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1. Let p be a prime number greater than 5. Let $f(p)$ denote the number of infinite sequences $a_1, a_2, a_3, \ldots$ such that $a_n \in \{1, 2, \ldots, p-1\}$ and
$$a_n a_{n+2} \equiv 1 + a_{n+1} \pmod{p}$$
for all $n \geq 1$. Prove that $f(p)$ is congruent to 0 or $2 \pmod{5}$.

   (quadratic residue from number theory) (2022 A3)

2. Let A be an $m \times n$ matrix with rational entries. Suppose that there are at least $m+n$ distinct prime numbers among the absolute values of the entries of A. Show that the rank of A is at least 2.

3. Alan and Barbara play a game in which they take turns filling entries of an initially empty 2008 $\times$ 2008 array. Alan plays first. At each turn, a player chooses a real number and places it in a vacant entry. The game ends when all the entries are filled. Alan wins if the determinant of the resulting matrix is nonzero; Barbara wins if its is zero. Which player has a winning strategy?

4. Let f be a three-times differentiable function (defined on $\mathbb{R}$ and real-valued) such that f has at least five distinct real zeros. Prove that the function
$$f + 6f' + 12f'' + 8f'''$$
has at least two distinct real zeros. (2015 B1)

5. Let $A, B \in M_n(\mathbb{R})$, with $A \neq B$, such that $A^3 = B^3$ and $A^2B = AB^2$. Is it possible that the matrix $A^2 + B^2$ is invertible? (1991 A2)

6. If A and B are square matrices of the same size such that $ABAB = 0$, does it follow that $BABA = 0$?

7. Does there exist a non-zero polynomial $f(x)$ for which $xf(x-1) = (x+1)f(x)$ for all x?

8. Let $(F_n)_n$ be the Fibonacci sequence. Prove the identity
$$F_{n+1}F_{n-1} - F_n^2 = (-1)^n,$$
for all $n \geq 1$.

9. If $P(x)$ is a polynomial of degree n such that
$$P(k) = \frac{k}{k+1} \quad \text{for } k = 0, \ldots, n,$$
determine $P(n+1)$.

10. A repunit is a positive integer which looks like 111 . . . 1. Find all polynomials f with real coefficients such that if n is a repunit, then so is $f(n)$.