Putnam 10/7

Analysis: sequences, inequalities

Cauchy-Schwartz:

$(\sum_{k=1}^{n} a_k^2)(\sum_{k=1}^{n} b_k^2) \geq (\sum_{k=1}^{n} a_k b_k)^2$ for vectors a, b in $\mathbf{R}^n$

Triangle Inequality:

$\|x+y\| \leq \|x\| + \|y\|, \| \|x\| - \|y\| \| \leq \|x-y\|$

Arithmetic Mean - Geometric Mean Inequality

If $x_1, x_2, \ldots, x_n$ are nonnegative real numbers, then $\frac{x_1+x_2+\ldots+x_n}{n} \geq \sqrt[n]{x_1 x_2 \ldots x_n}$

1. Prove that for all numbers $x \in \mathbf{R}, 2^x + 3^x - 4^x + 6^x - 9^x \leq 1$.

2. If $a_1 + a_2 + \cdots + a_n = n$, prove that $a_1^4 + \cdots + a_n^4 \geq n$.

3. If x and y are positive real numbers, then $x^y + y^x > 1$.

4. Find all positive integers $n, k_1, \ldots, k_n$ such that $k_1 + \cdots + k_n = 5n - 4$ and $\frac{1}{k_1} + \cdots + \frac{1}{k_n} = 1$.

5. If a, b, c are positive numbers, prove that $9a^2b^2c^2 \leq (a^2b + b^2c + c^2a)(ab^2 + bc^2 + ca^2)$

6.(AM-GM) Show that all real roots of the polynomial P(x) = $x^5 - 10x + 35$ are negative.

Recurrence Relations

Consider the sequence $(u_n)_n$ with $u_0 = u_1 = u_2 = 1$, and det $\begin{pmatrix} u_{n+3} & u_{n+2} \\ u_{n+1} & u_n \end{pmatrix}$ = n!, where $n \geq 0$. Prove that u_n is an integer for all n.

1. Find a formula for the general term of the sequence 1, 2, 2, 3, 3, 3, 4, 4, 4, 4, 5, 5, 5, 5, 5, . .

2. Consider the sequences $(a_n)_n$ and $(b_n)_n$ defined by $a_1 = 3, b_1 = 100, a_{n+1} = 3^{a_n}, b_{n+1} = 100^{b_n}$. Find the smallest number m for which $b_m > a_{100}$.

3. Define the sequence $(a_n)_{n\geq 0}$ by $a_0 = 0, a_1 = 1, a_2 = 2, a_3 = 6$ and $a_{n+4} = 2a_{n+3} + a_{n+2} - 2a_{n+1} - a_n$, for n $\geq$ 0. Prove that n divides a_n for all $n \geq 1$.

Limits

A sequence $(x_n)_n$ converges to a limit L iff for all $\varepsilon > 0$ there exists some $n = n(\varepsilon)$ such that for all $N \geq n, \|x_N - L\| < \varepsilon$.

One property: For sequences where $a_n \leq b_n \leq c_n$ and a and c converge to L, then b also converges to L.

1. Let $(a_n)_n$ be a sequence of real numbers with the property that for any $n \geq 2$ there exists an integer k, $\frac{n}{2} \leq k < n$, such that $a_n = \frac{a_k}{2}$. Prove that $\lim_{n\to\infty} a_n = 0$.

2. Let $(x_n)_n$ be a sequence of positive integers such that x_{x_n} = n^4 for all $n \geq 1$. Is it true that $\lim_{n\to\infty} x_n = \infty$?

Mean Value Theorem

1. Prove that not all zeros of the polynomial P(x) = $x^4 - \sqrt{7}x^3 + 4x^2 - \sqrt{22}x + 15$ are real.

2. Find all real solutions to the equation $6^x + 1 = 8^x - 27^{x-1}$.

3. f : [a, b] $\to$ **R** be a function, continuous on [a, b] and differentiable on (a, b). Prove that if there exists c $\in$ (a, b) such that $\frac{f(b)-f(c)}{f(c)-f(a)} < 0$ then there exists $\xi \in (a,b)$ such that $f'(\xi) = 0$

A1 Determine all ordered pairs of real numbers (a,b) such that the line $y = ax + b$ intersects the curve $y = \ln(1 + x^2)$ in exactly one point.

A2 Let n be an integer with $n \geq 2$. Over all real polynomials $p(x)$ of degree n, what is the largest possible number of negative coefficients of $p(x)^2$?

A3 Let p be a prime number greater than 5. Let $f(p)$ denote the number of infinite sequences $a_1, a_2, a_3, \ldots$ such that $a_n \in \{1, 2, \ldots, p-1\}$ and $a_n a_{n+2} \equiv 1 + a_{n+1} \pmod{p}$ for all $n \geq 1$. Prove that $f(p)$ is congruent to 0 or 2 $\pmod{5}$.

A4 Suppose that $X_1, X_2, \ldots$ are real numbers between 0 and 1 that are chosen independently and uniformly at random. Let $S = \sum_{i=1}^{k} X_i/2^i$, where k is the least positive integer such that $X_k < X_{k+1}$, or $k = \infty$ if there is no such integer. Find the expected value of S.

A5 Alice and Bob play a game on a board consisting of one row of 2022 consecutive squares. They take turns placing tiles that cover two adjacent squares, with Alice going first. By rule, a tile must not cover a square that is already covered by another tile. The game ends when no tile can be placed according to this rule. Alice's goal is to maximize the number of uncovered squares when the game ends; Bob's goal is to minimize it. What is the greatest number of uncovered squares that Alice can ensure at the end of the game, no matter how Bob plays?

A6 Let n be a positive integer. Determine, in terms of n, the largest integer m with the following property: There exist real numbers $x_1, \ldots, x_{2n}$ with $-1 < x_1 < x_2 < \cdots < x_{2n} < 1$ such that the sum of the lengths of the n intervals

$$[x_1^{2k-1}, x_2^{2k-1}], [x_3^{2k-1}, x_4^{2k-1}], \ldots, [x_{2n-1}^{2k-1}, x_{2n}^{2k-1}]$$

is equal to 1 for all integers k with $1 \leq k \leq m$.