which m we sum.
- Properties of the Fourier coefficients (which we will show later are Whittaker functions):
  - $\widetilde{\phi}_m(u \cdot z) = \psi_m(u)\widetilde{\phi}_m(z)$
  - $D\widetilde{\phi}_m = \lambda_D \widetilde{\phi}_m$ for all $D \in \mathfrak{D}^n$, where λ_D is a Harish-Chandra character
  - $\int_{\Sigma_{\frac{\sqrt{3}}{2}, \frac{1}{2}}} |\widetilde{\phi}_m(z)|^2 \mathrm{d}^* z < \infty$.
- Proof of properties:
  - The substitution $u \mapsto u \cdot u'$, for $u' \in U_n(\mathbb{R})$, does not change the measure.
  - Follows by definition of a Maass form.
  - Follows from Cauchy-Schwarz, ϕ is automorphic, and that ϕ is L^2.

5.3 Fourier expansions on $SL(n, \mathbb{Z})$

- Every Maass form for $\mathrm{SL}(n,\mathbb{Z})$ has the Fourier expansion

$$\phi(z) = \sum_{\gamma \in U_{n-1}(\mathbb{Z})\backslash \mathrm{SL}(n-1,\mathbb{Z})} \sum_{m_1 \neq 0} \sum_{m_2=1}^{\infty} \cdots \sum_{m_{n-1}=1}^{\infty} \widetilde{\phi}_{m_1,\dots,m_{n-1}} \left(\begin{pmatrix} \gamma & \\ & 1 \end{pmatrix} z \right).$$

The sum is independent of the choice of representatives γ. Recall that

$$\widetilde{\phi}_{(m_1,\dots,m_{n-1})}(z) = \int_0^1 \dots \int_0^1 \phi(u \cdot z) e^{-2\pi i (m_1 u_{1,2} + \dots + m_{n-1} u_{n-1,n})} \mathrm{d}^* u$$

where $\mathrm{d}^* u = \prod_{i=1}^n \mathrm{d}u_{i,i+1}$.

- Idea of proof: Inductively construct the Fourier expansion. Use standard Fourier expansion to get expansion with variables determined by the $n-1$ variables in the last column; i.e. let

$$v = \begin{pmatrix} 1 & & & & v_1 \\ & 1 & & & v_2 \\ & & \ddots & & \vdots \\ & & & 1 & v_{n-1} \\ & & & & 1 \end{pmatrix}$$

and

$$\widehat{\phi}_m(z) = \int_0^1 \dots \int_0^1 \phi(vz) e^{-2\pi \langle v, m \rangle} \mathrm{d}^* v.$$

Since ϕ is periodic when multiplying by v, we get

$$\phi(z) = \sum_{m \in \mathbb{Z}^{n-1}} \widehat{\phi}_m(z).$$

Rewrite this sum in terms of gcd of variables in last column and representatives of $\mathrm{SL}(n-1,\mathbb{Z})$ by $P(n-1,\mathbb{Z})$, where $P(n-1,\mathbb{Z})$ is matrices whose last row is e_{n-1}. Nothing corresponding to $m_{n-1} = 0$ because ϕ is a cuspform. Repeat inductively on all columns. m_1 also has negative coefficients because $\mathrm{SL}(1,\mathbb{Z})$ treats the orbits a and $-a$ separately.

5.4 Whittaker functions for $\mathbf{SL}(n, \mathbb{R})$

- A $\mathrm{SL}(n,\mathbb{Z})$ Whittaker function of type $v = (v_1, \dots, v_{n-1}) \in \mathbb{C}^{n-1}$ associated to a character $\psi : U_n(\mathbb{R}) \to \mathbb{C}$ is a smooth function $W : \mathfrak{h}^n \to \mathbb{C}$ such that
  - $W(uz) = \psi(u) W(z)$ for any $u \in U_n(\mathbb{R})$
  - $DW(z) = \lambda_D W(z)$ for any $D \in \mathfrak{D}^n$
  - $\int_{\Sigma_{\frac{\sqrt{3}}{2}, \frac{1}{2}}} |W(z)|^2 \mathrm{d}^* z < \infty$.
- In particular, note that the Fourier coefficient defined before

$$\widetilde{\phi}_m(z) = \int_0^1 \dots \int_0^1 \phi(u \cdot z) \overline{\psi}(u) \prod_{1 \leq i < j \leq n} \mathrm{d}u_{i,j}$$

is a Whittaker function.

5.5 Jacquet's Whittaker function

- Goal: Construct non-trivial (zero nowhere) Whittaker functions for rank n.
- Let $m = (m_1, \ldots, m_{n-1})$ (corresponding to Fourier frequency) and $v = (v_1, v_2, \ldots, v_{n-1})$ corresponding to the type of the Maass form. Alternatively, let $\alpha = (\alpha_1, \ldots, \alpha_n)$ be the Langlands parameters.
- Notation: for the upper triangular matrix u, we denote
$$u_i = u_{n-i,n-i+1}.$$
- Let $\psi_m : U_n(\mathbb{R}) \to \mathbb{C}$ to be the character
$$\psi_m(u) = e^{2\pi i(m_1 u_1 + \cdots + m_{n-1}u_{n-1})}.$$
Note that this has the reverse coefficients as expected in Section 5.3; i.e. the summation for Fourier expansion will be reversed.
- Let $z = xy$ and suppose all of the m_i are nonzero, we define the Jacquet Whittaker function $\mathfrak{h}^n \to \mathbb{C}$
$$W(z; v, \psi_m) = \int_{U_n(\mathbb{R})} I_v(wuz)\overline{\psi_m(u)}\mathrm{d}^*u,$$
where
$$w = \begin{pmatrix} & & & 1 \\ & & 1 & \\ & \iddots & & \\ 1 & & & \end{pmatrix}$$
and the integral is integrated with respect to all $u_{i,j}$ from $-\infty$ to ∞.
In terms of Langlands parameters, we have
$$W_\alpha(z) = \int_{U_n(\mathbb{R})} I(wuz, \alpha)\overline{\psi_m(u)}\mathrm{d}^*u.$$
Note that this exactly matches the construction for $\mathbb{H}$.
- Remark: Dorian's book uses w_n ($-1^{\lfloor n/2 \rfloor}$ in the top right corner), the long element of the Weyl group. This is equivalent because of the wedge product definition of $I_v(s)$, using that $e_j w_n = e_j w$ for all $j > 1$. We will use the original definition to show the functional equation.
- If $\mathrm{Re}(v_i) > 1/n$ for all i and $m_i \neq 0$ for all i, then:
  - W converges absolutely and uniformly on compact subsets of $\mathfrak{h}^n$
  - W has meromorphic continuation for all $v \in \mathbb{C}^{n-1}$
  - W is an $SL(n, \mathbb{Z})$-Whittaker function of type v and character ψ_m
  - $$W(z; v, \psi_m) = c_{v,m} W(Mz; v, \psi_{m_1/|m_1|, \ldots, m_{n-1}/|m_{n-1}|}) = c_{v,m}\psi_m(x) W(My; v, \psi_{1,\ldots,1}),$$
  where
  $$c_{v,m} = \prod_{i=1}^{n-1} |m_i|^{\left(\sum_{j=1}^{n-1} b_{i,j} v_j\right) - i(n-i)}$$
  and
  $$M = \begin{bmatrix} |m_1 m_2 \ldots m_{n-1}| & & & \\ & \ddots & & \\ & & |m_1| & \\ & & & 1 \end{bmatrix}.$$

- Remark: From the Fourier expansion, we only care about m where $m_1, \ldots, m_{n-2}$ are positive. By the above properties, it is sufficient to care only about $m = (1, \ldots, 1, \pm 1)$.
- Proof idea:
  - Proof that W is a Whittaker function:
    * $W(az) = \psi_m(a)W(z)$: Change of variable. Also proves part of second equation of fourth point.
    * $DW = \lambda_D W$: Use that $DI_v = \lambda_D I_v$.
    * Assume that integral converges absolutely and uniformly on compacts to an $\mathcal{L}^2$ function.
  - Proof of first equation of fourth point: Make the changes of variables in the integral from $wuMz$ to $wMuz$ to $wMw \cdot wuz$. This gives the correct constant $c_{v,m}$; see Broughan 2009, Theorem 6.1 for more details.
  - Proof of second equation of fourth point: Let δ_j be the identity matrix, except with $\varepsilon_j = m_j/|m_j|$ at the $n-j$th row. Replace u with $\delta_j \delta_j u$, then do a change of variable from $u \to \delta_j u$ and use that $\delta_{n-j} w = w\delta_j$, and since all the matrices in the integral are diagonal and $\delta_{n-j} \in O(n, \mathbb{R})$, δ_{n-j} can be ignored. Repeat for all j.
  - Proof of absolute convergence/meromorphic continuation for $n = 2$: Absolute convergence follows from computing the integral, which converges for $\mathrm{Re}(v) > 1/2$. Meromorphic continuation: Follows from $K_v = K_{-v}$.

5.6 The exterior power of a vector space

- Let $\otimes^\ell(\mathbb{R}^n)$ be the space of ℓth tensor products of the vector space $\mathbb{R}^n$. Formally, we define

  $$\Lambda^\ell(\mathbb{R}^n) = \otimes^\ell(\mathbb{R}^n)/\mathfrak{a}_\ell$$

  where $\mathfrak{a}_\ell$ is the vector subspace generated by all elements $v_1 \otimes \cdots \otimes v_\ell$ where $v_i = v_j$ for some $i \neq j$.
- In other words, we have the set of $v_1 \wedge \cdots \wedge v_\ell$ with the rules $v \wedge v = 0$, $v \wedge w = -w \wedge v$, and $(a_1 v_1 + a_2 v_2) \wedge w = a_1 v_1 \wedge w + a_2 v_2 \wedge w$.
- On $\otimes^\ell(\mathbb{R}^n)$, we have the (canonical) inner product

  $$\langle v, w \rangle_{\otimes^\ell} = \prod_{i=1}^{\ell} \langle v_i, w_i \rangle .$$

- Let $e_1, \ldots, e_n$ be the canonical basis for $\mathbb{R}^n$. Then letting

  $$a = \sum_{1 \leq i_1, \ldots, i_\ell \leq n} a_{i_1, \ldots, i_\ell} e_{i_1} \wedge \cdots \wedge e_{i_\ell},$$

  we define $\phi_\ell : \Lambda^\ell(\mathbb{R}^n) \to \otimes^\ell(\mathbb{R}^n)$ such that

  $$\phi_\ell(a) = \frac{1}{\ell!} \sum_{1 \leq i_1, \ldots, i_\ell \leq n} a_{i_1, \ldots, i_\ell} \sum_{\sigma \in S_\ell} \mathrm{Sign}(\sigma) \cdot e_{i_{\sigma(1)}} \otimes \cdots \otimes e_{i_{\sigma(\ell)}}.$$

  This is a well-defined injection, and hence $\Lambda^\ell(\mathbb{R}^n)$ can be viewed as a subspace of $\otimes^\ell(\mathbb{R}^n)$. We then define the inner product on Λ^ℓ to be

  $$\langle v, w \rangle_{\Lambda^\ell} = \langle \phi_\ell(v), \phi_\ell(w) \rangle_{\otimes^\ell} .$$

- We define the action of $\mathrm{SL}(n, \mathbb{R})$ on $\Lambda^\ell(\mathbb{R}^n)$ via

  $$v \circ g = (v_1 \cdot g) \wedge \cdots \wedge (v_\ell \cdot g)$$

  and similarly for $\otimes^\ell$.

- For $k \in O(n,\mathbb{R})$, $\langle v,w\rangle_{\Lambda^\ell} = \langle v\circ k, w\circ k\rangle_{\Lambda^\ell}$, and $\|v\| = \sqrt{\langle v,v\rangle_{\Lambda^\ell}} = \|v\circ k\|$. Proof: Prove the same properties from $\otimes^\ell$, and then apply ϕ_ℓ.
- For any upper triangular matrix u,
$$(e_{n-\ell}\wedge\cdots\wedge e_n)\circ u = e_{n-\ell}\wedge\cdots\wedge e_n.$$
- Cauchy-Schwarz: $|\langle v,w\rangle_{\Lambda^\ell}|^2 \le \langle v,v\rangle_{\Lambda^\ell}\cdot\langle w,w\rangle_{\Lambda^\ell}$, and $\|v\wedge w\|_{\Lambda^\ell} \le \|v\|_{\Lambda^\ell}\|w\|_{\Lambda^\ell}$. Proof: Use that $\|v\|_{\Lambda^\ell}^2 = \sum_{i_1,\ldots,i_\ell}|a_{i_1,\ldots,i_\ell}|^2$, and apply normal Cauchy-Schwarz.

5.7 Construction of the I_v function using wedge products

- We can write
$$I_v(z) = \left(\prod_{i=0}^{n-2}\|(e_{n-i}\wedge\cdots\wedge e_n)\circ z\|^{-nv_{n-i-1}}\right)\cdot|\det(z)|^{\sum_{i=1}^{n-1} iv_{n-i}},$$
and hence we can write W in terms of a wedge product.
- Check that operations inside are invariant under $SO(n,\mathbb{R})$ and $\mathbb{R}^*$, so the operation is well-defined. Moreover, use that x is upper triangular to get that $I_v^*(z) = I_v^*(y)$. Finish by doing the computation on y.
- Why is this helpful? Shows that we can choose to use w instead of w_n in the definition of the Jacquet-Whittaker function. Also can be used to explicitely compute the Whittaker function for the $\mathrm{SL}(n,3)$ case:

$$W(y;v,\psi_m) = y_1^{v_1+2v_2}y_2^{2v_1+v_2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(y_1^2y_2^2+u_1^2y_2^2+(u_1u_2-u_3)^2)^{-3v_1/2} \cdot(y_1^2y_2^2+u_2^2y_1^2+u_3^2)^{-3v_2/2}e^{-2\pi i(m_1u_1+m_2u_2)}\,\mathrm{d}u_1\,\mathrm{d}u_2\,\mathrm{d}u_3$$

5.8 Convergence of Jacquet's Whittaker function

- This section is incorrect; equation 5.8.2 is the wrong direction.
- TODO: Maybe see this paper.
- Heuristic: The integral will be on the order of something like the product of
$$\int(1+u_{j,j+1}^2+\cdots+u_{j,n}^2)^{-\frac{n}{2}\sum_{i=1}^{n-j}\mathrm{Re}v_i}\,\mathrm{d}u$$
for all j, and the integral converges if $\frac{n}{2}\sum_{i=1}^{n-j}\mathrm{Re}v_i > \frac{n-j}{2}$, or if $\sum_{i=1}^{n-j}\mathrm{Re}v_i > \frac{n-j}{n}$ for all j. Hence the convergence is for $\mathrm{Re}v_j > \frac{1}{n}$.

5.9 Functional equations of Jacquet's Whittaker function

- Everything stated here will be in terms of Langlands parameters, but it is possible to translate everything in terms of spectral parameters. It just is really annoying.
- Multiplicity one: Due to Shalika, there is only one Whittaker function of type v and character ψ up to constant multiple (implied here is the growth condition).
- Because in the Fourier expansion, we care only when $m_1,\ldots,m_{n-2}>0$, and we can relate $W(z;v,\psi_m)$ to $W(Mz;v,\psi_{1,\ldots,1,\pm1})$ by a constant, it suffices only to consider the following normalized Whittaker functions:
$$W_\alpha^{\pm}(z) = \prod_{1\le j<k\le n}\frac{\Gamma\left(\frac{1+\alpha_j-\alpha_k}{2}\right)}{\pi^{\frac{1+\alpha_j-\alpha_k}{2}}}\int_{U_n(\mathbb{R})}I(wug,\alpha)\overline{\psi_{1,\ldots,1,\pm1}(u)}\mathrm{d}^*u.$$

- Properties: It is an absolutely convergent integral for $\mathrm{Re}(\alpha_i - \alpha_{i+1}) > 0$ and has holomorphic continuation to all $\alpha \in \mathbb{C}^n$ with $\sum \alpha_i = 0$.
- Functional equation: For any permutation α' of α,
$$W_\alpha^\pm = W_{\alpha'}^\pm.$$
In other words, there is no abuse of notation regarding the Langlands parameters.
- Proof idea: It suffices to consider adjacent swaps of α_i. Let α' be the permutation of α swapping α_i and α_{i+1}. Consider
$$\sigma_i = \begin{pmatrix} I_{n-i-1} & & & \\ & 0 & 1 & \\ & 1 & 0 & \\ & & & I_{i-1} \end{pmatrix}.$$
Letting $w_i = \sigma_i^{-1} w$, every $u \in U_n(\mathbb{R})$ can be written in the form
$$u = (w_i^{-1} n_i w_i) n_i',$$
where $n_i \in N_i$ is the set of matrices with 1s on the diagonal, real number at position $(n-i, n-i+1)$, and zeros elsewhere, and $n_i' \in N_i'$ is the subgroup of $U_n(\mathbb{R})$ with a zero at the position $(i, i+1)$. Hence, we can write
$$W_\alpha^\pm(z) = \int_{U_n(\mathbb{R})} I_\alpha(wug)\overline{\psi}(u)\mathrm{d}^*u = \int_{N_i'} \left(\int_{N_i} I_\alpha(\sigma_i n_i (w_i n_i' z))\overline{\psi}(n_i)\,\mathrm{d}n_i \right) \overline{\psi}(n_i')\,\mathrm{d}n_i'.$$
The inner integral is a Whittaker function over $\mathrm{SL}(2,\mathbb{R})$, whose function equation is independent of choice of $w_i n_i' z$, and we remark that
$$I_\alpha(\sigma_i n_i) = (u^2+1)^{\frac{1}{2}(\alpha_i - \alpha_{i+1})},$$
where u is the nonzero element of n_i. This thus resembles the Bessel function, and swapping α_i and α_{i+1} changes α to α' and changes by the requisite constant.

5.10 Degenerate Whittaker functions

- It is possible to construct Whittaker using other elements of the Weyl group (elements of $\mathrm{SL}(n,\mathbb{Z})$ with exactly one 1 or -1 in each row or column) instead of w_n, as the only key property used was that I_v was an eigenfunction.
- However, these Whittaker functions will not contain all of the $u_{i,j}$. You can define the Whittaker functions by integrating only over the variables that appear.

6 Automorphic forms and L-functions for $SL(3,\mathbb{Z})$

6.1 Whittaker functions and multiplicity one for $SL(3,\mathbb{Z})$

- Recall that we had the Whittaker functions
$$W_\alpha(z, \psi_m) = \int_0^\infty \int_0^\infty \int_0^\infty I_\alpha(wuz)\overline{\psi_m(u)}\mathrm{d}^*u,$$
with u a unipotent matrix (integrating over all $u_{i,j}$),
$$w = \begin{pmatrix} & & 1 \\ & 1 & \\ 1 & & \end{pmatrix},$$

and

$$I_\alpha(z) = \prod_{i=1}^{3}\left(\prod_{j=1}^{3-i} y_j\right)^{\alpha_i+\rho_i}$$

with $\rho_i = \frac{n+1}{2} - i$.

- We have completed Whittaker function

$$W_\alpha^*(z,\psi_{1,1}) = \frac{\Gamma\left(\frac{1+\alpha_1-\alpha_2}{2}\right)\Gamma\left(\frac{1+\alpha_2-\alpha_3}{2}\right)\Gamma\left(\frac{1+\alpha_1-\alpha_3}{2}\right)}{\pi^{\frac{3+\alpha_1+\alpha_2-2\alpha_3}{2}}}\int_0^\infty\int_0^\infty\int_0^\infty I_\alpha(wuz)e^{-2\pi i(u_1+u_2)}\mathrm{d}^*u$$

  with the functional equation by permuting the α_i (and similarly for $\psi_{1,-1}$).

- We have the integral representation

$$W_\alpha^*(y,\psi_{1,1}) = 8y_1^{1+\alpha_2/2}y_2^{1-\alpha_2/2}\int_0^\infty K_{\frac{\alpha_1-\alpha_3}{2}}(2\pi y_1\sqrt{1+u^{-2}})K_{\frac{\alpha_1-\alpha_3}{2}}(2\pi y_2\sqrt{1+u^2})u^{-3\alpha_2/2}\frac{\mathrm{d}u}{u}$$

  (via long computation involving computing coefficients of $\mathrm{SL}(3,\mathbb{Z})$ Eisenstein series). Note that $W_\alpha(y,\psi_{1,1}) = W_\alpha(y,\psi_{1,-1})$, so this also takes care of $\psi_{1,-1}$.

- Taking the double Mellin transform gives

$$\widetilde{W_\alpha}(s) := \int_0^\infty\int_0^\infty W_\alpha^*(z,\psi_{1,1})y_1^{s_1-1}y_2^{s_2-1}\frac{\mathrm{d}y_1}{y_1}\frac{\mathrm{d}y_2}{y_2} = \frac{\pi^{-s_1-s_2}}{4}G(s_1,s_2)$$

  with

$$G(s_1,s_2) = \frac{\Gamma\left(\frac{s_1-\alpha_1}{2}\right)\Gamma\left(\frac{s_1-\alpha_2}{2}\right)\Gamma\left(\frac{s_1-\alpha_3}{2}\right)\Gamma\left(\frac{s_2+\alpha_1}{2}\right)\Gamma\left(\frac{s_2+\alpha_2}{2}\right)\Gamma\left(\frac{s_2+\alpha_3}{2}\right)}{\Gamma\left(\frac{s_1+s_2}{2}\right)}.$$

- One can explicitly show a multiplicity one result for GL(3) Whittaker functions: If $\Psi_\alpha(z)$ is a Whittaker function of type α associated to character ψ, and

$$\int_0^\infty\int_0^\infty y_1^{\sigma_1}y_2^{\sigma_2}|\Psi_v(y)|\frac{\mathrm{d}y_1\,\mathrm{d}y_2}{y_1y_2}$$

  converges for σ_1,σ_2 significantly large, then $\Psi_\alpha(z)$ is a constant multiple of $W_\alpha(z,\psi)$.

- Proof idea: Let Δ_1,Δ_2 be the $\mathrm{GL}(3,\mathbb{R})$-invariant differential operators, and let $\widetilde{\psi_\alpha}(s)$ be the double Mellin transform. Show that $\frac{\widetilde{\psi_\alpha}(s)}{\widetilde{W_\alpha}(s)}$ is invariant by taking $(s_1,s_2)\to(s_1+2,s_2)$ and (s_1,s_2+2) by looking at the action of Δ_1,Δ_2 on each term, hence giving a Fourier expansion, then use the growth condition of the Γ functions to show that this expansion must be identically zero.

6.2 Maass forms for $\mathrm{SL}(3,\mathbb{Z})$

- Same as Section 9.1.
- We have the Fourier expansion

$$\psi(z) = \sum_{\gamma\in U_2(\mathbb{Z})\backslash\mathrm{SL}(2,\mathbb{Z})}\sum_{m_1=1}^\infty\sum_{m_2\neq 0}\frac{A(m_1,m_2)}{|m_1m_2|}W_\alpha\left(\begin{pmatrix}|m_1m_2| & & \\ & m_1 & \\ & & 1\end{pmatrix}\begin{pmatrix}\gamma & \\ & 1\end{pmatrix}z,\psi_{1,\frac{m_2}{|m_2|}}\right).$$

- We also know that $\frac{A(m_1,m_2)}{|m_1m_2|}$ is bounded, independent of the choice of γ.

6.3 Dual and symmetric Maass forms

- Same as section 9.2.
- Let
$$\delta_0 = \begin{pmatrix} -1 & & \\ & 1 & \\ & & 1 \end{pmatrix},$$
and for any δ matrix consisting of only 1 and -1 on the diagonals and 1 in the bottom right entry, let
$$T_\delta \phi = \phi(\delta z).$$
Note that T_δ sends Maass forms to Maass forms.

  Recall that a Maass form ϕ is symmetric if $T_\delta \phi = \pm\phi$ for all δ. Specifically, ϕ is even if $T_{\delta_0}\phi = \phi$ and odd if $T_{\delta_0}\phi = -\phi$. Every Maass form can be expressed as a linear combination of symmetric Maass forms.
- For $n = 3$, all symmetric (and hence all) Maass forms are even, since $-I_3$ has determinant -1 and lies in Z.
- Recall that if ϕ is a Maass form of type $\alpha = (\alpha_1, \alpha_2, \alpha_3)$, then the dual Maass form
$$\widetilde{\phi}(z) := \phi((z^{-1})^T)$$
is a Maass form of type $(-\alpha_3, -\alpha_2, -\alpha_1)$. Moreover, the (m_1, m_2, m_3)th coefficient of $\widetilde{\phi}$ is the (m_3, m_2, m_1)th coefficient of ϕ.

6.4 Hecke operators for $SL(3, \mathbb{Z})$

- Same as Section 9.3.
- We have the Hecke operators
$$T_n f(z) = \frac{1}{n} \sum_{\substack{abc=n \\ 0\le c_1, c_2 < c \\ 0 \le b_1 < b}} f\left(\begin{pmatrix} a & b_1 & c_1 \\ 0 & b & c_2 \\ 0 & 0 & c \end{pmatrix} z \right).$$
- By construction (from double coset decompositions), the Hecke operators commute.
- With respect to the Petersson inner product, the Hecke operators are normal; i.e. T_n^* is a Hecke operator (and hence commutes with T_n). The Hecke ring will be the ring of all Hecke operators, T_δs, and the $\mathrm{GL}(3, \mathbb{R})$ differential operators.
- Multiplicativity relations: We have that for an Heckeeigen Maass form, if $A(1, 1) = 0$, then $f \equiv 0$, and otherwise, normalizing such that $A(1, 1) = 1$, we have that $T_n f = A(n, 1)f$, and
  - $A(m_1 m_1', m_2 m_2') = A(m_1, m_2)A(m_1', m_2')$ if $(m_1 m_2, m_1' m_2') = 1$.
  - $A(n, 1)A(m_1, m_2) = \sum_{\substack{d_0 d_1 d_2 = n \\ d_1 \mid m_1 \\ d_2 \mid m_2}} A\left(\frac{m_1 d_0}{d_1}, \frac{m_2 d_1}{d_2}\right).$
- Proof idea: Explicitly compute the result of $T_n f$ using the definition, then induction on prime factors.
- Moreover, if ϕ is a Heckeeigen Maass form, then
$$A(m_1, m_2) = \overline{A(m_2, m_1)}.$$
- Proof: Use that T_n^* is also a Hecke operator, and use its double coset decomposition.

6.5 The Godement-Jacquet L-function

- Same as section 10.4.
- We have the L-function

$$L_f(s)=\sum_{n=1}^{\infty}A(n,1)n^{-s}=\prod_p\left(1-A(p,1)p^{-s}+A(1,p)p^{-2s}-p^{-3s}\right)^{-1}$$

  where the Euler product is derived from the multiplicativity relations. This is absolutely convergent for $\mathrm{Re}(s)>2$, and has holomorphic continutation to all of s.

  Similarly, we have the dual L-function

$$L_{\widetilde{f}}(s)=\sum_{n=1}^{\infty}A(1,n)n^{-s}.$$

- Functional equation: Let

$$\Lambda_f(s)=\pi^{-3s/2}\Gamma\left(\frac{s-\alpha_1}{2}\right)\Gamma\left(\frac{s-\alpha_2}{2}\right)\Gamma\left(\frac{s-\alpha_3}{2}\right)L_f(s),$$

  where $(\alpha_1,\alpha_2,\alpha_3)$ are the Langlands parameters of Maass form f. Then we have functional equation

$$\Lambda_f(s)=\Lambda_{\widetilde{f}}(1-s).$$

  In particular, note that $\widetilde{f}$ has Langlands parameters $(-\alpha_3,-\alpha_2,-\alpha_1)$.

- Proof of functional equation: Two possible approaches:
  - Show that the functional equation of a Maass form must be the same as the Eisensteins series, which we compute in Section 10. This proof doesn't use any information about the arithmetic values of the Fourier coefficients, just the analytic properties of Whittaker functions.
  - Direct proof involving double Mellin transforms (Hoffstein-Murty, 1989). Can get L-function out of double Mellin transform of the Maass form. Do a lot of manipulations involving Maass form and its dual, and use the fact that we know the double Mellin transform of the Whittaker function is the product of six gammas over one.

6.6 Bump's double Dirichlet series

- The double Dirichlet series

$$\sum_{m_1=1}^{\infty}\sum_{m_2=1}^{\infty}\frac{A(m_1,m_2)}{m_1^{s_1}m_2^{s_2}}$$

  has meromorphic continuation to all of s and has functional equations.

- Specifically, we have that

$$\sum_{m_1=1}^{\infty}\sum_{m_2=1}^{\infty}\frac{A(m_1,m_2)}{m_1^{s_1}m_2^{s_2}}=\frac{L_f(s_1)L_{\widetilde{f}}(s_2)}{\zeta(s_1+s_2)}.$$

- Proof idea: Use the multiplicativity relation

$$A(m_1,1)A(1,m_2)=\sum_{d|m_1,d|m_2}A\left(\frac{m_1}{d},\frac{m_2}{d}\right).$$

- These functional equations can be computed using the functional equation of $L_f(s)$ from before, or explicitely computed by retrieving the double Dirichlet series with double Mellin transforms.
- Can be generalized to $\mathrm{GL}(n)$ via multiple Dirichlet series.

9 The Godement-Jacquet L-function

9.1 Maass forms for SL$(n, \mathbb{Z})$

- Recall that we showed that a Maass form ϕ had a Fourier expansion

$$\phi(z) = \sum_{\gamma \in U_{n-1}(\mathbb{Z}) \backslash \mathrm{SL}(n-1,\mathbb{Z})} \sum_{m_1=1}^{\infty} \cdots \sum_{m_{n-2}=1}^{\infty} \sum_{m_{n-1} \neq 0} \widetilde{\phi}_{(m_1,\ldots,m_{n-1})}\left(\begin{pmatrix} \gamma & \\ & 1 \end{pmatrix} z\right),$$

where

$$\widetilde{\phi}_{(m_1,\ldots,m_{n-1})}(z) = \int_0^1 \cdots \int_0^1 \phi(u \cdot z)\overline{\psi_m(u)}\mathrm{d}^*u,$$

where here d^*u is an integral over all the u_{ij} in the unipotent matrix u.

- We know that $\widetilde{\phi}_{(m_1,\ldots,m_{n-1})}$ is a Whittaker function. By the multiplicity one theorem for SL$(n, \mathbb{Z})$ and identities about Whittaker functions, we can write the Fourier expansion in the form

$$\phi(z) = \sum_{\gamma \in U_{n-1}(\mathbb{Z}) \backslash \mathrm{SL}(n-1,\mathbb{Z})} \sum_{m_1=1}^{\infty} \cdots \sum_{m_{n-2}=1}^{\infty} \sum_{m_{n-1} \neq 0} \frac{A(m_1, \ldots, m_{n-1})}{\prod_{k=1}^{n-1} |m_k|^{k(n-k)/2}} W_\alpha\left(M \begin{pmatrix} \gamma & \\ & 1 \end{pmatrix} z, \psi_{1,\ldots,1,\frac{m_{n-1}}{|m_{n-1}|}}\right),$$

where the $A(m_1, \ldots, m_{n-1}) \in \mathbb{C}$ are the Fourier coefficients. The choice of normalization comes from the constant in the identity

$$W_\alpha(z, \psi_m) = c_{\alpha,m} W_\alpha(Mz, \psi_{1,\ldots,1,\pm 1});$$

$c_{\alpha,m}$ is of the form $c_\alpha \prod_{k=1}^{n-1} |m_k|^{-k(n-k)/2}$, where c_α depends only on α and n.

- Now, since

$$\frac{A(m_1, \ldots, m_{n-1})}{\prod_{k=1}^{n-1} |m_k|^{k(n-k)/2}} W_\alpha\left(My, \psi_{1,\ldots,1,\frac{m_{n-1}}{|m_{n-1}|}}\right) = \int_0^1 \cdots \int_0^1 \phi(z)\overline{\psi_m(x)}\mathrm{d}^*x,$$

fixing some choice constant choice of y and using that ϕ is bounded, we conclude that the

$$\frac{A(m_1, \ldots, m_{n-1})}{\prod_{k=1}^{n-1} |m_k|^{k(n-k)/2}}$$

are bounded.

9.2 Dual and symmetric Maass forms

- Symmetric Maass forms: Let δ be a matrix of the form

$$\begin{pmatrix} \delta_1 \ldots \delta_{n-1} & & & & \\ & \ddots & & & \\ & & \delta_1\delta_2 & & \\ & & & \delta_1 & \\ & & & & 1 \end{pmatrix}$$

where each of the δ_i is ± 1. We define the operator

$$T_\delta \phi(z) = \phi(\delta z) = \phi(\delta z \delta),$$

which takes xy to $x'y$, where x' consists of $\delta_i x_i$ on the offdiagonal.

A Maass form is symmetric if $T_\delta \phi = \pm\phi$ for all T_δ, where the choice of $\pm$ depends on δ.

- By linear algebra, every Maass form is a linear combination of symmetric Maass forms, so it suffices to only consider aymmetric Maass forms.
- Let
$$\delta_0 = \begin{pmatrix} -1 & & & & \\ & 1 & & & \\ & & \ddots & & \\ & & & 1 & \\ & & & & 1 \end{pmatrix}.$$
We say that ϕ is even if $T_{\delta_0}\phi = \phi$, and odd if $T_{\delta_0}\phi = -\phi$.
- If n is odd, then all Maass forms are even. In particular, this means that
$$A(m_1, \ldots, m_{n-2}, m_{n-1}) = A(m_1, \ldots, m_{n-2}, -m_{n-1}).$$
Proof idea: ϕ is invariant by $\mathrm{SL}(n, \mathbb{Z})$ and $-I_n$, which has determinant -1, so $T_\delta \phi = \phi$ for all δ. Then looking at T_{δ_0}, only the x_{n-1} variable changes to $-x_{n-1}$, so the corresponding coefficients that must match are $A(m_1, \ldots, m_{n-1})$ and $A(m_1, \ldots, -m_{n-1})$.
- If n is even, then for a symmetric Maass form, we have
$$A(m_1, \ldots, m_{n-2}, m_{n-1}) = \pm A(m_1, \ldots, m_{n-2}, -m_{n-1}),$$
depending on if ϕ is odd or even.

  Proof idea: The δ matrices are determined by $\mathrm{SL}(n, \mathbb{Z})$ and T_{δ_0}; hence ϕ is symmetric iff $T_{\delta_0}\phi = \pm\phi$. (Typo on page 266.)

  Looking at T_{δ_0}, only the x_{n-1} variable changes to $-x_{n-1}$, so the corresponding coefficients when looking at that must match are $A(m_1, \ldots, m_{n-1})$ and $A(m_1, \ldots, -m_{n-1})$, with a $\pm$ depending on the sign of $T_{\delta_0}\phi = \pm\phi$; i.e. when ϕ is even or odd.
- Using the T_δ, we can interpret $A(m_1, \ldots, m_{n-1})$ for symmetric Maass forms, even when the m_i are negative. Note in particular that letting δ be the matrix with the $\delta_i = \frac{m_i}{|m_i|}$, then we can take $A(m_1, \ldots, m_{n-1})$ to be the $(|m_1|, \ldots, |m_{n-1}|)$-th coefficient of $T_\delta \phi$, multiplied by $\det \delta$ if ϕ is odd and 1 if ϕ is even.
- Dual Maass form: (Typos in book.) For ϕ a Maass form of Langlands parameters α, then we define the dual Maass form
$$\widetilde{\phi}(z) = \phi((z^{-1})^T),$$
a Maass form of Langlands parameters $\alpha' = (-\alpha_n, -\alpha_{n-1}, \ldots, -\alpha_1)$. Moreover, if ϕ is symmetric, then the $(m_1, \ldots, m_{n-1})$th coefficient of $\widetilde{\phi}$ is
$$\pm A(m_{n-1}, \ldots, m_1),$$
where the sign is $(-1)^{n-1+\lfloor n/2 \rfloor}$ if ϕ is odd, and 1 if ϕ is even. Equivalently, the sign is -1 if $4 \mid n$ and ϕ is odd, and 1 otherwise.
- Proof idea: $\widetilde{\phi}(\gamma z) = \widetilde{\phi}(z)$ because $\mathrm{SL}(n, \mathbb{Z})$ is preserved by inverse transpose.

  By $\mathrm{SL}(n, \mathbb{Z})$ left invariance and $O(n, \mathbb{R})$ right invariance, we have that
$$\widetilde{\phi}(z) = \phi(w(z^{-1})^T w^{-1}),$$
where
$$w = \begin{pmatrix} & & & (-1)^{\lfloor n/2 \rfloor} \\ & & 1 & \\ & \cdots & & \\ 1 & & & \end{pmatrix}$$

is the long element of the Weyl group for $\mathrm{SL}(n,\mathbb{Z})$. In particular, we note that $w(z^{-1})^T w^{-1}$ has a Iwasawa decomposition of the form $x'y'$, where y' consists of the elements $y_1 \dots y_{n-1}, y_2 \dots y_{n-1}, \dots, y_{n-1}, 1$, and x' consists of 1s on the diagonal, $-x_i$ on the offdiagonal for $2 \le i \le n-1$ and $(-1)^{\lfloor n/2 \rfloor+1} x_1$. One can show by induction (on what the remaining x' terms look like) that

$$\int_{\mathrm{SL}(n,\mathbb{Z}) \cap U) \backslash U} \widetilde{\phi}(z) \, \mathrm{d}u = 0$$

for all U from the cuspidal condition.

In particular, note that the y' reverses the order of the y_i, which in Langlands parameters corresponds to reversing the order of the parameters and negating them. Thus $I_{\alpha'}(z) = I_\alpha(w(z^{-1})^T w^{-1})$, and thus $\widetilde{\phi}$ has Langlands parameters α'.

Finally, the $(m_1, \dots, m_{n-1})$th coefficient of $\widetilde{\phi}$, by the work above, computed by looking at

$$\int_0^1 \cdots \int_0^1 \widetilde{\phi}(z) e^{-2\pi i (m_1 x_1 + \cdots + m_{n-1} x_{n-1})} \mathrm{d}^* x$$

corresponds to $A(-m_{n-1}, -m_{n-2}, \dots, -m_2, (-1)^{\lfloor n/2 \rfloor+1} m_1)$ of ϕ, giving the desired result.

9.3 Hecke operators for $\mathrm{SL}(n,\mathbb{Z})$

- Recall the Hecke theory discussed in Section 3.10. Let $G = \mathrm{GL}(n,\mathbb{R})$, $\Gamma = \mathrm{SL}(n,\mathbb{Z})$, and $X = \mathfrak{h}^n$. In this case, we have the that the matrices

$$\begin{pmatrix} m_0 \dots m_{n-1} & & & \\ & \ddots & & \\ & & m_0 m_1 & \\ & & & m_0 \end{pmatrix},$$

where the m_i are all positive integers lies in the commensurator. Let Δ be the semigroup generated by all of these matrices. Since Δ is preserved under transpose, the Hecke ring generated by Δ is commutative.

- Define the set

$$S_n = \left\{ \begin{pmatrix} c_1 & c_{1,2} & \dots & c_{1,n} \\ & c_2 & \dots & c_{2,n} \\ & & \ddots & \vdots \\ & & & c_n \end{pmatrix} \mid c_\ell \ge 1, \prod_{\ell=1}^n c_\ell = N, 0 \le c_{i,\ell} < c_\ell \right\}.$$

Then we have the double coset partition

$$\bigcup_{m_0^n m_1^{n-1} \dots m_{n-1} = N} \Gamma \begin{pmatrix} m_0 \dots m_{n-1} & & & \\ & \ddots & & \\ & & m_0 m_1 & \\ & & & m_0 \end{pmatrix} \Gamma = \bigcup_{\alpha \in S_N} \Gamma \alpha.$$

Proof idea: Use Hermite and Smith normal form.

- We can use the S_N to define Hecke operators:

$$T_N f(z) = \frac{1}{N^{\frac{n-1}{2}}} \sum_{\substack{\prod_{\ell=1}^n c_\ell = N \\ 0 \le c_{i,\ell} < c_\ell}} f\left(\begin{pmatrix} c_1 & c_{1,2} & \dots & c_{1,n} \\ & c_2 & \dots & c_{2,n} \\ & & \ddots & \vdots \\ & & & c_n \end{pmatrix} \cdot z \right).$$

- With respect to the standard inner product, the Hecke operators are no longer self-adjoint. However, the adjoint of a Hecke operator is itself a Hecke operator, and commutes with the original Hecke operator, so the operator is normal. Mathematically, we have that T_N^* is associated to the double coset union
$$\bigcup_{m_0^n m_1^{n-1}\dots m_{n-1}=N} \Gamma \begin{pmatrix} N\cdot m_0^{-1} & & & \\ & N\cdot(m_0m_1)^{-1} & & \\ & & \ddots & \\ & & & N\cdot(m_0\dots m_{n-1})^{-1} \end{pmatrix} \Gamma.$$
Proof idea: Apply a change of variable in the integral for $\langle T_N f, g\rangle$.

- To form the full Hecke ring, we take ring of operators generated by the T_δ, the $\mathrm{GL}(n,\mathbb{R})$-invariant differential operators, and then T_n.

- As expected, we can use the Hecke operators to gain information about the Fourier coefficients of Hecke-eigen Maass forms.

- For a Heckeeigen Maass form ϕ, if $A(1,\dots,1)=0$, then $\phi=0$. Otherwise, choosing to normalize to let $A(1,\dots,1)=1$, we have that
$$T_m\phi = A(m,1,\dots,1)\phi,$$
with multiplicativity relations
$$A(m_1,\dots,m_{n-1})A(m_1',\dots,m_{n-1}') = A(m_1m_1',\dots,m_{n-1}m_{n-1}')$$
if $\gcd(m_1\cdots m_{n-1}, m_1'\cdots m_{n-1}')=1$, and
$$A(m,1,\dots,1)A(m_1,\dots,m_{n-1}) = \sum_{\substack{\prod_{\ell=1}^n c_\ell = m \\ c_i|m_i}} A\left(\frac{m_1c_n}{c_1}, \frac{m_2c_1}{c_2}, \dots \frac{m_{n-1}c_{n-2}}{c_{n-1}}\right).$$

- Proof idea: Manually compute what the $(m_1,\dots,m_{n-1})$ coefficient of $T_m\phi$ looks like as a sum over the c matrices. Do a change of variable to swap from cx to $x'c$ and working through the relationship gives the equation
$$\lambda_m A(m_1,\dots,m_{n-1}) = \sum_{\substack{\prod_{\ell=1}^n c_\ell = m \\ c_i|m_i}} A\left(\frac{m_1c_n}{c_1}, \frac{m_2c_1}{c_2}, \dots \frac{m_{n-1}c_{n-2}}{c_{n-1}}\right).$$
First assume $A(1,\dots,1)\neq 0$, with normalization $A(1,\dots,1)=1$. We directly get that $T_m\phi = A(m,1,\dots,1)\phi$ and the relation
$$A(m,1,\dots,1)A(m_1,\dots,m_{n-1}) = \sum_{\substack{\prod_{\ell=1}^n c_\ell = m \\ c_i|m_i}} A\left(\frac{m_1c_n}{c_1}, \frac{m_2c_1}{c_2}, \dots \frac{m_{n-1}c_{n-2}}{c_{n-1}}\right).$$
Now, to prove multiplicativity, we use the above relation. One can inductively show
$$A(p^{K_1}m_1, p^{K_2}m_2,\dots,p^{K_{n-1}}m_{n-1}) = A(p^{K_1},p^{K_2},\dots,p^{K_{n-1}})A(m_1,m_2,\dots,m_{n-1})$$
for $p\nmid m_i$ by applying the relation to
$$A(p^{K_0},1,\dots,1)A(p^{K_1}m_1,p^{K_2}m_2,\dots,p^{K_{n-1}}m_{n-1}),$$
first proving it for p^{K_1} only, then p^{K_1} and p^{K_2}, etc. See Goldfeld's paper for more details.
Now, if $A(1,\dots,1)=0$, then all of the $A(m,1,\dots,1)$ are 0, and inductively using the prime power idea, one can show that all of the $A(m_1,\dots,m_{n-1})$ are 0.

- In addition, if ϕ is an eigenform of the full Hecke ring, then
$$A(m_{n-1},\dots,m_1) = \overline{A(m_1,\dots,m_{n-1})}.$$

- Proof idea: Use that the double coset decomposition of T_n^*.

9.4 The Godement-Jacquet L-function

- Given a Hecke-eigen Maass form f, we have the Godement-Jacquet L-function

$$L_f(s) = \sum_{m=1}^{\infty} A(m,1,\ldots,1)m^{-s},$$

absolutely convergent for $\mathrm{Re}(s) > \frac{n+1}{2}$.

- We have an Euler product

$$L_f(s) = \prod_p \phi_p(s),$$

where

$$\phi_p(s) = \sum_{k=0}^{\infty} \frac{A(p^k,1,\ldots,1)}{p^{ks}}.$$

By using the multiplicativity relations, one can show that

$$\phi_p(s) = \Big(1 - A(p,\ldots,1)p^{-s} + A(1,p,\ldots,1)p^{-2s} - \cdots + (-1)^{n-1}A(1,\ldots,p)p^{-(n-1)s} + (-1)^n p^{-ns}\Big)^{-1}$$

- For GL(2), taking the Mellin transform of the Maass form along the y-axis gives the L-function, which we then can use to get the functional equation. We want to do something similar for GL(n), but the $\begin{pmatrix} \gamma & \\ & 1 \end{pmatrix}$ terms cause issues.

- Inductively, we can show that

$$\int_0^1 \cdots \int_0^1 f(\widehat{u}z)e^{-2\pi i(u_1+\cdots+u_{n-2})}\mathrm{d}^*\widehat{u}$$

$$= \sum_{m\neq 0} \frac{A(1,\ldots,1,m)}{|m|^{\frac{n-1}{2}}} e^{2\pi i m x_{n-1}} e^{2\pi i(x_1+\cdots+x_{n-2})} W_\alpha\left(\begin{pmatrix} |m| & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix} y, \psi_{1,\ldots,1}\right),$$

where $\widehat{u}$ is an integral over a unipotent matrix, except for the u_n element (top left offdiagonal element). The induction is by column (right to left).

- Taking the Mellin transform of this formula gives the L function (of the dual modular form, possibly with a minus sign), along with the integral of a Whittaker function. For $n = 3$, we can compute explicitely everything in terms of Gamma functions and Bessel functions to compute the functional equation.

- For higher n, we will get the functional equation from the functional equation of the Eisenstein series.

- The functional equation: For an even Maass form f,

$$\Lambda_f(s) = \pi^{-ns/2} \prod_{i=1}^{n} \Gamma\left(\left(\frac{s-\alpha_i}{2}\right)\right) L_f(s),$$

then

$$\Lambda_f(s) = \Lambda_f(1-s).$$

10 Langlands Eisenstein series

10.1 Parabolic subgroups

- To each partition
$$n = n_1 + \cdots + n_r$$
we associate the parabolic subgroup $P_{n_1,\ldots,n_r}$
$$P_{n_1,\ldots,n_r} = \begin{pmatrix} \mathfrak{m}_{n_1} & & & * \\ & \mathfrak{m}_{n_2} & & \\ & & \ddots & \\ & & & \mathfrak{m}_{n_r} \end{pmatrix},$$
where each $\mathfrak{m}_{n_i} \in \mathrm{GL}(n_i, \mathbb{R})$. Here r is called the rank.
- Two parabolic subgroups are said to be associate if their partitions are permutations of each other. We denote $\Sigma(P, P')$ to be the set of all permutations that permute the partition of P into P' (assuming they are associate).

10.2 Langlands decomposition of parabolic subgroups

- A parabolic subgroup can be decomposed into the form
$$P_{n_1,\ldots,n_r} = N_{n_1,\ldots,n_r} \cdot M_{n_1,\ldots,n_r},$$
where
$$N_{n_1,\ldots,n_r} = \begin{pmatrix} I_{n_1} & & & * \\ & I_{n_2} & & \\ & & \ddots & \\ & & & I_{n_r} \end{pmatrix}$$
is the unipotent radical and
$$M_{n_1,\ldots,n_r} = \begin{pmatrix} \mathfrak{m}_{n_1} & & & \\ & \mathfrak{m}_{n_2} & & \\ & & \ddots & \\ & & & \mathfrak{m}_{n_r} \end{pmatrix}$$
is the Levi component.
- The Levi component can be further decomposed into
$$M_{n_1,\ldots,n_r} = A_{n_1,\ldots,n_r} \cdot M'_{n_1,\ldots,n_r},$$
where
$$A_{n_1,\ldots,n_r} = \begin{pmatrix} t_1 I_{n_1} & & & \\ & t_2 I_{n_2} & & \\ & & \ddots & \\ & & & t_r I_{n_r} \end{pmatrix}$$
with all $t_i > 0$ is the connected center of $M_{n_1,\ldots,n_r}$., and
$$M'_{n_1,\ldots,n_r} = \begin{pmatrix} \mathfrak{m}_{n_1} & & & \\ & \mathfrak{m}_{n_2} & & \\ & & \ddots & \\ & & & \mathfrak{m}_{n_r} \end{pmatrix}$$
with $\det(m_i) = \pm 1$.

- These two together comprise the Langlands decomposition, which is a generalization of the Iwasawa decomposition.
- One can view the parabolic subgroups as stabilizers of flags on $\mathbb{R}^n$.

10.3 Bruhat decomposition

- Let the Weyl group of $\mathrm{GL}(n, \mathbb{R})$ be W_n, and B_n be the group of invertible upper triangular matrices. Then we have the Bruhat decomposition
$$\mathrm{GL}(n, \mathbb{R}) = B_n W_n B_n.$$
- Idea of proof: Left multiplication by B_n corresponds to row operations from lower rows to higher rows, and right multiplication by B_n corresponds to column operations from left columns to right columns. One can inductively reduce the bottom row to one 1 and 0s with right multiplication, then use left multiplication to make all the rows above 0 above the 1.
- One can compute explictely the Bruhat decomposition of a $g \in \mathrm{GL}(n, \mathbb{R})$. For any $\lambda = (\ell_1, \ldots, \ell_k) \in \mathbb{Z}^k$, let $M_\lambda(g)$ be the $k \times k$ minor formed by columns $\ell_1, \ldots, \ell_k$ and the last k rows. Moreover, for $w \in W_n$, let ω be the permutation associated to w such that $we_i = e_{\omega^{-1}(i)}$ for basis column vector e_i.

  Then the Bruhat decomposition is precisely
$$g = u_1 c w u_2,$$
  with $u_1, u_2 \in U_n$ (unipotent), $w \in W_n$, and c a diagonal matrix, with
$$c = \begin{pmatrix} \varepsilon/c_{n-1} & & & & \\ & c_{n-1}/c_{n-2} & & & \\ & & \ddots & & \\ & & & c_2/c_1 & \\ & & & & c_1, \end{pmatrix}$$
  with $\varepsilon = \det(w)\det(g)$, and
$$c_i = M_{\omega(n), \omega(n-1), \ldots, \omega(n-i+1))}(g).$$
  In particular, note that u_1, u_2 are not necessarily unique in this form of the decomposition.

10.4 Minimal, maximal, and general parabolic Eisenstein series

- The minimal parabolic subgroup corresponds to the partition $n = 1 + 1 + \cdots + 1$, and the maximal parabolic subgroup corresponds to the partition $n = (n-1) + 1$.
- For definitions, see notes from Dorian's class. Recall that our power function here already includes ρ, but the power function for Dorian's class doesn't include the ρ, so it must be included.
- The power function $|g|_{\mathcal{P}}^s$ is left-invariant by $\mathcal{P} \cap \Gamma$ by the Langlands decomposition. Thus, the summation over $(\mathcal{P} \cap \Gamma)\backslash\Gamma$ is well-defined. Convergence of the Eisenstein series comes from convergence of the Borel Eisenstein series, plus that the power function for a parabolic subgroup is equivalent to the standard power function by applying the Iwasawa decomposition to each $\mathrm{GL}(n_i)$ block then changing the powers in the standard power function.

10.5 Eisenstein series twisted by Maass forms

- Same thing, see notes from Dorian's class. Only difference is the addition of the induced Maass form.

10.6 Fourier expansion of the minimal parabolic Eisenstein series

- We are interested in computing the $m = (m_1, \ldots, m_{n-1})$th Fourier coefficient of $E_P(z, s)$. There are two approaches - one by direct computation using the Bruhat decomposition, and one by Hecke operators. This section discuss the direct comptuation approach.
- The object of interest is the Fourier coefficient

$$\mathcal{E}_m(z,s) = \int_{U(\mathbb{Z})\backslash U(\mathbb{R})} E_P(uz,s)\overline{\psi_m(u)}\mathrm{d}^*u.$$

- One can partition the summation of $(P \cap \Gamma)\backslash\Gamma$ by the element w in the Bruhat decomposition. In particular, every $\gamma \in (P \cap \Gamma)\backslash\Gamma$ can be expressed in the form

$$b_1 c w b_2 \ell,$$

where $b_1 \in (w^{-1}U(\mathbb{Q})w) \cap U(\mathbb{Q})$, $b_2 \in (w^{-1}U(\mathbb{Q})^T w) \cap U(\mathbb{Q})$, c is a diagonal matrix, $\ell \in w^{-1}P^T w \cap P$, and w lies in the Weyl group. Rearranging the summation gives a product of a Whittaker function times the integral of $\overline{\psi_m}$. The Whittaker function will be the Jacquet-Whittaker function when w is precisely the long element. Moreover, when the m_i are all nonzero, the integral of the character will be 0 unless w is the long element.

- In other words, for m_i all nonzero, the only contribution comes from the long Weyl element, which gives the Jacquet Whittaker function (times a constant).

10.7 Meromorphic continuation and functional equation of maximal parabolic Eisenstein series

- For $P = P_{n-1,1}$, we have Eisenstein series

$$E_P(z,s) = \sum_{\gamma \in (P\cap\Gamma)\backslash\Gamma} |\det(\gamma z)|^{s+\rho}$$

and completed Eisenstein series

$$E_P^*(z,s) = \pi^{-ns/2}\Gamma(ns/2)\zeta(ns)E_P(z,s).$$

- In particular, $E_P(z, s)$ has meromorphic continuation to all of $s \in \mathbb{C}$, with get functional equation

$$E_P^*(z,s) = E_P^*((z^{-1})^T, 1-s).$$

In particular, E_P^* is holomorphic in s except for simple poles at $s = 0, 1$.

- Proof idea: Proof is similar to proof of Riemann zeta functional equation. Letting $f_u(x) = e^{-\pi(x_1^2+x_2^2+\cdots+x_n^2)u}$, one can show that

$$E_P^*(z,s) = |\det(z)|^{s+\rho}\int_0^\infty \left(\sum_{(a_1,\ldots,a_n)\in\mathbb{Z}^n} f_u((a_1,\ldots,a_n)\cdot z) - f((0,\ldots,0))\right) u^{ns/2}\frac{\mathrm{d}u}{u}$$

by using the definition of the Gamma integral.

Splitting the integral into $[0, 1]$ and $[1, \infty]$ and applying the Poisson summation formula

$$\sum_{(a_1,\ldots,a_n)\in\mathbb{Z}^n} f((a_1,\ldots,a_n)z) = \frac{1}{|\det(z)|}\sum_{(a_1,\ldots,a_n)\in\mathbb{Z}^n} \widehat{f}((a_1,\ldots,a_n)(z^{-1})^T)$$

gives the desired functional equation.

10.8 The L-function associated to a minimal parabolic Eisenstein series

- Here, we use the Hecke operators to compute the coefficients of the Eisenstein series – in this case we'll consider the minimal parabolic Eisenstein series.

- Recall that we have the Hecke operators

$$T_m f = \frac{1}{m^{\frac{n-1}{2}}} \sum_{\substack{\prod_{i=1}^n c_i = m \\ 0 \le c_{i,\ell} < c_\ell}} f\left(\begin{pmatrix} c_1 & & & c_{i,j} \\ & c_2 & & \\ & & \ddots & \\ & & & c_n \end{pmatrix} z\right).$$

- By the same work as for Maass forms, one can get a Fourier expansion for Eisenstein series of the form (here note that the m_i are essentially reversed to follow the structure of Chapter 5). Also note that $E(z,s)$ is normalized here such that $A(1,\ldots,1) = 1$ – this will be important later for the completed Eisenstein series:

$$E(z,s) = C(z,s)$$
$$+ \sum_{\gamma \in U_{n-1}(\mathbb{Z})\backslash\Gamma} \sum_{m_1=1}^{\infty} \cdots \sum_{m_{n-1}\neq 0} A((m_1,\ldots,m_{n-1}),s) W\left(\begin{pmatrix} |m_1 m_2 \ldots m_{n-1}| & & & \\ & \ddots & & \\ & & |m_1| & \\ & & & 1 \end{pmatrix} \begin{pmatrix} \gamma & \\ & 1 \end{pmatrix} z, s, \psi_{1,1,\ldots,\frac{m_{n-1}}{|m_{n-1}|}}\right)$$

where the $C(z,s)$ correspond to terms with some $m_i = 0$. This follows because we showed previously that integrating the Eisenstein series against the character ψ_m, when all the m_i are nonzero, the only contribution is the Jacquet-Whittaker function.

- We are interested in computing the exact value of the coefficients $A((m,1,\ldots,1),s)$. We show that it is sufficient to compute the value of the coefficients on the power function $|g|^{s+\rho}$.

- Since the Hecke operators are Γ-invariant, it is sufficient to compute the action of T_m on the power function to get the action of T_m on the Eisenstein series. In particular, by applying the definition directly,

$$\begin{aligned} T_m |g|^{s+\rho} &= \frac{1}{m^{\frac{n-1}{2}}} \left(\sum_{\substack{c_1\ldots c_n = m \\ 0 \le c_{i,\ell} < c_\ell}} \prod_{i=1}^n c_i^{s_i+\rho_i} \right) |g|^{s+\rho} \\ &= \left(\sum_{c_1\ldots c_n = m} \prod_{i=1}^n c_i^{s_i+\rho_i+(i-1)-\frac{n-1}{2}} \right) |g|^{s+\rho} \\ &= \left(\sum_{c_1\ldots c_n = m} \prod_{i=1}^n c_i^{s_i} \right) |g|^{s+\rho}, \end{aligned}$$

where we use that $\rho_i = \frac{n+1}{2} - i$.

- Since the power function is an eigenfunction of all the Hecke operators, so is the Eisenstein series, so by the same logic as for Maass forms, one concludes that T_m gives the $A((m,1,\ldots,1))$ coefficients of the Eisenstein series, with the same multiplicativity relations (The computation for Maass forms to get the multiplicativity relation is the exact same as for the Eisenstein series, as the only thing used is the Fourier expansion.)

- Note that by symmetry, the coefficient

$$A((m,1,\ldots,1),s) = \sum_{c_1\ldots c_n = m} \prod_{i=1}^n c_i^{s_i}$$

remains unchanged upon permutations of s. By the multiplicativity relations, this then holds for all Fourier coefficients of the Eisenstein series. Then, we can defined completed Eisenstein series

$$E^*(z,s) = \left(\prod_{1\le j<k\le n} \pi^{-\frac{1+s_j-s_k}{2}} \Gamma\left(\frac{1+s_j-s_k}{2}\right) \zeta(1+s_j-s_k) \right) (E(z,s) - C(z,s)),$$

(the extra zeta factors here are because of our previous choice to normalize $E(z,s)$ such that it had $A(1,\dots,1)=1$), which then satsifies the functional equation

$$E^*(z,s) = E^*(z,s')$$

for every permutation s' of s. The same equation can be shown for $C(z,s)$.

- Remark: The computation of $A(1,\dots,1)$ arises from the Bruhat decomposition and a Kloosterman sum computation – see Chapter 11.
- Now for the Eisenstein series $E(z,v)$, we can define the L function

$$\begin{aligned} L_{E(*,v)}(s) &= \sum_m A((m,1,\dots,1),v) m^{-s} \\ &= \sum_m \left(\sum_{c_1\dots c_n = m} \prod_{i=1}^n c_i^{v_i} \right) m^{-s} \\ &= \sum_m \left(\sum_{c_1\dots c_{n-1}|m} \prod_{i=1}^{n-1} c_i^{v_i} \right) \left(\frac{m}{c_1\cdots c_{n-1}} \right)^{v_n} m^{-s} \\ &= \sum_{c_1,\dots,c_{n-1},m} \left(\prod_{i=1}^n c_i^{v_i} \right) m^{v_n} (mc_1\dots c_{n-1})^{-s}, \end{aligned}$$

where in the third line we perform the substitution $m \mapsto mc_1\dots c_{n-1}$. Hence we conclude that

$$L_{E(*,v)}(s) = \prod_{i=1}^n \zeta(s-v_i).$$

- We get completed L function

$$\Lambda_{E(*,v)}(s) = \left(\prod_{i=1}^n \pi^{-\frac{s-v_i}{2}} \Gamma\left(\frac{s-v_i}{2}\right) \right) L_{E(*,v)}(s) = \pi^{-\frac{ns}{2}} \left(\prod_{i=1}^n \Gamma\left(\frac{s-v_i}{2}\right) \right) L_{E(*,v)}(s),$$

so we have functional equation

$$\Lambda_{E(*,v)}(s) = \Lambda_{\widetilde{E}(*,v)}(1-s)$$

arising from the functional equation of the Riemann zeta function, where $\widetilde{E}$ is the dual Eisenstein series defined by

$$\widetilde{E}(z,v) = E((z^{-1})^T, v)$$

(which hence takes the Langlands parameters $(v_1,\dots,v_n) \mapsto (-v_n,\dots,-v_1)$).

- Note that this matches the functional equation for an even $\mathrm{SL}(n,\mathbb{Z})$ Maass form - in general, the functional equation for the minimal parabolic can be used as a heuristic for the functional equations of other objects. According to Dorian, "The proof of the functional equation of L-functions associated to Maass forms of GL(n) was first obtained by Godement and Jacquet using a generalized Poisson summation formula and the methods in Tate's thesis." For more details, see Dorian's book with Hundley or his paper with Jacquet.

10.9 Fourier coefficients of Eisenstein series twisted by Maass forms

- One can perform the same computation for the Borel Eisenstein series as for any general Langlands Eisenstein series to get the Fourier coefficients – the only difference is now the different power function.
- Now, consider any Langlands Eisenstein series twisted by Hecke-Maass forms. Let $P = P_{n_1,\dots,n_r}$ be the corresponding partition, with $s = (s_1, \dots, s_r)$ the parameters such that $\sum n_i s_i = 0$. Let ϕ_i be a $\mathrm{SL}(n_i)$ Maass form, and let $\Phi(z) = \prod_{i=1}^r \phi(m_{n_i})$ be the corresponding induced Maass form, all normalized with $A((1,\dots,1))$ coefficients equal to 1. Then recall that we have Langlands Eisenstein series

$$E_P(z,s,\phi) = \sum_{(P\cap\Gamma)\backslash\Gamma} \Phi(\gamma z)|\gamma z|_P^{s+\rho},$$

where $\rho_i = \frac{n-n_i}{2} - n_1 - \cdots - n_{i-1}$.

- One can show that $E_P(z,s,\phi)$ is an eigenfunction of the Hecke operators, with eigenvalue

$$\lambda_m(s) = \left(\sum_{C_1\dots C_r=m} \prod_{i=1}^r A_i(C_i)C_i^{s_i}\right),$$

where $A_i(C_i)$ is the eigenvalue of T_{C_i} applied to ϕ_i.

- Proof idea: Again, since the Hecke operators are Γ-invariant, it suffices to consider the action of the Hecke operators on the Maass form times the power function. We have that

$$T_m \prod_{i=1}^r \phi_i(m_{n_i}(z))|\det(m_{n_i}(z))|^{s_i+\rho_i} = m^{-\frac{n-1}{2}} \sum_{\substack{c_1\dots c_n=m\\0\le c_{i,\ell}<c_\ell}} \prod_{i=1}^r \phi_i(m_{n_i}(cz))|\det(m_{n_i}(cz))|^{s_i+\rho_i}.$$

We can break up c into blocks of size n_i, of determinant C_i, and treat each individual block as its own Hecke operator; the elements above each $n_i \times n_i$ block on the diagonal do not affect the computation. In particular, $\mathfrak{m}_{n_i}(cy)$ corresponds to the n_i block acting on z. Let $\eta_1 = 0$ and $\eta_i = n_1 + \cdots + n_{i-1}$. Note in particular that $\rho_i + \eta_i = \frac{n-n_i}{2}$. One can show, in particular, that

$$\begin{aligned}
&m^{-\frac{n-1}{2}} \sum_{\substack{c_1\dots c_n=m\\0\le c_{i,\ell}<c_\ell}} \prod_{i=1}^r \phi_i(m_{n_i}(cz))|\det(m_{n_i}(cz))|^{s_i+\rho_i}\\
=&m^{-\frac{n-1}{2}} \sum_{C_1\dots C_r=m} \sum_{c_i \text{ above the diagonal in the } C_i \text{ block}} \prod_{i=1}^r C_i^{\eta_i}\phi_i(m_{n_i}(C_{n_i}z))|\det(m_{n_i}(C_{n_i}z))|^{s_i+\rho_i}\\
=&m^{-\frac{n-1}{2}} \sum_{C_1\dots C_r=m} \prod_{i=1}^r C_i^{\eta_i} C_i^{\frac{n_i-1}{2}} C_i^{s_i+\rho_i} T_{C_i}\phi_i(m_{n_i}(z))|\det(m_{n_i}(z))|^{s_i+\rho_i}\\
=&\left(\sum_{C_1\dots C_r=m} \prod_{i=1}^r A_i(C_i)C_i^{s_i}\right)\phi_i(m_{n_i}(z))|\det(m_{n_i}(z))|^{s_i+\rho_i},
\end{aligned}$$

applying all of the identities of ρ_i and η_i. We conclude that (assuming first coefficient equal to 1)

$$A((m,1,\dots,1),s) = \lambda_m(s) = \sum_{C_1\dots C_r=m} \prod_{i=1}^r A_i(C_i)C_i^{s_i}.$$

10.10 The constant term

- Rather than a singular constant term in the expansion of the Eisenstein series, we can define one for each parabolic subgroup.

- Let $E_P(s,z,\phi)$ be a Langlands Eisenstein series as defined before, and let P' be another parabolic subgroup with Langlands decomposition $P' = N'M'$ (in particular, recall that N' is the unipotent portion). Then we define the constant term of E_P along the parabolic P' to be

$$\int_{N'(\mathbb{Z})\backslash N'(\mathbb{R})} E_P(uz, s, \phi)\mathrm{d}^*u,$$

where $N'(\mathbb{Z}) = N'(R) \cap \mathrm{SL}(n,\mathbb{Z})$.

- Remark: Note that this integral is the exact same as the one we did for Maass cusp forms; in particular, Maass (cusp) forms ϕ are those such that the constant term of ϕ with respect to every parabolic subgroup is 0.
- One can show that the constant term of E_P along P' is 0 if P has lower rank than P', or if P and P' are same rank but are not associate.
- Idea of proof: Doing computation involving the Bruhat decomposition gives an integral of a cusp form over essentially the intersection of N' and P - if P has lower rank or is associate, then some nonzero elements will lie in their intersection and contribute 0.

10.11 The constant term of SL$(3,\mathbb{Z})$ Eisenstein series twisted by SL$(2,\mathbb{Z})$-Maass forms

- For the Eisenstein series $E_{P_{2,1}}(z,s,\phi)$, we have the constant terms
  - 0 for $P_{1,1,1}$
  - $2(y_1^2y_2)^{s+1/2}\phi(z_2)$ for $P_{2,1}$
  - $2y_1^{1/2-s}y_2^{1-2s}\frac{\Lambda_\phi(\lambda-1)}{\Lambda_\phi(\lambda)}\phi(z_1)$

  where

$$\Lambda_\phi(s) = \pi^{-s}\Gamma\left(\frac{s+\varepsilon+\alpha}{2}\right)\Gamma\left(\frac{s+\varepsilon-\alpha}{2}\right)L_\phi(s)$$

  if ϕ has Langlands parameters $(\alpha, -\alpha)$, and $\lambda = 3s+1$ if ϕ is even and $3s+2$ if ϕ is even.

10.12 An application of the theory of Eisenstein series to the non-vanishing of L-functions on the line Re$(s) = 1$

- In the constant terms of the (twisted) Eisenstein series, L-functions appear in the coefficients. Thus we can use information about the Eisenstein series to infer information about the L-functions.
- This method can be extended to explicit zero-free regions for general automorphic L-functions. Here we specifically, we show that L-functions have no zeros on the line $\mathrm{Re}(s) = 1$ for GL(2) and GL(3).
- In the $n = 2$ case, the Riemann zeta functions appear in the constant term of the coefficients. Thus, we can use the $n = 2$ Eisenstein series to conclude information about the non-vanishing of the Riemann zeta function on $\mathrm{Re}(s) = 1$.
- Proof idea: Suppose $\zeta(1+it_0) = 0$ for some t_0. Then the constant term of $E^*(z,s)$ is

$$2\zeta^*(1+2s)y^{s+1/2} + 2\zeta^*(2s)y^{1/2-s}$$

  where ζ^* is the completed Riemann ζ function. Thus, $E^*(z, it_0/2)$ has 0 as a constant term (as $\zeta^*(1-it_0) = \overline{\zeta^*(1+it_0)} = 0$). Hence, since Whittaker functions are rapidly decaying, and is non-zero/non-constant by taking $y \to \infty$. Thus, $E^*(z, it_0/2)$ is a Maass (cusp) form with Langlands parameters $(it_0/2, -it_0/2)$. However, by the spectral decomposition, since Eisenstein series are orthogonal to cusp forms (but $E^*(z, it_0/2)$ is itself is a cusp form),

$$\langle E^*(z, it_0/2), E^*(z, it_0/2)\rangle = 0,$$

which is a contradiction as $E^*(z, it_0/2)$ is nonzero. Thus, we conclude that $\zeta(s)$ has no zeros on $\mathrm{Re}(s) = 1$.

- One can do something similar for a Maass form ϕ by considering

$$E^*_{P_{2,1}}(z, s\phi) = \Lambda_\phi(\lambda) E_{P_{2,1}}(z, s, \phi)$$

and considering a specific value of s such that the constant term vanishes assuming $L_\phi(s)$ vanishes on $\mathrm{Re}(s) = 1$.