Columbia Putnam Seminar

12/2/24

1 Probability

- Definition of (discrete) probability: We have sample space Ω consisting of elements ω. Subsets of Ω are events. We have a probability function $\mathbb{P}$ such that $\mathbb{P}(\omega) \geq 0$ and $\sum_{\omega \in \Omega} \mathbb{P}(\omega) = 1$.
  - Example: For rolling a fair die, the sample space could be $\Omega = \{1, 2, 3, 4, 5, 6\}$. $\mathbb{P}$ applied to each element is $1/6$. An event of rolling an even number would be represented by $\{2, 4, 6\}$.
- Principle of Inclusion - Exclusion: $\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B)$
- Independence: If events A and B are independent, $\mathbb{P}(A \cap B) = \mathbb{P}(A)\mathbb{P}(B)$.
- Bayes' Rule: $\mathbb{P}(A|B) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)}$.
- Random variable: A random variable X is a function $X : \Omega \to \mathbb{R}$.
  - Example: A random variable representing the square of the result of a dice roll could be $X(\omega) = \omega^2$, for the sample space $\Omega = \{1, 2, 3, 4, 5, 6\}$.
- Expectation: $\mathbb{E}[X] = \sum_{\omega \in \Omega} \mathbb{P}(\omega) X(\omega)$. $\mathbb{E}[f(X)] = \sum_{\omega \in \Omega} \mathbb{P}(\omega) f(X(\omega))$.
- Linearity of expectation: $\mathbb{E}[X+Y] = \mathbb{E}[X] + \mathbb{E}[Y]$, even when X and Y are dependent random variables!
- Indicator variable: A variable 1_A associated to an event A, which is 1 if A happens and 0 if A doesn't happen. Note that $\mathbb{E}[1_A] = \mathbb{P}(A)$. Often useful to split a problem into the correct indicator variables to solve the problem.
- Continuous probability: We have the probability density function

  $$f_X(x) = \frac{\mathrm{d}}{\mathrm{d}x} \mathbb{P}(X \leq x).$$

  Note that $f_X(x) \geq 0$, but can be arbitrarily large. We have the following properties:
  - $\mathbb{P}(a \leq X \leq b) = \int_a^b f_X(x)\,\mathrm{d}x$. In particular, $\int_{-\infty}^{\infty} f_X(x)\,\mathrm{d}x = 1$.
  - $\mathbb{E}[X] = \int_{-\infty}^{\infty} x f_X(x)\,\mathrm{d}x$
  - $\mathbb{E}[g(X)] = \int_{-\infty}^{\infty} g(x) f_X(x)\,\mathrm{d}x$.
- When solving continuous probability problems, its useful to think of the following analogy between discrete and continuous probability:

  $$\mathbb{P}(X = x) \sim f_X(x)$$
  $$\sum \mathbb{P}(X = x) \sim \int f_X(x)\,\mathrm{d}x\,.$$
- Geometric Probability - represent the sample space as a geometric object, areas correspond to probabilities

2 Easier Problems

1. A bag contains 5 blue and 5 red marbles. If marbles are chosen uniformly at random from the bag without replacement, what is the probability that the third marble is red?
2. Jen enters a lottery by picking 4 distinct numbers from $S = \{1, 2, 3, \cdots, 9, 10\}$. 4 numbers are randomly chosen from S. She wins a prize if at least two of her numbers were 2 of the randomly chosen numbers, and wins the grand prize if all four of her numbers were the randomly chosen numbers. What is the probability of her winning the grand prize given that she won a prize?
3. A husband and wife agree to meet at a street corner between 4 and 5 o'clock to go shopping together. The one who arrives first will await the other for 15 minutes, and then leave. What is the probability that the two meet within the given time interval, assuming that they can arrive at any time with the same probability?
4. N numbers are chosen independently and uniformly at random from $[0, 1]$. What is the expected value of the largest of the N numbers?
5. Three points, A, B, and C, are selected independently and uniformly at random from the interior of a unit square. Compute the expected value of $\angle ABC$.
6. You pick 5 numbers uniformly at random from $\{1, 2, \dots, 10\}$. Compute the expected number of distinct numbers you pick.
7. 10 freshmen and 10 sophomores randomly line up for a photo. Let X be the number of places in line where a freshman is standing next to a sophomore. What is the expected value of X?
8. Suppose X is a nonnegative random variable such that

$$\mathbb{P}(X > r + s | X > r) = \mathbb{P}(X > s)$$

for all positive real numbers r and s. What is the distribution for X; i.e. what is $\mathbb{P}(X > k)$ for any positive real number k?

3 Putnam-Level Problems

1. At the Berkeley Mart for Technology, every item has a real-number cost independently and uniformly distributed from 0 to 2022. Sumith buys different items at the store until the total amount he spends strictly exceeds 1. Compute the expected value of the number of items Sumith buys
2. Suppose that the plane is tiled with an infinite checkerboard of unit squares. If another unit square is dropped on the plane at random with position and orientation independent of the checkerboard tiling, what is the probability that it does not cover any of the corners of the squares of the checkerboard?
3. Shanille O'Keal shoots free throws on a basketball court. She hits the first and misses the second, and thereafter the probability that she hits the next shot is equal to the proportion of shots she has hit so far. What is the probability she hits exactly 50 of her first 100 shots?
4. You have coins $C_1, C_2, \dots, C_n$. For each k, C_k is biased so that, when tossed, it has probability $1/(2k + 1)$ of falling heads. If the n coins are tossed, what is the probability that the number of heads is odd? Express the answer as a rational function of n.
5. Two real numbers x and y are chosen at random in the interval (0,1) with respect to the uniform distribution. What is the probability that the closest integer to x/y is even? Express the answer in the form $r + s\pi$, where r and s are rational numbers.
6. Suppose that $X_1, X_2, \dots$ are real numbers between 0 and 1 that are chosen independently and uniformly at random. Let $S = \sum_{i=1}^{k} X_i/2^i$, where k is the least positive integer such that $X_k < X_{k+1}$, or $k = \infty$ if there is no such integer. Find the expected value of S.