Putnam Seminar 3 10/14

1 Formulas

$(1-x)^{-n} = \sum\limits_{i=0}^{\infty} \binom{n+i-1}{i} x^i$

$\sum\limits_{i=0}^{n} ar^i = \frac{a-ar^{n+1}}{1-r}$

2 Easier Generating Functions

1. What is the coefficient of x^{10} in the binomial $(1+x)^{-2}$
2. Find the coefficient of x^{17} in $(1+x+x^2+x^3+x^4+x^5+x^6+x^7+x^8+x^9)^3$
3. An integer between 1000 and 9999, inclusive, is called balanced if the sum of its two leftmost digits equals the sum of its two rightmost digits. How many balanced integers are there? [2003 AIME I #9]
4. Consider polynomials $P(x)$ of degree at most 3, each of whose coefficients is an element of $\{0,1,2,3,4,5,6,7,8,9\}$. How many such polynomials satisfy $P(-1) = -9$? [2018 AMC 12B #22]
5. On each face of two dice some positive integer is written. The two dice are thrown and the numbers on the top face are added. Determine whether one can select the integers on the faces so that the possible sums are $2,3,4,5,6,7,8,9,10,11,12,13$, all equally likely? [1993 Baltic Way #15]

3 Harder Generating Functions

1. For nonnegative integers n and k, define $Q(n,k)$ to be the coefficient of x^k in the expansion of $(1+x+x^2+x^3)^n$. Prove that
$$Q(n,k) = \sum_{j=0}^{k} \binom{n}{j}\binom{n}{k-2j},$$
where $\binom{a}{b}$ is the standard binomial coefficient. (Reminder: For integers a and b with $a \geq 0$, $\binom{a}{b} = \frac{a!}{b!(a-b)!}$ for $0 \leq b \leq a$, with $\binom{a}{b} = 0$ otherwise.) [1992 Putnam B2]
2. Let k be a positive integer and let $m = 6k-1$. Let
$$S(m) = \sum_{j=1}^{2k-1} (-1)^{j+1} \binom{m}{3j-1}.$$
for example with $k = 3$,
$$S(17) = \binom{17}{2} - \binom{17}{5} + \binom{17}{8} - \binom{17}{11} + \binom{17}{14}.$$
Prove that $S(m)$ is never zero. $\left[\text{As usual, } \binom{m}{r} = \frac{m!}{r!(m-r)!}.\right]$ [1983 Putnam A4]
3. Let S be the set of sequences of length 2018 whose terms are in the set $\{1,2,3,4,5,6,10\}$ and sum to 3860. Prove that the cardinality of S is at most
$$2^{3860} \cdot \left(\frac{2018}{2048}\right)^{2018}.$$
[2018 Putnam B6]

4 Potential Proof Problems

1. Prove that $11^{n+2} + 12^{2n+1}$ is divisible by 133 for all natural numbers, n.
2. Prove that the number of partitions of n into distinct parts is equal to the number of partitions of n into odd parts.
3. Show that every positive rational number can be written as a quotient of products of factorials of (not necessarily distinct) primes. For example,
$$\frac{10}{9} = \frac{2! \cdot 5!}{3! \cdot 3! \cdot 3!}.$$
[2009 Putnam B1]