Putnam 11-11

November 25, 2024

1 Identities

$\sin^2 a + \cos^2 b = 1$
$\sin(a+b) = \sin a \cos b + \sin b \cos a$
$\cos(a+b) = \cos a \cos b - \sin a \sin b$
$\sin^2 a = \frac{1-\cos(2a)}{2}$
$\cos^2 a = \frac{1+\cos(2a)}{2}$
$\sin a + \sin b = 2\sin\left(\frac{a+b}{2}\right)\cos\left(\frac{a-b}{2}\right)$
$\cos a + \cos b = 2\cos\left(\frac{a+b}{2}\right)\cos\left(\frac{a-b}{2}\right)$
$e^{ix} = \cos x + i \sin x$

2 Easy Problems

1. Find the range of the function $f : \mathbb{R} \to \mathbb{R}$, $f(x) = (\sin x + 1)(\cos x + 1)$
2. Compute the value of the sum

   $$\frac{1}{1+\tan^3 0^\circ} + \frac{1}{1+\tan^3 10^\circ} + \frac{1}{1+\tan^3 20^\circ} + \frac{1}{1+\tan^3 30^\circ} + \frac{1}{1+\tan^3 40^\circ} + \frac{1}{1+\tan^3 50^\circ} + \frac{1}{1+\tan^3 60^\circ} + \frac{1}{1+\tan^3 70^\circ} + \frac{1}{1+\tan^3 80^\circ}.$$

   [2010 Math Prize for Girls #15]
3. Prove that $\sin^3 18^\circ + \sin^2 18^\circ = \frac{1}{8}$. [1995 Baltic Way #7]
4. Let x and y be positive real numbers such that $x^2 + y^2 = 1$ and $(3x - 4x^3)(3y - 4y^3) = -\frac{1}{2}$. Compute $x + y$. [2012 HMMT-F Guts #18]

3 Medium Problems

1. Prove that $\cos\frac{\pi}{7} - \cos\frac{2\pi}{7} + \cos\frac{3\pi}{7} = \frac{1}{2}$. [1963 IMO #5]
2. Let $z = x + iy$ be a complex number with x and y rational and with $|z| = 1$. Show that the number $|z^{2n} - 1|$ is rational for every integer n. [1973 Putnam B2]

3. Prove that for every natural number n, and for every real number $x \neq \frac{k\pi}{2^t}$ ($t = 0, 1, \ldots, n$; k any integer)

$$\frac{1}{\sin 2x} + \frac{1}{\sin 4x} + \cdots + \frac{1}{\sin 2^n x} = \cot x - \cot 2^n x$$

[1966 IMO #4]

4. Let $a_0 = \pi/2$, and let $a_n = \sin(a_{n-1})$ for $n \geq 1$. Determine whether

$$\sum_{n=1}^{\infty} a_n^2$$

converges. [2020 Putnam A3]

4 Hard Problems

1. Prove

$$\frac{1}{\cos 0^\circ \cos 1^\circ} + \frac{1}{\cos 1^\circ \cos 2^\circ} + \cdots + \frac{1}{\cos 88^\circ \cos 89^\circ} = \frac{\cos 1^\circ}{\sin^2 1^\circ}.$$

[1992 USAMO #2]

2. Compute the value of

$$\sin(6^\circ) \cdot \sin(12^\circ) \cdot \sin(24^\circ) \cdot \sin(42^\circ) + \sin(12^\circ) \cdot \sin(24^\circ) \cdot \sin(42^\circ).$$

[ARML I-10]

3. Determine the greatest possible value of $\sum_{i=1}^{10} \cos(3x_i)$ for real numbers $x_1, x_2, \ldots, x_{10}$ satisfying $\sum_{i=1}^{10} \cos(x_i) = 0$. [2018 Putnam A3]