Sieves Talk - Aditya Ghosh

p_n : n^{th} prime.

Twin prime conjecture : $\liminf_{n\to\infty}(p_{n+1} - p_n) = 2$

GPY : (Goldston - Pintz - Yildirum) (2005)

$$\liminf_{n\to\infty} \frac{p_{n+1} - p_n}{\log n} = 0$$

$p_n \sim n \log n$

So Avg gap. $\approx \log n$

Zhang : (2013) : $\liminf_{n\to\infty} (p_{n+1} - p_n) \leq 7 \times 10^7$

Polymath : ≤ 4600

Maynard (2013) : ≤ 600

Polymath : ≤ 246

Admissible ~~sequences~~ sets

{ eg: $\{0,3\}$ not admissible, $p \,;\, p+3 \pmod 2$)

DEF: $H = \{h_1, \dots, h_k\}$ admissible if

$\forall$ primes p, $\exists\, n_p \in \mathbb{N}$ st $n_p \not\equiv h_i \pmod p$ $\forall\, h_i$

[Prime k-tuples conjecture]: H be admissible.
Then $\exists$ infinitely many n st $\{n+h_1, \dots, n+h_k\}$ all primes

| More generally, you can ask for linear forms
$a_i n + h_i$ or polynomials in n.
That basically gives you Goldbach's conjecture
& a lot of conj. in this field.

You can see easily this $\Rightarrow$ Twin primes conj (as $0,2$ admissible)

$\Rightarrow$ $\liminf_n (p_{n+m} - p_n) \le (1+o(1))\, m \log m$

|| GPY's method: Studying approximations to this k-tuple conjecture ||

Primes in AP (Bombieri - Vinagradov thm)

$(a,q)=1$,

$$\pi(x;q,a) = \#\{p \le x : p \equiv a \pmod q\} \approx \frac{\pi(x)}{\phi(q)}$$

$$E_q := \sup_{a \text{ st } (a,q)=1} \left| \pi(x;q,a) - \frac{\pi(x)}{\phi(q)} \right|$$

(error for each residue class)

DEF: (Level of distribution) Primes have l.o.d. θ if

$\forall A > 0$,

$$\sum_{q < x^\theta} E_q \ll_A \frac{x}{(\log x)^A}$$

Thm: (Bombieri - Vinagradov) The primes have level of distribution θ $\forall \theta < 1/2$

{ "RH true on average" - avg over mod q.

|| ~~E~~ Elliot-Halberstam conj: true for $\forall \theta < 1$ ||

θ large : Strong level of equidistribution

θ small : Small " ...

Overview of the Sieve

1

|| Primes in AP (Bomb-Vin thm) ||

(Arithmetic result)

2 Sieve Method (GPY sieve)
(Non-Arithmetic)

(i) Optimization problem : Choice of weight (similar to Selberg sieve)

(ii) Combinatorial problem : Dense admissible sets H

Zhang : Improves step 1

Maynard : (2)(i) More efficient sieve.
Multi-dimensional parameters for weight (Since we have multiple elements in H)

(Very flexible, can be generalized to other subsets of primes, doesn't depend on $EH(\theta)$ as much)

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GPY Sieve

$H = \{h_1, \ldots, h_k\}$ admissible.

$$S = \frac{\displaystyle\sum_{N < n \leq 2N} \#\{i \mid n + h_i \text{ prime}\}\, w_n}{\displaystyle\sum_{N < n \leq 2N} w_n}$$

w_n : Choice of weights ≥ 0

Analogous to what we did with the Selberg sieve

Think of it as a probability measure

& we're looking at expected no. of primes in this sequence.

Key obs: (i) If $S > m$, then $\exists\, n$ st term n contributes $> m$

(ii) Thus $\exists$ at least $m+1$ of the $\{n + h_1, \ldots, n + h_k\}$ which are primes.

(iii) If $S > m$ for all large N, then

$$\liminf\, (p_{n+m} - p_n) \leq h_k - h_1 < \infty$$

We need $S > 1$ for bounded gaps.

~~Lem: $(h_1, \ldots, h_k)$ k-tuple in $[0, x]$ admissible~~

~~Then~~

$\cancel{(1 + o(1)) \frac{x}{\log x} \leq k \leq \frac{x}{\log x} (2 + o(1))}$

$\cancel{h_k - h_1 \geq \left(\tfrac{1}{2} + o(1)\right) k \log k}$

Choice of w_n

Morally,

(a) w_n small when few among the $n+h_i$ are primes
(b) " large " a lot of " "
(c) Find good upper bound for $\sum_{N<n\leq 2N} w(n)$

good lower bound for $\sum_{N<n\leq 2N} w(n)\#\{i \mid n+h_i \text{ prime}\}$

Guess 1 : $\prod_{j=1}^{k} \mathbb{1}_{n+h_j \text{ prime}}$

Hard to find lower / upper bds in (c)

Guess 2 : Sieves! (Selberg-like sieve)

~~Selberg Sieve~~: $w_n = \left(\sum_{\substack{d \mid (n+h_1)\cdots(n+h_k) \\ d < R}} \lambda_d \right)^2$

(weights highly concentrated on primes.)

R small, so we care about small prime factors.

w_n small when $(n+h_1)\cdots(n+h_k)$ has many small prime factors.

Choice of λ_d : Mimic Selberg's weight,

(i) Standard choice (inspired by other problems):

$$\lambda_d = \mu(d) \underbrace{\log(R/d)^k}_{\text{cutoff}}$$

- If k large, then $S \approx \theta$
  & $EH(\theta)$
  (primes have level θ)

  So we just fail to prove bdd gaps with $\theta = 1-\varepsilon$
  (Most optimistic estimate)

(ii) GPY choice: Optimize cutoff function f, $\lambda_d = \mu(d) f(d)$
Then, $S \approx 2\theta$.

So unconditionally ($\theta = 1/2 - \varepsilon$), just failed.

Zhang : Made $EH(\theta)$ for θ just a bit bigger than $1/2$
& he could prove bdd gaps!!

[ARITHMETIC RESULT]

8.

Maynard's work

Generalizes sieve wts w_n to higher dimensions. parameters.

$$w_n = \sum_{\substack{d_1,\dots,d_k \\ d_i \mid n+h_i \\ \prod d_i < R}} \lambda_{d_1,\dots,d_k} \approx \mu\left(\prod_{i=1}^{k} d_i\right) f(d_1,\dots,d_k)$$

This helps us look at divisors of $n+h_i$ separately.
Essentially an optimization problem!
(Use heuristics & calc of variations)

Given $\mathcal{H} = \{h_1,\dots,h_k\}$ admissible

$$M_k = \sup_F \frac{J(F)}{I(F)}$$

J, I mimic the sums from before
Integrals.

If $M_k > 2m/\theta$, then $\exists$ infinitely many integers n s.t. at least $m+1$ of $n+h_i$ are primes.

Finding F.

$$F(t_1,\dots,t_k) = \begin{cases} \prod_{i=1}^{k} g(k t_i) & (t_1,\dots,t_k) \in \mathcal{R}_k \text{ (support)} \\ 0 & \text{else} \end{cases}$$

function g

k dim $\to$ 1 dim optimization problem.

~~H~~ ~~M_k~~ ~~$\gg$~~ ~~$\log$~~

In summary,

(i) $M_k > \log k - 2\log\log k - 2$ if k large

(ii) If $M_k > 2m/\theta$, $\exists$ inf. many primes st at least $m+1$ of the $n+h_i$ are primes.

Lem: (i) $\exists$ admissible set of size k contained in $[0,H]$, $H \approx k\log k$
(take first k primes $p > k$)

(ii) We can take $\theta < 1/2$ by Bomb Vin.

$$\boxed{\text{Thm: } \liminf_n (p_{n+m} - p_n) \leq c\, m^3 e^{4m}}$$

Small k

$$F = \begin{cases} P(t_1,\dots,t_k) & (t_1,\dots,t_k) \in R_k \\ 0 & \text{else} \end{cases}$$

P symmetric poly
(Stone-Weierstrass)
We can compute the integrals easily.

Prop: (i) $M_{105} > 4$ $(k = 105)$ $(m = 1)$

(ii) $\exists$ admissible set of size 105 in $[0, 600]$ "H" Brute (Computational)

Thus $\liminf_{n\to\infty} p_{n+1} - p_n \leq 600$.

Application of ~~[illegible]~~ Maynard's work

$M_k \to \infty$ as $k \to \infty$, so we have this result for any $\theta > 0$.
Makes it very flexible. Don't need $\theta = 1/2 - \varepsilon$.
So we can look at other subsets of primes, with weaker dist.

Consequences:

(i) $d_n = p_{n+1} - p_n$. Then $\exists$ arbitrarily long strings of increasing gaps $d_n < d_{n+1} < \dots < d_{n+m}$ in primes.

Similarly for decreasing

(ii) $\exists$ arbitrarily large sets of primes with any pair differing in at most 2 decimal places.