Ch. 1:

Most Basic Sieve: Eratosthenes — cross out $p\mathbb{Z}$ up to let X for $p \le \sqrt{X}$.

$$\pi(x) - \pi(\sqrt{x}) + 1 = \sum_{p \mid d \Rightarrow p \le \sqrt{x}} \mu(d) \left[\tfrac{x}{d} \right]$$

Delete $[\cdot]$:

$$= x \sum \mu(d)/d + R$$

R: by big info, trivial $\le 2^{\pi(\sqrt{x})}$

huge anyway, $\left(\sum \mu(d)/d \sim \frac{2e^{-\gamma}}{\log x} \right)$

Improve by limiting p.

z is the "sifting level"

Given a function $a(n): \mathbb{N} \to \mathbb{R}_{\ge 0}$, we consider

$$S(A, z) = \sum_{\substack{n \le X \\ (n, P_z) = 1}} a(n) = ??$$

$P_z : \prod_{\substack{p \in \mathcal{P} \\ p < z}} p$

$\boxed{\mu(n)}$: $\sum_{\substack{d \mid n \\ d \mid P(z)}} \mu(d) = \begin{cases} 0 & \text{otherwise} \\ 1 & \text{coprime} \end{cases}$

Recall: $(1-1)^n = 0$

$$\Rightarrow S(A, z) = \sum_{d \mid P(z)} \mu(d) \sum_{d \mid n} a_n = \sum_{d \mid P(z)} \mu(d) A_d(x)$$

We consider

$$A_d(x) = g(d) \langle \sim \rangle + r_d(x)$$

($g(d)$: $\mathbb{P}$; $\langle \sim \rangle \approx X = \sum a_n$)

Twin primes: $a(p-2) = 1$, $A_d(x) = \pi(x; d, 2)$, $X = \pi(x)$, $g(d) = 1/\varphi(d)$

$$\Rightarrow r_d(x) = \pi(x; d, 2) - \frac{\pi(x)}{\varphi(d)} \text{ "} = O\left(x^{1/2} (\log x)^A\right) \text{ (Bombieri–Vinogradov)}$$

$1/2$ raised to $\frac{1}{2} + \frac{1}{200}$ (Zhang) in some cases

Add 1.4 after Ch. 5

From A, create $\mathcal{B}$ satisfying the sieve axioms for $x = 1$ and density g.

Let:

$$h = \frac{g(p^\alpha) - g(p^{\alpha+1})}{1 - g(p)} \quad (= g \text{ when } h \text{ is multiplicative})$$

Assume: $f(n) = h(n) \cdot n$ satisfies

$$\sum_{n \in (d\mathbb{Z}) \le x} f(n) = \hat{f}(d) x + O\left(\tilde{f}(d) (x/d)^\theta\right)$$

for funny $\tilde{f}$ and $\hat{f}$.

$\hat{f}(d) = g(d) H^{-1}$, $H = \prod (\log p)(1 - \frac{1}{p})$

$\tilde{f}(d) = \sum_{abc = d} f(a) b^{-\xi} c^{-\xi}$

let $b_n = \prod_p (1-g(p))(1-\frac{1}{p})^{-1} f(n) \Rightarrow r_d(x) \ll \tilde{f}(d)(x/d)^{\theta}$

$\sum_{d \le D} r_d(x) \ll x^{\theta} D^{1-\theta}$

$\Rightarrow D \ll x(\log x)^{-A}$ will do

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$$S(A,z) \sim \prod_{p \in P(z)} (1-g(p)) =: V(z)$$

Not so nice situation where we can't just have $g(d)$:

$$\zeta_A(s) = \sum_{d|n} a_n n^{-s} \overset{\text{Tauberian}}{\Rightarrow} \sum_{d|n} a_n \left[\sum_j g(d) \times (c_j + h_j(d))(\log(x/d))^j \right]$$

no mult

A more mild modification: $g(d) \sum_{\substack{n \le x \\ (n,d)=1}} a_n + r_d(x)$

$\mu(d)$ is an example of a sifting weight. It is a "canonical choice".

~~sifted sum~~ $= 1 * \lambda$ want: λ have finite support up to D.

Another technique: $S^- \le S^\wedge \le S^+$ TODO: cite

$$S^\wedge(A,z) = X \cdot V(D,z) + R^\wedge(D,z)$$

$$V(D,z) = \sum_{d|P(z)} \lambda_d g(d) \qquad R^\wedge(D,z) = \sum_{d|P(z)} \lambda_d r_d$$

* The idea is that V abstracts away A but $R^\wedge$ doesn't.
* ~~Want $|\lambda_d| \le 1$ or $|\lambda_d| \le \tau_k(d)$~~ Want $|\lambda_d| \le 1$ or $\tau_{10}(d)$
* $\mu(d)$ gives cancellations, but λ_d generally won't.
* If it does, get $R(M,N,z) = \sum_{mn|P(z)} \alpha_m \beta_n r_{mn}$ where α, β achieve the bounds we want. c.f. Friedlander-Iwaniec

If $R^\wedge(D,z) = O(A(x)(\log x)^{-B})$, say D is a "level of dist"

If $D = D(x) = x^{\alpha - \varepsilon}$, say A has exponent of distribution α.

* For $A = \{p-2\}$, $\alpha = 1/2$ by Bombieri-Vinogradov.

Example Bank

(1) Basics (+ a twist)

$\mathcal{A} = \{m^2+1\}$ $\mathcal{P} = \{p \equiv 1 (4)\}$

$\{\mathcal{A} \text{ squarefree}\}_{\leq x} = \sum_{d | P(x)} \mu(d) A_{d^2}(x)$

$\sum_{d^2 | n} \mu(d) = \begin{cases} 1 & \square\text{-free} \\ 0 & \text{not} \end{cases}$

Splitting into classes $m \equiv v \ (d^2)$, we have

$$A_{d^2}(x) = \frac{\#\{v^2+1 \equiv 0 \ (d^2)\} \text{ or } \rho(v)}{d^2} \underbrace{\sqrt{x}}_{x^{1/2}} + O(\rho(d^2))$$

$\rho(d^2) = \tau(d)$

$\Rightarrow$ only beats the error for $d \leq x^{1/4}$.

Consider $m^2+1 = d^2 k \leq x$

$m^2 - kd^2 = -1$ $\Rightarrow$ need Pell's equation

The solutions to Pell's equation grows exponentially, so $O(\log x)$ of them

For $d > x^{1/4}$ $(=: D(x))$

Combining these estimates:

$$\#\{m^2+1 \leq x \, ; \, \square\text{-free}\} = \underbrace{c}_{\prod_{p \equiv 1 (4)} (1 - \frac{2}{p^2})} x^{1/2} + O(x^{1/3} \log x)$$

APPENDIX

Affine sieve — multidimensional sieves

Hardy-Littlewood Conjecture (model) $b \in \mathbb{Z}^n$, $\{\lambda \in \Gamma : \prod(\lambda_i + b_i) \text{ has } \leq n \text{ prime factors}\}$

James Maynard $b \in \mathbb{Z} : (b, 2b, 3b, \ldots)$

Question here: replace ∞ w/ Zariski-dense; ask things that sound like:

"When is $\{\gamma \in \Gamma \mid f(\gamma) \text{ has } \leq r \text{ prime factors}\}$"

$\Gamma \subset GL_n$, $\overline{\Gamma} = G \leftarrow$ in dense?

Fundamental Thm: $\text{Hom}(G^\circ, G_m) = 1$ is enough

"Thm 8" : $G = \langle \Omega \rangle$; $\text{Cay}(\pi_q(\Gamma), \pi_q(\Omega))$ is a family of exp. $\Leftrightarrow$ G is perfect