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Induction

In regular induction, the induction hypothesis assumes $p(k)$ is true to prove that $p(k+1)$ is true. For strong induction, in the induction hypothesis, use the fact that for all j, $n_0 \leq j < k+1$, $p(j)$ is true to show that $p(k+1)$ is true.

1. For m a positive integer and n an integer greater than 2, define $f_1(n) = n, f_2(n) = n^{f_1(n)}, ..., f_{i+1}(n) = n^{f_i(n)}, ...$ Prove that $f_m(n) < n!!! < f_{m+1}(n)$, where the term in the middle has m factorials.

2. Let F_n be the nth Fibonacci number. Prove that F_{5n} is divisible by 5 for every $n \geq 1$.

3. Prove that two consecutive Fibonacci numbers are always relatively prime.

Invariants and Monovariants

An invariant is a property about a mathematical object or a family of mathematical objects that is unchanged under certain operations. The key is to find some invariant.

Ex: The Euclidean area is invariant under linear maps with a matrix of determinant = 1.

A monovariant may change but only in one direction(constantly increasing or constantly decreasing)

1. A real number is written in each square of an n × n chessboard. We can perform the operation of changing all signs of the numbers in a row or a column. Prove that by performing this operation a finite number of times we can produce a new table for which the sum of each row or column is positive.

2. A deck of n cards, labelled 1, . . . , n is shuffled by looking at the label of the top card, k, then reversing the order of the top k cards. Show that this eventually places card 1 on top.

3. At first, a room is empty. Each minute, either one person enters or two people leave. After exactly 3^{2024} minutes, could the room contain $3^{1000}+2$ people

4. If 127 people play in a singles tennis tournament, prove that at the end of the tournament the number of people who have played an odd number of games is even.

5. Passengers P_1,..., P_n enter a plane with n seats. Each passenger has a different assigned seat. The first passenger sits in the wrong seat. Thereafter, each passenger either sits in their seat if unoccupied or otherwise sits in a random unoccupied seat. What is the probability that the last passenger sits in their own seat?

Pigeonhole Principle

If $a > b$ pigeons are placed into b boxes, then there exists at least one box with at least two pigeons.

1. Recall that a regular icosahedron is a convex polyhedron having 12 vertices and 20 faces; the faces are congruent equilateral triangles. On each face of a regular icosahedron is written a nonnegative integer such that the sum of all 20 integers is 39. Show that there are two faces that share a vertex and have the same integer written on them.

2. Show that there is a multiple of 2024 whose digits are only 0 and 1

3. Prove that there is some power of 2 that begins with 2024

4. A chess player trains by playing at least one game per day, but, to avoid exhaustion, no more than 12 games a week. Prove that there is a group of consecutive days in which he plays exactly 20 games.

Further reading if interested: Ordered sets, Vieta Jumping(Highly unlikely to be relevant to Putnam)