WEEK 2: FRIEDLANDER-IWANIEC, CHAPTERS 2, 4, §5.5-10

ALAN ZHAO

1. Basic Setup

Recall that we are estimating $A(x) = \sum_{n\leq x} a_n$ by looking at $X_d = \sum_{d|n} a_n$, the latter of which is generally estimated by considering $A_d = g(d)X + r(d)$, where g is some set of weights coming close to being multiplicative and X well approximates $A(x)$. Recall that we also have some distinguished primes $\mathcal{P}$ to avoid, up to some height z (denoted $\mathcal{P}(z)$). The Möbius μ-function lets us express this coprimality as a convolution, but it has the downside of being hard to estimate. We will choose some replacements λ_n for $\mu(n)$, supported up to height D, the *level of distribution*. If $D(x) = x^\alpha$, then call α the *exponent of distribution*. Recall further from the Legendre form that we expect $\sum_{(d,\mathcal{P}(z))=1} X_d$ to be something like $X \cdot \prod(1 - g(p)) \approx G = \sum_{d|\mathcal{P}(z)} \lambda_d g(d)$.

In what follows, we will be heavily using convolutions of arithmetic functions. If this is confusing, just know that every convolution takes the form $C * \left(\sum_{cd=n}, \int\right) f(c)g(d)$ in some appropriate ambient group. In our case, we have Möbius inversion: convoluting by 1 is inverse to convoluting by μ.

2. Correction to First Week

There was a problem in the first week about how I explained Zhang's result on "raising the level" or "breaking through the square root barrier". Indeed, I meant that α goes from $1/2$ to $1001/2000$. The level of distribution D must satisfy that $R = \sum_{d|\mathcal{P}(z)} \lambda_d r_d \ll A(x)(\log x)^{-A}$. Of course, raising the exponent of distribution is non-trivial (and even that it has one!).

3. Dirichlet Hyperbola Trick

3.1. **Basic Example.**

The function $\tau(n)$ returns the number of positive divisors of n. Then an over count gives

$$\sum_{n\leq x} \tau(n) = \sum_{d\leq x} \lfloor x/d \rfloor = x\log(x) + O(x).$$

We know that divisors come in pairs, as solutions to $cd = n$, so

$$\sum_{n\leq x} \tau(n) = 2\sum_{d\leq\sqrt{x}} \lfloor x/d \rfloor - \lfloor \sqrt{x} \rfloor^2 = x(\log x + 2\gamma - 1) + O(\sqrt{x}).$$

The equation $cd = n$ can be viewed as the level set of a convolution, and apparently the technique above applies more generally to convolutions.

Date: February 5, 2024.

3.2. **Another Example.** Let $r(n) = 4\sum_{d|n}\chi_4(d)$. We use the trick above, but this time there's not a direct pairing, but rather we get

$$\sum_{n\le x} r(n) = 4\sum_{d\le\sqrt{x}} \chi_4(d)\lfloor x/d\rfloor 4\sum_{c\le\sqrt{x}}\sum_{d\le x/c}\chi_4(d) + O(\sqrt{x}).$$

Our target is the first summation on the RHS. Splitting into integral and fractional parts we get that

$$\sum_{d\le\sqrt{x}} \frac{\chi_4(d)}{d} = L(1,\chi_4) + O(x^{-1/2})$$

from which, after applying $L(1,\chi_4) = \pi/4$, we get the estimate

$$\sum_{n\le x} r(n) = \pi x + O(\sqrt{x}).$$

3.3. **A Harder Example.** I have not studied this well. But the start is

$$\sum_{n\le x} \tau(n^2+1) = 2\sum_{d\le x}\sum_{n\le x, d|n^2+1} 1 - \sum_{cd=n^2+1} 1 + O(x) = 2x\sum_{d\le x}\frac{\rho(d)}{d} + O(x).$$

From the generating series for $\rho(d)$ as $\zeta(2s)^{-1}\zeta(s)L(s,\chi_4)$, one has $\rho(d) = \sum_{a^2bc=d}\mu(a)\chi_4(c)$ and we can get $\sum_{d\le x}\frac{\rho(d)}{d} = \frac{L(1,\chi_4)}{\zeta(2)}\log x + O(1)$. Hence,

$$\sum_{n\le x} \tau(n^2+1) = \frac{3}{\pi}x\log x + O(x).$$

4. The Von Magnoldt Function

It is the convolution Λ of μ and the logarithm. Also, convoluting the constant function with Λ gives the logarithm. We can use this function to detect large prime divisors. Let y be the cutoff and set

$$S(x,y) = \sum_{n\le x} a_n\left(\sum_{d|n, d>y}\Lambda(d)\right).$$

Note that ignoring $d > y$ means we get a regular convolution $A'(x) = \sum a_n \log n = A(x)(\log x + O(1))$, where the bound holds for many sequences. So, $S(x,y)$ is the difference between $A'(x)$ and the contribution of low primes. Using the forms in §1, and noting a few natural approximations, we achieve

Proposition 4.1. *Suppose the remainder terms satisfy $\sum_{d\le y}\Lambda(d)r_d(x) \ll A(x)$ for a power of x between 0 and 1. Then,*

$$S(x,y) \ll_\theta A(x)(\log(x) + O(1)).$$

If x is large enough to outpace the $O(1)$, we get that some n has a primary factor larger than x^θ.

5. Further Generalities About Sieves

We want some regularity on the weights $g(p)$. Typically one has

$$\sum_{p\leq z} g(p)\log(p) = \kappa \log z + O(1).$$

Partial summation gives

$$\sum_{p\leq z} g(p) = \kappa \log\log z + \alpha + O((\log z)^{-1})$$

(κ is the sieve dimension) whence a formula of Mertens gives

$$\prod_{p\leq z}(1-g(p)) = e^{-\gamma\kappa}H(\log z)^{-\kappa}(1+O((\log z)^{-1}).$$

We will want V to grow at most like a logarihtmic power: $V(w)/V(z) \ll (\log_w(z))^\kappa$ and hence $g(p) \leq 1-K^{-1}$. Usually $K = 1+\frac{L}{\log 2}$ works. If this K is too large, cut out some small primes from $\mathcal{P}$ so that you can replace 2 with something larger, and then handle these small primes some other way. You can also instead enlarge κ. Increasing it by 1 adds a $(\log z)$ to the denominator.

If you have this logarithmic growth of V, you get nice bounds.

Lemma 5.1. *Let h be continuous, non-negative, and non-decreasing. Let $\Delta = \sum_{y\leq p<z} g(p)h(p)V(p)$.*

$$\Delta \leq -KV(z) = \int_y^z h(w)d(\log_w z)^\kappa + (K-1)h(z)V(z),$$

$$\Delta \leq -V(z)\int_y^z h(w)d(\log_w(z))^\kappa + \left(1-\frac{1}{K}\right)h(z)V(y),$$

$$\Delta_{without\ V} \leq \kappa\int_y^z \frac{h(w)}{w\log w}dw + h(z)\log K.$$

5.1. Monotonicity.

As a precursor, we will be taking upper and lower bound sieves with weights λ_d^+ and λ_d^- respectively. It turns out that in the multiplicative weight case, the signs refer to the signs of the corresponding $\theta = 1 * \lambda$ when things are not coprime to $\mathcal{P}$. It says nothing about the signs of the λ themselves.

By inversion,

$$\lambda_d = \sum_{mn=d}\mu(m)\theta_n$$

whence

$$G = \sum_{m,n}\mu(m)g(mn)\theta_n.$$

If g is multiplicative, we can eventually derive that $G \geq V$ (resp. $G \leq V$) when $\theta_n^+ \geq 0$ (resp. $\theta_n^- \leq 0$) for n not coprime to the sifting set. Admitting all of this, we have the following

Proposition 5.2. *With assumptions as above, we have*

$$\sum_{(d,q)=1}\lambda_d^+ g(d) \leq (1-g(q))^{-1}\sum_d \lambda_d^+ g(d)$$

and the reverse for the minus case.

We can also fiddle with the sieve weights.

Proposition 5.3. *If $g'(d) \geq g(d)$, one has*

$$G' \geq G \prod \left(\frac{1 - g'(p)}{1 - g(p)} \right).$$

6. Quick Comments on Composition

Composition of sieves with weights λ' and λ'' just multiplies the θ's involved. Also,

$$G' * G'' = \sum_{(d_1,d_2)=1} \lambda'_{d_1} \lambda''_{d_2} g'(d_1) g''(d_2)$$

which is useful when the two sequences are linearly dependent. See §14.7.