Selberg sieve

Rudimentary analysis.

Start with standard sieve.

$$S(A, P, z) = \sum_{n \le x} \sum_{d \mid (n, P(z))} \mu(d) \longrightarrow x \prod_{p<z}\left(1-\frac{1}{p}\right) + O\left(x (\log z)^2 \exp\left(-\frac{\log x}{\log z}\right)\right)$$

(Brun's sieve)

Selberg's idea
$$\sum_{d \mid m} \mu(d) = \begin{cases} 1, & m = 1 \\ 0, & m \neq 1. \end{cases}$$

So if (λ_d) sequence of real numbers with $\lambda_1 = 1$, then

$$\sum_{d \mid m} \mu(d) \le \left(\sum_{d \mid m} \lambda_d\right)^2$$

So,
$$S(A, P) \le \sum_{n \le x} \left(\sum_{d \mid (n, P(z))} \lambda_d\right)^2$$

$$= \sum_{n \le x} \left(\sum_{d_1, d_2 \mid (n, P(z))} \lambda_{d_1} \lambda_{d_2}\right)$$

$$= \sum_{d_1, d_2 \mid P(z)} \lambda_{d_1} \lambda_{d_2} \Bigg(\sum_{\substack{n \le x \\ [d_1, d_2] \mid n}} 1\Bigg) \longrightarrow \frac{x}{[d_1, d_2]} + O(1).$$

lcm

$$= x \sum_{d_1, d_2 \mid P(z)} \frac{\lambda_{d_1} \lambda_{d_2}}{[d_1, d_2]} + O\left(\sum_{d_1, d_2 \mid P(z)} |\lambda_{d_1}| |\lambda_{d_2}|\right)$$

λ_d arbitrary, so take $\lambda_d = 0$ for $d > z$.

Then $S(\mathcal{A}, P) \leq x \underbrace{\sum_{\substack{d_1, d_2 \\ \leq z}} \frac{\lambda_{d_1}\lambda_{d_2}}{[d_1, d_2]}}_{\text{quadratic form.}} + O\left(\sum_{d_1, d_2 \leq z} |\lambda_{d_1}||\lambda_{d_2}|\right)$.

Try to minimize this

Recall $[d_1, d_2](d_1, d_2) = d_1 d_2$.

& $\sum_{\delta | d} \phi(\delta) = d$.

$$\sum_{d_1, d_2 \leq z} \frac{\lambda_{d_1}\lambda_{d_2}}{[d_1, d_2]} = \sum_{d_1, d_2 \leq z} \frac{\lambda_{d_1}\lambda_{d_2}(d_1, d_2)}{d_1 d_2}$$

$$= \sum_{d_1, d_2 \leq z} \frac{\lambda_{d_1}\lambda_{d_2}}{d_1 d_2} \left(\sum_{\delta | (d_1, d_2)} \phi(\delta)\right)$$

$$= \sum_{\delta \leq z} \phi(\delta) \left(\sum_{\substack{d_1, d_2 \leq z \\ \delta | (d_1, d_2)}} \frac{\lambda_{d_1}\lambda_{d_2}}{d_1 d_2}\right)$$

$$= \sum_{\delta \leq z} \phi(\delta) \left(\sum_{\substack{d \leq z \\ \delta | d}} \frac{\lambda_d}{d}\right)^2.$$

$\| u_\delta$.

Effectively diagonalized quadratic form.

$$\sum_{\delta \le z} \phi(\delta) u_\delta^2 .$$

Note $u_\delta = 0$ for $\delta > z$.

Explicitly, $$u_\delta = \sum_{\substack{d \le z \\ \delta \mid d}} \frac{\lambda_d}{d}.$$

By dual Möbius inversion

$$\frac{\lambda_\delta}{\delta} = \sum_{\delta \mid d} \mu\left(\frac{d}{\delta}\right) u_d .$$

So, $$\sum_{\delta \le z} \mu(\delta) u_\delta = 1. \qquad (\lambda_1 = 1)$$

Using this, we have

$$1 = \left| \sum_{\delta \le z} \mu(\delta) u_\delta \right|^2 = \left| \sum_{\delta \le z} \frac{\mu(\delta)}{\sqrt{\phi(\delta)}} u_\delta \sqrt{\phi(\delta)} \right|^2$$

$$\le \left(\sum_{\delta \le z} \frac{\mu^2(\delta)}{\phi(\delta)} \right) \cdot \left(\sum_{\delta \le z} u_\delta^2 \phi(\delta) \right)$$

So, $$\sum_{\delta \le z} \phi(\delta) u_\delta^2 \ge \frac{1}{\sum_{\delta \le z} \frac{\mu^2(\delta)}{\phi(\delta)}} = \frac{1}{V(z)}$$

w/ equality when $\boxed{u_\delta = \frac{\mu(\delta)}{\phi(\delta) V(z)}}$

So minimized at $\nearrow$ w/ min. value $\frac{1}{V(z)}$

$$\hookrightarrow \sum_{\delta \le z} \frac{\mu^2(\delta)}{\phi(\delta)} = V(z)$$

Thus, $\boxed{\lambda_\delta = \delta \sum_{\substack{d \le z \\ \delta | d}} \frac{\mu(\frac{d}{\delta}) \mu(d)}{\phi(d) V(z)}}$. (from dual Möbius inversion)

So, $$S(A, P) \le \frac{x}{V(z)} + O\left(\sum_{d_1, d_2 \le z} |\lambda_{d_1}| |\lambda_{d_2}| \right).$$

For the error term,

$$V(z) \lambda_\delta = \delta \sum_{\substack{d \le z \\ \delta | d}} \frac{\mu(\frac{d}{\delta}) \mu(d)}{\phi(d)}$$

$$= \delta \sum_{t \le \frac{z}{\delta}} \frac{\mu(t) \mu(\delta t)}{\phi(\delta t)}$$

$$= \delta \sum_{\substack{t \le \frac{z}{\delta} \\ (t, \delta) = 1}} \frac{\mu^2(t) \mu(\delta)}{\phi(\delta) \phi(t)}$$

$$\frac{\delta}{\phi(\delta)} = \prod_{p \mid \delta} \left(1 + \frac{1}{p-1}\right)$$

So, $$V(z)\lambda_\delta = \mu(\delta) \prod_{p \mid \delta} \left(1 + \frac{1}{p-1}\right) \sum_{\substack{t < \frac{z}{\delta} \\ (t,\delta)=1}} \frac{\mu^2(t)}{\phi(t)}.$$

partial products are $\frac{1}{\phi(d)}$ for different square-free $d \mid \delta$, so

$$\prod_{p \mid \delta} \left(1 + \frac{1}{p-1}\right) \sum_{t \le \frac{z}{\delta}} \frac{\mu^2(t)}{\phi(t)} \le V(z).$$

So, $$|V(z)|\,|\lambda_\delta| \le |V(z)|$$
$$\Rightarrow |\lambda_\delta| \le 1.$$

Thus, error term is $O(z^2)$.

Thus, we have

<u>Thm</u> $$S(\mathcal{A}, P) \le \frac{x}{V(z)} + O(z^2).$$

$$V(z) \rightarrow \sum_{d \le z} \frac{\mu^2(d)}{\phi(d)}.$$

Let's use this. $z < x$.

$$\pi(x) - \pi(z) = S(\mathcal{A}, P)$$
$$\pi(x) \le S(\mathcal{A}, P) + z.$$

Let's lower bound $V(z) = \sum_{d \leq z} \frac{\mu^2(d)}{g(d)}$

$$\sum_{d \leq z} \frac{\mu^2(d)}{g(d)} \geq \sum_{d \leq z} \frac{\mu^2(d)}{d} = \sum_{\substack{d \leq z \\ d \text{ sq.free}}} \frac{1}{d}$$

$$= \underbrace{\sum_{d \leq z} \frac{1}{d}}_{\log z + O(1).} - \underbrace{\sum_{\substack{d \leq z \\ d \text{ not sq. free}}} \frac{1}{d}}$$

$$\sum_{\substack{d \leq z \\ d \text{ not sq. free}}} \frac{1}{d} \leq \frac{1}{4} \sum_{d \leq \frac{z}{4}} \frac{1}{d} \sim \frac{1}{4} \log z + O(1).$$

So $V(z) \gg \log z$.

Thus, ~~$S(\mathcal{A})$~~ $\pi(x) \ll \frac{x}{\log z} + z^2$.

Choose $z = \sqrt{\frac{x}{\log x}}$.

$$\pi(x) \ll \frac{x}{\frac{1}{2}\log x - \frac{1}{2}\log\log x} + \frac{x}{\log x}$$

$$\ll \frac{x}{\log x} \quad \text{(Chebycheff's upper bound)}$$

$\square$

<u>Rmk</u> Main term is $\frac{x}{V(z)} = O\left(\frac{x}{\log x}\right)$

& so is $x \prod_{p < z} \left(1 - \frac{1}{p}\right) = O\left(\frac{x}{\log x}\right)$

But here, the error term is much smaller.

(General Thm for Selberg sieve).

f mult. func. w/ $f(p) > 1$ for all primes $p \in P$
& for d sq. free made of primes in P,

$$\# A_d = \frac{X}{f(d)} + R_d \quad \text{for some } X > 0$$
$$\phantom{\# A_d = \frac{X}{f(d)} + {}} \underset{\mathbb{R}}{\cap}$$

$$f(n) = \sum_{d \mid n} f_1(d).$$

unique by Möbius inversion

$$f_1(n) = \sum_{d \mid n} \mu\left(\frac{n}{d}\right) f(d)$$

Set $$V(z) := \sum_{\substack{d \le z \\ d \mid P(z)}} \frac{\mu^2(d)}{f_1(d)}.$$

Then, $$S(A, P, z) \le \frac{X}{V(z)} + O\left(\sum_{\substack{d_1, d_2 \le z \\ d_1, d_2 \mid P(z)}} |R_{[d_1, d_2]}|\right)$$

Lemma d_1, d_2 sq. free

$$f([d_1, d_2]) \cdot f((d_1, d_2)) = f(d_1) f(d_2)$$

Pf: Trivial. $\square$

Again, λ_d s.t. $\lambda_1 = 1$, $\lambda_d = 0$ for any $d \geq z$.

Let $D(a) = \prod_{\substack{p \in P \\ a \in A_p}} p$.

$D(a) = 1$ if $a \notin A_p$ for any $p \in P$.

Then,

$$\sum_{\substack{d \mid P(z) \\ a \in A_d}} \mu(d) = \sum_{d \mid (P(z), D(a))} \mu(d) \leq \Big(\sum_{d \mid (P(z), D(a))} \lambda_d \Big)^2$$

$$= \Big(\sum_{\substack{d \mid P(z) \\ a \in A_d}} \lambda_d \Big)^2.$$

Now

$$S(A, P, z) = \sum_{d \mid P(z)} \mu(d) \sum_{a \in A_d} 1.$$

$$= \sum_{a \in A} \Big(\sum_{\substack{d \mid P(z) \\ a \in A_d}} \mu(d) \Big) \leq \sum_{a \in A} \Big(\sum_{\substack{d_1, d_2 \mid P(z) \\ a \in A_{[d_1, d_2]}}} \lambda_{d_1} \lambda_{d_2} \Big)$$

$$= \sum_{d_1, d_2 \leq z} \lambda_{d_1} \lambda_{d_2} \# A_{[d_1, d_2]}$$

$$\# A_{[d_1,d_2]} = \frac{X}{f([d_1, d_2])} + R_{[d_1, d_2]}$$

So $S(A, P, z) \leq X \sum_{\substack{d_1, d_2 \leq z \\ d_1, d_2 \mid P(z)}} \frac{\lambda_{d_1} \lambda_{d_2}}{f([d_1, d_2])} + O\Big(\sum_{\substack{d_1, d_2 \leq z \\ d_1, d_2 \mid P(z)}} |\lambda_{d_1}| |\lambda_{d_2}| |R_{[d_1, d_2]}| \Big)$

As before, & using lemma, diagonalize quadratic form

$$\sum_{\substack{\cancel{d_1, d_2 \leq z} \\ d_1, d_2 \mid P(z)}} \frac{\lambda_{d_1} \lambda_{d_2}}{f([d_1, d_2])} = \sum_{\substack{d_1, d_2 \leq z \\ d_1, d_2 \mid P(z)}} \frac{\lambda_{d_1} \lambda_{d_2}}{f(d_1) f(d_2)} f((d_1, d_2)).$$

$$= \sum_{\substack{\delta \leq z \\ \delta \mid P(z)}} f_1(\delta) \sum_{\substack{d_1, d_2 \leq z \\ d_1, d_2 \mid P(z) \\ \delta \mid (d_1, d_2)}} \frac{\lambda_{d_1} \lambda_{d_2}}{f(d_1) f(d_2)}$$

$$= \sum_{\substack{\delta \leq z \\ \delta \mid P(z)}} f_1(\delta) \Big(\underbrace{\sum_{\substack{d \leq z \\ d \mid P(z) \\ \delta \mid d}} \frac{\lambda_d}{f(d)}}_{=u_\delta} \Big)^2.$$

Again, $u_\delta = 0$ for $\delta > z$

$$\sum_{\substack{\delta \le z \\ \delta | P(z)}} \mu(\delta) u_\delta = 1$$

$\downarrow$

b/c dual Möbius inversion gives

$$\frac{\lambda_\delta}{f(\delta)} = \sum_{\substack{d | P(z) \\ \delta | d}} \mu\left(\frac{d}{\delta}\right) u_d .$$

Again by Cauchy-Schwarz,

$$\sum_{\substack{\delta \le z \\ \delta | P(z)}} f_1(\delta) u_\delta^2 \text{ minimized at}$$

$$u_\delta = \frac{\mu(\delta)}{f_1(\delta) V(z)}$$

w/ mini. value $\frac{1}{V(z)}$

where $V(z) = \sum_{\substack{d \le z \\ d | P(z)}} \frac{\mu^2(d)}{f_1(d)}$

Note $f_1(p) = f(p) - 1 > 0$

$\Rightarrow f_1(d) > 0$ for all $d | P(z)$

$\hookrightarrow$ sq. free

Now ~~V(z)~~ bound $V(z)\lambda_\delta$ as before

use $\frac{f(p)}{f_1(p)} = \frac{f_1(p)+1}{f_1(p)} = 1 + \frac{1}{f_1(p)}$.

& get $|\lambda_\delta| \leq 1$ for $\delta < z$ & $\delta \mid P(z)$.

Thus the error term is

$$O\Big(\sum_{\substack{d_1, d_2 \leq z \\ d_1, d_2 \mid P(z)}} |R_{[d_1, d_2]}|\Big)$$

$\square$.

Lower bounds for $V(z)$.

Let $\tilde{f}$ be the completely multiplicative func.
w/ $\tilde{f}(p) = f(p)$, $\forall$ p prime (all primes)

Set $\bar{P}(z) = \prod_{\substack{p \notin \mathcal{P} \\ p < z}} p$.

Claim: (i) $V(z) \geq \sum_{\substack{\delta \leq z \\ p \mid \delta \Rightarrow p \mid P(z)}} \frac{1}{\tilde{f}(\delta)}$.

(ii) $f(\bar{P}(z)) V(z) \geq f_1(\bar{P}(z)) \sum_{\delta \leq z} \frac{1}{\tilde{f}(\delta)}$

Pf: (i) Note if d sq. free,
$$\frac{f(d)}{f_1(d)} = \prod_{p \mid d}\left(1 - \frac{1}{f(p)}\right)^{-1} = \prod_{p \mid d} \sum_{n \geq 0} \frac{1}{f(p)^n} = \sum_{\substack{k \\ p \mid k \Rightarrow p \mid d}} \frac{1}{\tilde{f}(k)}$$

So $V(z) = \sum_{\substack{d \le z \\ d | P(z)}} \frac{\mu^2(d)}{f(d)} \left(\sum_{\substack{k \\ p|k \Rightarrow p|d}} \frac{1}{\tilde{f}(k)} \right)$.

$$\geq \sum_{\substack{s \le z \\ p|s \Rightarrow p | P(z)}} \frac{1}{\tilde{f}(s)}.$$

$\square$

(ii) $\frac{f(\bar{P}(z))}{f_1(\bar{P}(z))} V(z) = \prod_{\substack{p \notin P \\ p < z}} \left(1 - \frac{1}{f(p)}\right)^{-1} \sum_{\substack{d \le z \\ d | P(z)}} \frac{\mu^2(d)}{f(d)} \left(\sum_{\substack{k \\ p|k \Rightarrow p|d}} \frac{1}{\tilde{f}(k)} \right)$

$$= \prod_{\substack{p \notin P \\ p < z}} \left(\sum_{n \geq 0} \frac{1}{f(p)^n} \right) \text{ "}$$

$$\geq \sum_{s \le z} \frac{1}{\tilde{f}(z)}.$$

$\square$

Brun-Titchmarsh Thm

$(a, k) = 1$.

$\pi(x, k; a) = \#\{p \le x \mid p \equiv a \pmod k\}$.

$k \le x^\theta \quad \theta < 1$.

For $\varepsilon > 0 \quad \exists x_0 > 0$ s.t.

$\pi(x, k; a) \le \frac{(2+\varepsilon) x}{\phi(k) \log(2x/k)}$ for all $x > x_0$.

For $z < x$

$$\pi(x;k,a) = \pi(z;k,a) + \#\{z \le p \le x \mid p \equiv a \pmod k\}.$$

$$\le z + \#\{n \le x \mid n \equiv a \pmod k,\ n \not\equiv 0 \pmod p\ \forall p < z,\ (p,k) =$$

$A := \{n \le x \mid n \equiv a \pmod k\}$

$\mathcal{P} := \{p \mid (p,k) = 1\}$

& $A_p := \{n \le x \mid n \equiv a \pmod k,\ n \equiv 0 \pmod p\}$ for all $p \in \mathcal{P}$.

$$P(z) := \prod_{\substack{p < z \\ (p,k)=1}} p.$$

$$S(A, \mathcal{P}, z) = \#\{n \le x \mid n \equiv a \pmod k,\ n \not\equiv 0 \pmod p\ \forall p \mid P(z)\}.$$

Note that $\#A_d = \dfrac{x}{kd} + O(1).$

$$\forall d$$

$$A_d = \bigcap_{p \mid d} A_p.$$

So, $X = \dfrac{x}{k}$, $f(d) = d$, $f_1(d) = \phi(d)$

& $R_d = O(1)$.

So, $S(A, \mathcal{P}, z) \le \dfrac{x}{kV(z)} + O(z^2).$

$$\hookrightarrow \sum_{\substack{d \le z \\ (d,k)=1}} \frac{\mu^2(d)}{\phi(d)}$$

Note $\frac{k}{\phi(k)} = \frac{\bar{P}(z)}{\phi(\bar{P}(z))} \overset{f(\bar{P}(z))}{\underset{f_1(\bar{P}(z))}{"}}$ (in this case, $\bar{P}(z) = \text{rad}(k)$)

if $z > k$

(& $\geq$ otherwise).

So, by Claim (ii) for lower bounds for $V(z)$,

we get $\frac{k}{\phi(k)} V(z) \geq \sum_{\delta \leq z} \frac{1}{\delta} \geq \log z + O(1)$.

Thus,

$$\pi(x, k; a) \leq z + S(A, P, z)$$

$$\leq \frac{x}{\phi(k)(\log z + O(1))} + O(z^2)$$

Taking $z = \left(\frac{2x}{k}\right)^{\frac{1}{2} - \varepsilon}$ gives the upperbound

$\square$

<u>Rmk for RH</u>

(Montgomery & Vaughan)

$$\pi(x+y; k, a) - \pi(x; k, a) \leq \frac{2y}{\phi(k)\log(\frac{y}{k})}$$

If we could improve 2, could show non-existence of Siegel zeroes.

Twin Primes

$\mathcal{A} = \{ n(n+2) : n < x \}$.

$\mathcal{A}_d = \{ a_n \in \mathcal{A} \mid d \mid a_n \} = \{ n(n+2) \mid n < x,\ d \mid n(n+2) \}$

$z \leq \sqrt{x}$.

Define $\rho(d) = | \{ n \ (\text{mod } d) : n(n+2) \equiv 0 \ (\text{mod } d) \}$

Note $\rho(p) = \begin{cases} 1, & p = 2 \\ 2, & \text{otherwise} \end{cases}$

$$|A_d| = \frac{x\rho(d)}{d} + O(\rho(d)).$$

$$= \left\lfloor \frac{x\rho(d)}{d} \right\rfloor.$$

So $f(d) = \dfrac{d}{\rho(d)}$, $R_d = \rho(d)$

Note $\rho(d) \leq 2^{\nu(d)}$ for d sq. free

$\nu(d)$ = # distinct prime factors of d.

Selberg error is

$$\sum_{\substack{d_1, d_2 < z \\ d_1, d_2 \mid P(z) \\ \text{sq. free}}} |R_{[d_1, d_2]}| \leq \sum_{\substack{d_1, d_2 < z \\ d_1, d_2 \mid P(z) \\ \text{sq. free}}} 2^{\nu([d_1, d_2])}$$

$$\leq \sum_{\substack{d_1, d_2 < z \\ d_1, d_2 \mid P(z) \ \text{sq. free}}} 2^{\nu(d_1)} 2^{\nu(d_2)} = \left(\sum_{\substack{d < z \\ d \ \text{sq. free}}} 2^{\nu(d)} \right)^2.$$

dimension

$$\sum_{\substack{d<z \\ d \text{ sq. free}}} 2^{\nu(d)} = \sum_{\substack{d<z \\ d \text{ sq. free}}} \tau(d)$$

$$= \sum_{ab<z} 1$$

$$= \sum_{a<z} \left\lfloor \frac{z}{a} \right\rfloor$$

$$= z \sum_{a<z} \frac{1}{a} + O(z)$$

$$= z\log(z) + O(z)$$

So, error is $\ll (z\log(z))^2$

Can show (using claim part (i)) that

$$V(z) \geq \frac{1}{4}(\log^2 z) + O(\log z)$$

$\rightarrow (\log z)^2$

So $V(z) \gg \log^2 z$.

Thus, $S(\mathcal{A}, \mathcal{P}, z) \ll \frac{x}{\log^2 z} + O((z\log z)^2)$

Taking $z = x^{\frac{1}{4}}$, get

$$S(\mathcal{A}, \mathcal{P}, z) \ll \frac{x}{\log^2 z} + \cancel{O}\, O\left(\frac{x^{1/2}\log^2 x}{2}\right)$$

$$\ll \frac{x}{\log^2 z}$$

Thus $\pi_2(x) < S(\mathcal{A}, \mathcal{P}, z) + z.$

$$\ll \frac{x}{\log^2 x} + x^{1/4}.$$

In particular, $\pi_2(x) << \dfrac{x}{\log^2 x}$.