Topics in Analytic Number Theory Notes

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These notes were taken in the Spring 2025 version of the Topics in Analytic Number Theory Class, taught by Dorian Goldfeld. If you spot any mistakes, please let me know.

Contents

1 Lecture 1 - 1/21/25

The real content will start on January 31st. There are colloquium talks on Thursday and next Tuesday - you are strongly recommmended to attend.
Some of the content in the course will follow his book *Automorphic Forms and L-Functions for the Group* $GL(n, \mathbb{R})$. When I refer to "Dorian's book" in the notes, this is the book I refer to.

1.1 History of Analytic Number Theory

- 1700s - Euler invents the zeta function $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$. Discovers the Euler product
$$\zeta(s) = \prod_p (1 - p^{-s})^{-1}.$$
Gets the functional equation for ζ in special cases (like for $s = i$).

- 1859 - Riemann gets the functional equation for all s; letting
$$\xi(s) = \pi^{-s/2}\Gamma(s/2)\zeta(s),$$
with
$$\Gamma(s) = \int_0^{\infty} e^{-u} u^s \frac{\mathrm{d}u}{u},$$
he proves the functional equation
$$\xi(s) = \xi(1 - s).$$
How? Riemann uses the known identity
$$\sum_{n=-\infty}^{\infty} e^{-\pi n^2 y} = \frac{1}{\sqrt{y}} \sum_{n=-\infty}^{\infty} e^{-\pi n^2/y}$$
then applies the Mellin transform: for a smooth function $f : \mathbb{R}_{\geq 0} \to \mathbb{C}$, the Mellin transform is $\widetilde{f}(s) = \int_0^{\infty} f(y) y^s \frac{\mathrm{d}y}{y}$. (This arises from a change of variable from the Fourier transform). More specifically, you have to take
$$\int_0^{\infty} \left(\sum_{n=-\infty}^{\infty} e^{-\pi n^2 y} - 1 \right) y^s \frac{\mathrm{d}y}{y} = \pi^{-s}\Gamma(s)\zeta(2s).$$

- Dirichlet, 1800s: Taking $\chi : (\mathbb{Z}/q\mathbb{Z})^* \to \mathbb{C}^*$, you get the Dirichlet L function $L(s, \chi) = \sum_{n=1}^{\infty} \chi(n)/n^s$. Shows that everything one can do with the zeta function can be applied to L-function. Can use them to show that there are infinitely many primes in an arithmetic progression.

- Hecke, early 1900s: Generalizes previous exponential sums to theta functions
$$\theta(z) = \sum_{n=-\infty}^{\infty} e^{2\pi i n^2 z},$$
where $z = x + iy \in \mathbb{H}$. This function turns out to be modular: for $\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \Gamma_0(4)$,
$$\theta\left(\frac{az + b}{cz + d}\right) = \varepsilon_d^{-1}\chi_c(d)\sqrt{cz + d}\,\theta(z),$$
where χ_c is a Dirichlet character mod c and $\varepsilon_d = 1$ if $d \equiv 1 \pmod 4$ and -1 if $d \equiv -1 \pmod 4$.

Hecke looks at modular functions: Recall that for $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{Z})$, a modular function f satisfies

$$f\left(\frac{az+b}{cz+d}\right) = (cz+d)^k f(z),$$

for $k \in \mathbb{Z}_{>0}$. This implies $f(z+1) = f(z)$, giving periodicity in the x direction. If f is holomorphic, we get a Fourier expansion

$$f(z) = \sum_{n=0}^{\infty} a_n e^{2\pi i n z}.$$

Hecke defines the Hecke L-function

$$L(s,f) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}.$$

We also have $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \in \mathrm{SL}(2,\mathbb{Z})$, taking $z \mapsto -1/z$. This corresponds to taking y to $1/y$. This gives a functional equation for Hecke L-functions, with a symmetry on the completed L-functions taking $s \to k - s$.

Moreover, using Hecke operators, Hecke was able to show that Hecke L-functions have a Euler product. Everything Hecke does can be generalized to subgroups of $\mathrm{SL}_2(\mathbb{Z})$.

- Gelfand, Piatetski-Shapiro: Replace the upper half plane with matrices: points $x + iy$ are replaced with $\begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix}$, where $x \in \mathbb{R}$ and $y > 0$, and examine functions of the matrices: $f(z)$ is replaced by $f\left(\begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix}\right)$ and $f\left(\frac{az+b}{cz+d}\right) = f\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix}\right)$. How can you do this? Will be explained later. They also introduced automorphic representations.

  This course will primarily focus on $\mathrm{SL}(n,\mathbb{Z})$, especially when $n \geq 3$. Hence the matrix approach becomes necessary.

- Jacquet-Godement: Introduced analogue of Hecke L-functions for cuspidal automorphic forms for higher rank. Lots of results due to Shalika-Jacquet-Piatetski-Shapiro.

- Eisenstein series: Selberg proves analytic continuation and functional equation (proof involves Fredholm operators). Langlands generalizes Selberg's proof to arbitrary reductive groups. We will talk about Eisenstein series for $\mathrm{SL}(n,\mathbb{Z})$ in this course.

1.2 Iwasawa decomposition for $\mathbf{GL}(n,\mathbb{R})$

Before we mentioned functions of matrices as a replacement for functions on $\mathbb{H}$. How do we make this work? Recall that a matrix $m \in M_n(\mathbb{R})$ is orthogonal if $m \cdot m^T = I$, or equivalently if all the rows/columns of m form an orthonormal basis. We denote the set of such matrices $O(n,\mathbb{R})$.

In particular, note that $O(2,\mathbb{R}) = \left\{\begin{pmatrix} \pm\cos t & \mp\sin t \\ \pm\sin t & \pm\cos t \end{pmatrix}\right\}$.

Theorem 1.1 (Iwasawa). *Every $g \in GL(n,\mathbb{R})$ is of the form*

$$g = xykd,$$

where

- *x is an upper triangular matrix with 1s on the diagonal, whose elements are denoted x_{ij}, all real.*
- *y is a diagonal matrix, with $y_1 y_2 \ldots y_{n-1}$ in the top left, $y_1 y_2 \ldots y_{n-2}$ in the next entry, going down to 1 in the bottom right, with all the $y_i > 0$.*
- *$k \in O(n,\mathbb{R}) = K$, where K is used to denoted the maximal compact group.*

- *d is a diagonal matrix with d_0 on all entries on the diagonal, with $d_0 \neq 0$.*

Example 1.2. *In the $GL(2,\mathbb{R})$ case, the Iwasawa decomposition $g = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix} kd$. Hence we can express*

$$\mathbb{H} = GL(2,\mathbb{R})/(O(2,\mathbb{R}) \cdot \mathbb{R}^*).$$

In general, we get the generalized upper half plane

$$\mathfrak{h}^n := GL(n,\mathbb{R})/(O(n,\mathbb{R}) \cdot \mathbb{R}^*).$$

Proof. Recall that a positive definite matrix is a matrix $m \in M(n,\mathbb{R})$ such that m is symmetric and $xmx^T > 0$ for all nonzero $x \in \mathbb{R}^n$, or equivalently m is symmetric and all its eigenvalues are positive. Moreover, note that for any $u \in \mathrm{GL}(n,\mathbb{R})$, uu^T is positive definite.
Consider any $g \in \mathrm{GL}(n,\mathbb{R})$.

Claim 1.3. *There exists upper triangular matrix u, lower triangular matrix ℓ and diagonal matrix d, such that $ugg^T = \ell d$.*

Proof. View this as solving for u. There are $n(n-1)/2$ parameters for u and $n(n-1)/2$ equations (the upper elements of ℓd need to be 0). This can be solved because gg^T is full rank. □

This gives that

$$gg^T = u^{-1}\ell d = d\ell^T(u^T)^{-1},$$

hence $\ell du^T = ud\ell^T$. Note that the LHS is an lower triangular matrix, and the right is an upper triangular matrix, so $ud\ell^T = d^*$, some diagonal matrix.
Further manipulation gives that $ugg^T = d^*(u^T)^{-1}$, so $ugg^Tu^T = d^*$. The LHS must be positive definite, d^* must consist of positive entries on the diagonal. Let a be its squareroot. Then we can write

$$(aug)(aug)^T = I.$$

Hence $aug \in O(n,\mathbb{R})$, and hence we get the decomposition. □

Here is an alternative proof using Gram-Schmidt:

Proof. Let $a_1,\ldots,a_n$ be the column vectors of $g^{-1} \in GL(n,\mathbb{R})$ and $q_1,\ldots,q_n$ be the outputs of the Gram-Schmidt process for the a_i. Let q be the matrix with the q_i as columns. The Gram-Schmidt process gives us an upper triangular matrix r such that

$$g^{-1} = qr.$$

Taking the inverse precisely gives the Iwasawa decomposition for g, as desired. □

2 Lecture 2 - 1/30/25

Last time, we talked about the Iwasawa decomposition. We defined

$$\mathfrak{h}^n = \mathrm{GL}(n,\mathbb{R})/(O(n,\mathbb{R})\cdot\mathbb{R}^*),$$

and showed that every $g \in \mathfrak{h}^n$ has the decomposition

$$g = xy = \begin{pmatrix} 1 & & & & \\ & 1 & & x_{ij} & \\ & & \ddots & & \\ & & & 1 & \\ & & & & 1 \end{pmatrix} \begin{pmatrix} y_1y_2\cdots y_{n-1} & & & & \\ & y_1y_2\cdots y_{n-2} & & & \\ & & \ddots & & \\ & & & y_1 & \\ & & & & 1 \end{pmatrix},$$

where the $x_{ij} \in \mathbb{R}$ and $y_j > 0$.

Example 2.1. *In the case $n = 2$, we have*

$$\mathfrak{h}^2 = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} x & y \\ 0 & 1 \end{pmatrix}.$$

This is isomorphic to the upper half place, with $z = x + iy$, $x \in \mathbb{R}$, $y > 0$.
This has complex structure, making it easier to study (holomorphic modular forms). However, $\mathfrak{h}^n$, for $n \geq 3$, has no complex structure.

2.1 GL$(n,\mathbb{Z})$ action on $\mathfrak{h}^n$

We have an action of $\mathrm{GL}(n,\mathbb{Z})$ acting on $\mathfrak{h}^n$, given via left-multiplication of matrices (modulo $O(n,\mathbb{R})\cdot\mathbb{R}^*$). This will be notated $\alpha \cdot g$, but sometimes I might be lazy and write it like pure multiplication.

Example 2.2. *Consider $\alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $g = \begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix}$, or equivalently $z = x+iy$. (We will use g to denote elements of $\mathfrak{h}^n$, rather than z in Dorian's book. We will reserve z for the classical $n = 2$ upper half plane approach.) Then $\alpha z = \frac{az+b}{cz+d}$, and similarly for $\alpha \cdot g$*

$$\alpha \cdot g = \begin{pmatrix} ay & a + bx \\ cy & c + dx \end{pmatrix},$$

which we then need to quotient by the right element of $O(n,\mathbb{R})\cdot\mathbb{R}^*$ to get back into $\mathfrak{h}^n$.

The theory of automorphic forms is all about functions

$$f : \mathrm{GL}(n,\mathbb{Z})\backslash\mathfrak{h}^n \to \mathbb{C}.$$

Equivalently, for $\alpha \in \mathrm{GL}(n,\mathbb{Z})$, $g \in \mathfrak{h}^n$, $k \in K = O(n,\mathbb{R})$, and $d = \begin{pmatrix} d_0 & & & \\ & d_0 & & \\ & & \ddots & \\ & & & d_0 \end{pmatrix}$, for $d_0 = 0$, we want functions

$$f(\alpha g k d) = f(g).$$

Example 2.3. *When $n = 2$, this is precisely the theory of modular forms.*
In this case, we have the standard fundamental domain for $SL(2,\mathbb{Z})\backslash\mathfrak{h}^2$

$$\left\{ z \in \mathfrak{h}^2 : |x| \leq 1/2, |z| \geq 1 \right\}.$$

What is the area of this region? It is precisely the integral

$$\int_{-1/2}^{1/2} \int_{\sqrt{1-x^2}}^{\infty} \frac{\mathrm{d}x\,\mathrm{d}y}{y^2} = \frac{\pi}{3}.$$

Here $\frac{\mathrm{d}x\mathrm{d}y}{y^2}$ *is the hyperbolic measure. It is an* ***invariant measure****: it is invariant under the action* $z \mapsto \frac{az+b}{cz+d}$*. How does one show this? Note that we can write*

$$\frac{\mathrm{d}}{\mathrm{d}z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)$$

and

$$\frac{\mathrm{d}}{\mathrm{d}\overline{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right),$$

so $\frac{\mathrm{d}}{\mathrm{d}z} = 1$ *and* $\frac{\mathrm{d}}{\mathrm{d}\overline{z}} = 0$*. Hence a holomorphic function can be defined as a function* $f : \mathbb{C} \to \mathbb{C}$ *such that* $\frac{\partial}{\partial \overline{z}} f = 0$*.*
Then we can express

$$\frac{\mathrm{d}x\,\mathrm{d}y}{y^2} = \frac{-i}{4}\frac{\mathrm{d}z \wedge \mathrm{d}\overline{z}}{\mathrm{Im}(z)^2},$$

where $\mathrm{d}z = \mathrm{d}x + i\,\mathrm{d}y$ *and* $\mathrm{d}\overline{z} = \mathrm{d}x - i\,\mathrm{d}y$*.*
Now, applying the action of $\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix} \in SL_2(\mathbb{R})$ *on the RHS and applying the quotient rule gives*

$$\frac{-i}{4}\frac{\mathrm{d}\frac{\alpha z+\beta}{\gamma z+\delta} \wedge \mathrm{d}\frac{\alpha\overline{z}+\beta}{\gamma\overline{z}+\delta}}{\mathrm{Im}\left(\frac{\alpha z+\beta}{\gamma z+\delta}\right)^2} = -\frac{i}{4}\frac{\frac{\mathrm{d}z}{(\gamma z+\delta)^2} \wedge \frac{\mathrm{d}\overline{z}}{(\gamma\overline{z}+\delta)^2}}{\frac{\mathrm{Im}(z)^2}{|\gamma z+\delta|^4}} = \frac{-i}{4}\frac{\mathrm{d}z \wedge \mathrm{d}\overline{z}}{\mathrm{Im}(z)^2},$$

hence the measure is invariant.

We'll want to generalize this idea to $\mathrm{GL}(n)$, but this approach doesn't generalize naturally, since we lack complex structure.

2.2 Invariant measure on $\mathfrak{h}^n$

We will want to integrate $\mathrm{GL}(n,\mathbb{Z})$ invariant functions over $\mathfrak{h}^n$, so we need to define an invariant measure. Let $g = xy \in \mathfrak{h}^n$.

Proposition 2.4. *The measure*

$$\mathrm{d}g = \left(\prod_{1\le i<j\le n} \mathrm{d}x_{ij}\right)\left(\prod_{k=1}^{n-1} y_k^{-k(n-k)-1}\,\mathrm{d}y_k\right)$$

is invariant under $g \mapsto \alpha g$ *with* $\alpha \in GL(n,\mathbb{R})$*.*

Proof. It suffices to prove that measure is invariant for a set of generators for $\mathrm{GL}(n,\mathbb{R})$. In particular, $\mathrm{GL}(n,\mathbb{R})$ is generated by matrices B_n, W_n, D_n, where B_n are upper triangular matrices, W_n is the Weyl group of $\mathrm{GL}(n,\mathbb{R})$ (the set of all matrices in $\mathrm{GL}(n,\mathbb{Z})$ with precisely one 1 in each column and row), and

$$D_n = \begin{pmatrix} a_1a_2\ldots a_{n-1} & & & & \\ & a_1a_2\ldots a_{n-2} & & & \\ & & \ddots & & \\ & & & a_1 & \\ & & & & 1 \end{pmatrix}$$

are diagonal matrices.

Remark 2.5. *Why this notation for the diagonal matrices? Since we quotient out by* $\mathbb{R}^*$*, we can have the lower right element be 1. The formulas are all nicer with the* a_i *written this way. (There's also intuition involving root systems that Dorian doesn't want to get into.)*

First, we check the invariance under the action by D_n. Let $\alpha = \begin{pmatrix} a_1a_2\ldots a_{n-1} & & & & \\ & a_1a_2\ldots a_{n-2} & & & \\ & & \ddots & & \\ & & & a_1 & \\ & & & & 1 \end{pmatrix}$.

For any $g = xy$, we can write $\alpha g = (\alpha x \alpha^{-1})(\alpha y)$, where $(\alpha x \alpha^{-1})$ is an upper triangular matrix with 1s on the diagonal and

$$(\alpha x \alpha^{-1})_{ij} = \left(\prod_{k=n-j+1}^{n-i} a_k\right) x_{ij}$$

for all $i < j$, and

$$(\alpha y)_{ii} = \prod_{k=1}^{n-i} (\alpha_k y_k).$$

Plugging everything in, the a_k will all cancel, giving the desired invariance.
Dorian leaves the invariance by the upper triangular matrices and Weyl elements to the reader. Alternatively, details can be found in his book (Section 1.5). □

2.3 Volume of fundamental domain

Let $\Gamma_n = \mathrm{SL}(n, \mathbb{Z})$.

Theorem 2.6 (Siegel, 1936)**.**

$$\mathit{Vol}(\Gamma_n \backslash \mathfrak{h}^n) = n2^{n-1} \prod_{\ell=2}^{n} \frac{\zeta(\ell)}{\mathit{Vol}(S^{\ell-1})},$$

where

$$\mathit{Vol}(S^{\ell-1}) = \frac{2(\sqrt{\pi})^\ell}{\Gamma\left(\frac{\ell}{2}\right)}.$$

The proof will require (a generalization of) the Poisson summation formula. Recall the standard Poisson summation formula:

Proposition 2.7 (Poisson summation)**.** *Let $f : \mathbb{R} \to \mathbb{C}$ be a smooth function (with some technical conditions, i.e. exponential decay). Then*

$$\sum_{n\in\mathbb{Z}} f(n) = \sum_{n\in\mathbb{Z}} \widehat{f}(n),$$

where $\widehat{f}(y) = \int_{-\infty}^{\infty} f(u)e^{-2\pi i y u}\, \mathrm{d}u$ is the Fourier transform.

Proof. Define the new function $G(x) = \sum_{n\in\mathbb{Z}} f(x+n)$. Note that $G(x+1) = G(x)$, so we have a Fourier expansion

$$G(x) = \sum_{k\in\mathbb{Z}} A_k e^{2\pi i k x}$$

where

$$A_k = \int_0^1 G(u)e^{-2\pi i u k}\, \mathrm{d}u\,.$$

Hence

$$\begin{aligned} G(x) &= \sum_{k\in\mathbb{Z}} \left(\int_0^1 \sum_{n\in\mathbb{Z}} f(u+n) e^{-2\pi i u k}\, \mathrm{d}u \right) e^{2\pi i k x} \\ &= \sum_{k\in\mathbb{Z}} \int_{-\infty}^{\infty} f(u) e^{-2\pi i u (k-x)}\, \mathrm{d}u\,, \end{aligned}$$

so we conclude that

$$\sum_{n\in\mathbb{Z}} f(x+n) = \sum_{k} \widehat{f}(k-x).$$

Substituting $x=0$ gives the result. □

In particular, we will need a GL(2) version of Poisson summation.

Proposition 2.8 (Poisson summation for $\mathrm{GL}(2,\mathbb{R})$)**.** *Consider a smooth, compactly supported function $f:\mathbb{R}^2/SO(2,\mathbb{R})\to\mathbb{C}$; i.e. $f((u,v)k)=f((u,v))$ for any $(u,v)\in\mathbb{R}^2$ and $k\in K=SO(2,\mathbb{R})$. Then we have*

$$\sum_{(m,n)\in\mathbb{Z}} f((m,n)\cdot g) = \sum_{(m,n)\in\mathbb{Z}^2} \widehat{f}((m,n)\cdot(g^T)^{-1}).$$

Here $\widehat{f}$ is the (double) Fourier transform

$$\widehat{f}((x,y)) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f((u,v))e^{-2\pi ixu}e^{-2\pi iyv}\,\mathrm{d}u\,\mathrm{d}v\,.$$

Proof. Consider $g\in\mathrm{SL}(2,\mathbb{R})$ of the form $g=\begin{pmatrix} y^{1/2} & xy^{-1/2} \\ 0 & y^{-1/2}\end{pmatrix}$. We define

$$F(g) := \sum_{(m,n)\in\mathbb{Z}^2} f((m,n)\cdot g) = \sum_{(m,n)\in\mathbb{Z}^2} f(my^{1/2}, mxy^{-1/2}+ny^{-1/2}),$$

and for fixed g and n, define

$$G_g(n) := \sum_{m\in\mathbb{Z}} f(my^{1/2}, mxy^{-1/2}+ny^{-1/2}).$$

By standard Poisson Summation (in n),

$$F(g) = \sum_{n\in\mathbb{Z}} G_g(n) = \sum_{n\in\mathbb{Z}} \widehat{G_g}(n).$$

Hence

$$F(g) = \sum_{(m,n)\in\mathbb{Z}} \widehat{f}(my^{1/2}, mxy^{-1/2}+ny^{-1/2}) = \sum_{(m,n)\in\mathbb{Z}^2} \int_{-\infty}^{\infty} f(my^{1/2}, mxy^{-1/2}+uy^{-1/2})e^{-2\pi iun}\,\mathrm{d}u\,,$$

where above the Fourier transform is taken only in the n variable.
We now do the same thing in the m variable. Define

$$H_g(m) := \sum_{n\in\mathbb{Z}} \int_{-\infty}^{\infty} f(my^{1/2}, mxy^{-1/2}+uy^{-1/2})e^{-2\pi iun}\,\mathrm{d}u\,.$$

Poisson summation again gives that

$$F(g) = \sum_{m\in\mathbb{Z}} H_g(m) = \sum_{m\in\mathbb{Z}} \widehat{H_g}(m).$$

Hence, we can write

$$F(g) = \sum_{m\in\mathbb{Z}}\sum_{n\in\mathbb{Z}} \int_{-\infty}^{\infty} f(vy^{1/2}, vxy^{-1/2}+uy^{-1/2})e^{-2\pi inu}e^{-2\pi imv}\,\mathrm{d}u\,\mathrm{d}v\,.$$

Making the transformation $u' = vy^{1/2}$ and $v' = vxy^{-1/2}+uy^{-1/2}$ finishes the proof. □

We'll get to Siegel's proof next time.

Remark 2.9. *Siegel's proof for the volume of the fundamental domain was generalized by Langlands in the paper The volume of the fundamental domain for some arithmetic subgroups of Chevalley groups, Proc AMS, 1965.*

3 Lecture 3 - 2/4/25

3.1 Fundamental Domains

Consider a topological space X and group G, with G acting on X. Recall that a (left) group action is a map $\circ : G \times X \to X$ such that $e \circ x = x$ for all x, and $(g_1 g_2) \circ x = g_1 \circ (g_2 \circ x)$.

Proposition 3.1. *$GL(n, \mathbb{Z})$ acts on $\mathfrak{h}^n = GL(n, \mathbb{R})/(O(n, \mathbb{R}) \cdot \mathbb{R}^*)$. If $\gamma \in GL(n, \mathbb{Z})$ and $g \in \mathfrak{h}^n$, $\gamma \circ g := \gamma \cdot g$ as matrix multiplication.*

Proof. This is clear. □

Note that

$$\mathfrak{h}^n = \mathrm{GL}(n, \mathbb{R})/(O(n, \mathbb{R}) \cdot \mathbb{R}^*) = \mathrm{SL}(n, \mathbb{R})/SO(n, \mathbb{R}).$$

Hence we can talk about the action of $\mathrm{SL}(n, \mathbb{Z})$ on $\mathfrak{h}^n = \mathrm{SL}(n, \mathbb{R})/SO(n, \mathbb{R})$ (via matrix multiplication). What is a fundamental domain for this action?

Recall that a fundamental domain for G acting on X, typically denoted $G\backslash X$), has the properties

- Every $x \in X$ is equivalent to some $y \in G\backslash X$, where $x = g \circ y$ for some $g \in G$.
- No two points in the fundamental domain are equivalent to each other.

In the $n = 2$ case, we have the standard fundamental domain

$$\mathrm{SL}(2, \mathbb{Z})\backslash\mathfrak{h}^2 = \left\{ z = x + iy \in \mathfrak{h}^2 \mid |x| \leq \frac{1}{2}, |z| \geq 1 \right\}.$$

To generalize this idea, we will consider a Siegel set:

$$\Sigma_{\frac{\sqrt{3}}{2}, \frac{1}{2}} = \left\{ x + iy \in \mathfrak{h}^2 \mid |x| \leq \frac{1}{2}, y \geq \frac{\sqrt{3}}{2} \right\}.$$

This set is bigger than the fundamental domain, but small enough to be a good approximation for analytic purposes. Specifically,

$$\bigcup_{\gamma \in \mathrm{SL}(2,\mathbb{Z})} \gamma \cdot \Sigma_{\frac{\sqrt{3}}{2}, \frac{1}{2}} = \mathfrak{h}^2.$$

Theorem 3.2 (Siegel)**.** *The Siegel set for $SL(n, \mathbb{Z})\backslash\mathfrak{h}^n$*

$$\Sigma_{\frac{\sqrt{3}}{2}, \frac{1}{2}} = \left\{ xy \in \mathfrak{h}^n \mid |x_{ij}| \leq \frac{1}{2}, y \geq \frac{\sqrt{3}}{2} \right\}$$

satisfies

$$\bigcup_{\gamma \in SL(n,\mathbb{Z})} \gamma \cdot \Sigma_{\frac{\sqrt{3}}{2}, \frac{1}{2}} = \mathfrak{h}^n.$$

The proof can be found in Dorian's book.

3.2 Volume of fundamental domain SL$(2, \mathbb{Z})\backslash\mathfrak{h}^2$

Last time we stated

Theorem 3.3 (Siegel, 1936)**.**

$$Vol(\Gamma_n\backslash\mathfrak{h}^n) = n2^{n-1} \prod_{\ell=2}^{n} \frac{\zeta(\ell)}{Vol(S^{\ell-1})},$$

where

$$Vol(S^{\ell-1}) = \frac{2(\sqrt{\pi})^\ell}{\Gamma\left(\frac{\ell}{2}\right)}.$$

The proof is inductive, so we'll want to prove the statement for $n = 2$.

Proof for $n = 2$. Let $K = O(2, \mathbb{R})$. Consider a smooth and compactly supported function $f : \mathbb{R}^2/K \to \mathbb{C}$. We can then define

$$F(g) = \sum_{(m,n)\in\mathbb{Z}^2} f((m, n) \cdot g),$$

where multiplication is taken as a row vector multiplied by a matrix. Since f is right-invariant by K, we have that

$$F(gk) = F(g)$$

for all $g \in \mathrm{GL}(2, \mathbb{R})$ and $k \in K$.

Claim 3.4. *$F(\gamma g) = F(g)$ for all $\gamma \in SL(2, \mathbb{Z})$.*

Proof. Let $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, \mathbb{Z})$. Since we want $g \in \mathrm{SL}(2, \mathbb{R})$, we take $g = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} y^{1/2} & 0 \\ 0 & y^{-1/2} \end{pmatrix}$. Then

$$\begin{aligned} F(\gamma g) &= \sum_{(m,n)} f\left((m, n) \begin{pmatrix} a & b \\ c & d \end{pmatrix} g\right) \\ &= \sum_{(m,n)} F\left((am + cn, bm + dn) \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} y^{1/2} & 0 \\ 0 & y^{-1/2} \end{pmatrix}\right) \\ &= \sum_{M,N} F\left((M, N) \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} y^{1/2} & 0 \\ 0 & y^{-1/2} \end{pmatrix}\right) = F(g) \end{aligned}$$

which proves the claim. Here there are no convergence issues because f has compact support. ☐

Next, letting $\Gamma = \mathrm{SL}(2, \mathbb{Z})$, consider

$$\int_{\Gamma\backslash\mathfrak{h}^2} F(g)\,\mathrm{d}g = \int_{\Gamma\backslash\mathfrak{h}^2} F\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} y^{1/2} & 0 \\ 0 & y^{-1/2} \end{pmatrix}\right) \frac{\mathrm{d}x\,\mathrm{d}y}{y^2}.$$

Again, this integral converges because f is compactly supported.
Note that we can write

$$\{(m, n) \mid m, n \in \mathbb{Z}\} = \{(0, 0)\} \cup \bigcup_{\substack{\ell=1 \\ \gamma\in\Gamma_\infty\backslash\Gamma}}^{\infty} \{\ell(0, 1)\gamma\},$$

where $\Gamma_\infty = \left\{\begin{pmatrix} 1 & r \\ 0 & 1 \end{pmatrix} \mid r \in \mathbb{Z}\right\}$. This follows because

$$\Gamma_\infty\backslash\Gamma = \left\{\begin{pmatrix} * & * \\ c & d \end{pmatrix} \mid (c, d) = 1\right\}.$$

Thus,

$$\begin{aligned} \int_{\Gamma\backslash\mathfrak{h}^2} F(g)\,\mathrm{d}g &= \int_{\Gamma\backslash\mathfrak{h}^2} F(0, 0)\,\mathrm{d}g + \int_{\Gamma\backslash\mathfrak{h}^2} \sum_{\ell=1}^{\infty} \sum_{\gamma\in\Gamma_\infty\backslash\Gamma} f(\ell(0, 1)\gamma g)\,\mathrm{d}g \\ &= F((0, 0)) \cdot \mathrm{Vol}(\Gamma\backslash\mathfrak{h}^2) + 2\int_{\Gamma_\infty\backslash\mathfrak{h}^2} \sum_{\ell=1}^{\infty} f((0, \ell) \cdot g)\,\mathrm{d}g, \end{aligned}$$

where the factor of 2 arises because $\begin{pmatrix} -1 & \\ & -1 \end{pmatrix}$ is in the stabilizer for $\mathfrak{h}^2$.

Hence

$$\begin{aligned}\int_{\Gamma\backslash\mathfrak{h}^2} F(g)\,\mathrm{d}g &= F((0,0))\cdot\mathrm{Vol}(\Gamma\backslash\mathfrak{h}^2) + 2\int_{\Gamma_\infty\backslash\mathfrak{h}^2}\sum_{\ell=1}^{\infty} f\left((0,\ell)\cdot\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}\begin{pmatrix}y^{1/2} & 0\\ 0 & y^{-1/2}\end{pmatrix}\right)\frac{\mathrm{d}x\,\mathrm{d}y}{y^2}\\ &= F((0,0))\cdot\mathrm{Vol}(\Gamma\backslash\mathfrak{h}^2) + 2\int_{\Gamma_\infty\backslash\mathfrak{h}^2}\sum_{\ell=1}^{\infty} f\left((0,\ell y^{-1/2})\right)\frac{\mathrm{d}x\,\mathrm{d}y}{y^2}\\ &= F((0,0))\cdot\mathrm{Vol}(\Gamma\backslash\mathfrak{h}^2) + 2\int_{x=0}^{1}\int_{y=0}^{\infty}\sum_{\ell=1}^{\infty} f\left((0,\ell y^{-1/2})\right)\frac{\mathrm{d}x\,\mathrm{d}y}{y^2}.\end{aligned}$$

Taking the transformations $y \mapsto \ell^2 y$ in the first line and then $y \to y^{-2}$ in the second line, we get

$$\begin{aligned}2\int_{x=0}^{1}\int_{y=0}^{\infty}\sum_{\ell=1}^{\infty} f\left((0,\ell y^{-1/2})\right)\frac{\mathrm{d}x\,\mathrm{d}y}{y^2} &= 2\int_{x=0}^{1}\int_{y=0}^{\infty}\sum_{\ell=1}^{\infty} f\left((0,y^{-1/2})\right)\frac{1}{\ell^2}\frac{\mathrm{d}x\,\mathrm{d}y}{y^2}\\ &= 4\zeta(2)\int_0^\infty f\left((0,y)\right)y\,\mathrm{d}y\,.\end{aligned}$$

Now, we convert to polar coordinates. Since f is right invariant by $k = \begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix} \in K$,

$$f((0,y)) = f((y\sin\theta, y\cos\theta))$$

for any θ.
Thus we get that

$$\begin{aligned}4\zeta(2)\int_{y=0}^{\infty} f\left((0,y)\right)y\,\mathrm{d}y &= \frac{2\zeta(2)}{\pi}\int_0^{2\pi}\int_{y=0}^{\infty} f\left((y\sin\theta, y\cos\theta)\right)y\,\mathrm{d}y\,\mathrm{d}\theta\\ &= \frac{2\zeta(2)}{\pi}\int_{\mathbb{R}^2} f(u,v)\,\mathrm{d}u\,\mathrm{d}v = \frac{2\zeta(2)}{\pi}\widehat{f}((0,0)).\end{aligned}$$

Hence we have shown that

$$\int_{\Gamma\backslash\mathfrak{h}^2} F(g)\,\mathrm{d}g = f((0,0))\mathrm{Vol}(\Gamma\backslash\mathfrak{h}^2) + \frac{2\zeta(2)}{\pi}\widehat{f}((0,0)).$$

Now, consider replacing f by $\widehat{f}$. By Poisson summation for $\mathrm{GL}(2,\mathbb{R})$,

$$\sum_{(m,n)\in\mathbb{Z}^2} f((m,n)g) = \sum_{(m,n)\in\mathbb{Z}^2}\widehat{f}((m,n)(g^T)^{-1}).$$

We can replace g by $(g^T)^{-1}$ in all of the computation above, and nothing would change. Hence, we get that

$$\int_{\Gamma\backslash\mathfrak{h}^2} F(g)\,\mathrm{d}g = \widehat{f}((0,0))\mathrm{Vol}(\Gamma\backslash\mathfrak{h}^2) + \frac{2\zeta(2)}{\pi}f((0,0)),$$

using that $\widehat{\widehat{f}}(x) = f(-x)$. Subtracting the two equations and solving for the volume gives the desired formula. □

Next time, we will finish the proof for general n.

4 Lecture 4 - 2/6/25

4.1 Proof of Siegel's Theorem

This time we will finish the proof of Siegel's theorem:

Theorem 4.1 (Siegel, 1936)**.**

$$Vol(\Gamma_n\backslash\mathfrak{h}^n) = n2^{n-1}\prod_{\ell=2}^{n}\frac{\zeta(\ell)}{Vol(S^{\ell-1})},$$

where

$$Vol(S^{\ell-1}) = \frac{2(\sqrt{\pi})^{\ell}}{\Gamma\left(\frac{\ell}{2}\right)}.$$

We proved it for $n = 2$ last time. Now we will finish the proof for $n > 2$ inductively.
We will use the Poisson summation formula for $\mathrm{GL}(n, \mathbb{R})$:

Proposition 4.2. *For a function $f : \mathbb{R}^n/K_n \to \mathbb{C}$, where $K_n = O(n, \mathbb{R})$, we have that*

$$\sum_{m\in\mathbb{Z}^n} f(m\cdot g) = \sum_{m\in\mathbb{Z}^n}\widehat{f}(m\cdot(g^T)^{-1}).$$

We showed this for $n = 2$; it can be generalized to higher n inductively.

Proof of Siegel's Theorem. For more details, one can check Dorian's book, section 1.6.
Let $\Gamma_n = \mathrm{SL}(n, \mathbb{Z})$. Recall that for $g \in \mathfrak{h}^n$, we write $g = xy$ with the usual notation for x and y. We want to this to lie in $\mathrm{SL}(n, \mathbb{R})/\mathrm{SO}(n, \mathbb{R})$, so we instead consider

$$y = \begin{pmatrix} y_1y_2\cdots y_{n-1}t & & & & \\ & y_1y_2\cdots y_{n-2}t & & & \\ & & \ddots & & \\ & & & y_1t & \\ & & & & t \end{pmatrix},$$

where $t = \left(\prod_{j=1}^{n-1} y_j^{n-j}\right)^{-1}$.
Emulating the proof for $n = 2$, let $f : \mathbb{R}^n/K_n \to \mathbb{C}$ be a smooth and compactly supported function. Again we define

$$F(g) = \sum_{m\in\mathbb{Z}^n} f(m\cdot g).$$

Then we can show $F(\gamma g) = F(g)$ for all $\gamma \in \mathrm{SL}(n, \mathbb{Z})$.

Definition 4.3. *The **mirabolic subgroup of GL**(n) is*

$$P_n = \left\{\begin{pmatrix} & & * & & \\ 0 & 0 & \dots & 0 & 1\end{pmatrix}\right\}$$

Then one can check that

$$F(g) = f((0,\ldots,0)) + \sum_{\ell=1}^{\infty}\sum_{\gamma\in P_n\backslash\Gamma_n} f(\ell\cdot e_n\cdot\gamma g),$$

where $e_n = (0, \ldots, 0, 1)$.
Now, we have that

$$\begin{aligned}\int_{\Gamma_n\backslash\mathfrak{h}^n} F(g)\,\mathrm{d}g &= f((0,\ldots,0))\mathrm{Vol}(\Gamma_n\backslash\mathfrak{h}^n) + \int_{\Gamma_n\backslash\mathfrak{h}^n}\sum_{\ell=1}^{\infty}\sum_{\gamma\in P_n\backslash\Gamma_n} f(\ell\cdot e_n\cdot\gamma g)\,\mathrm{d}g \\ &= f((0,\ldots,0))\mathrm{Vol}(\Gamma_n\backslash\mathfrak{h}^n) + 2\sum_{\ell=1}^{\infty}\int_{P_n\backslash\mathfrak{h}^n} f(\ell\cdot e_n\cdot g)\,\mathrm{d}g\,,\end{aligned}$$

where again the 2 appears because the diagonal element with -1 on the diagonal lies in the stabilizer for $\mathfrak{h}^n$. (TODO: Only for n even? This will change the formula slightly – will need to double check this.)
Now we can write

$$g = \begin{pmatrix} 1 & & & & \\ & 1 & & x_{ij} & \\ & & \ddots & & \\ & & & 1 & \\ & & & & 1 \end{pmatrix} \begin{pmatrix} y_1 y_2 \dots y_{n-1} t & & & & \\ & y_1 y_2 \dots y_{n-2} t & & & \\ & & \ddots & & \\ & & & y_1 t & \\ & & & & t \end{pmatrix} \begin{pmatrix} t^{\frac{1}{n-1}} I_{n-1} & \\ & t \end{pmatrix} \begin{pmatrix} t^{-\frac{1}{n-1}} I_{n-1} & \\ & t \end{pmatrix}$$

$$= \begin{pmatrix} 1 & & & & x_{1n} \\ & 1 & & & x_{2n} \\ & & \ddots & & \vdots \\ & & & 1 & x_{n-1,n} \\ & & & & 1 \end{pmatrix} \begin{pmatrix} g' & \\ & 1 \end{pmatrix} \begin{pmatrix} t^{-\frac{1}{n-1}} I_{n-1} & \\ & t \end{pmatrix}$$

where g' is the $n-1$ by $n-1$ matrix

$$g' = \begin{pmatrix} 1 & x_{12} & x_{13} & \dots & x_{1,n-1} \\ & 1 & x_{23} & \dots & x_{2,n-1} \\ & & \ddots & \ddots & \vdots \\ & & & 1 & x_{n-2,n-1} \\ & & & & 1 \end{pmatrix} \begin{pmatrix} y_1 y_2 \dots y_{n-1} t^{n/(n-1)} & & & \\ & y_1 y_2 \dots y_{n-2} t^{n/(n-1)} & & \\ & & \ddots & \\ & & & y_1 t^{n/(n-1)} \end{pmatrix} \in \mathfrak{h}^{n-1}$$

Recall that

$$\mathrm{d}g = \left(\prod_{1 \le i < j \le n} \mathrm{d}x_{ij} \right) \prod_{k=1}^{n-1} y_k^{-k(n-k)-1} \, \mathrm{d}y_k \, ,$$

and we have that

$$\mathrm{d}g' = \left(\prod_{1 \le i < j \le n-1} \mathrm{d}x_{ij} \right) \prod_{k=1}^{n-2} y_{k+1}^{-k(n-k-1)-1} \, \mathrm{d}y_{k+1} \, .$$

Computation thus gives us that

$$\mathrm{d}g = -\frac{n}{n-1} \, \mathrm{d}g' \left(\prod_{j=1}^{n-1} \mathrm{d}x_{j,n} \right) t^n \frac{\mathrm{d}t}{t}.$$

Now, to apply induction, we will want to relate $P_n \backslash \mathfrak{h}^n$ to $\Gamma_{n-1} \backslash \mathfrak{h}^{n-1}$.
Every $p \in P_n$ is of the form

$$p = \begin{pmatrix} \gamma & b \\ & 1 \end{pmatrix} = \begin{pmatrix} I_{n-1} & b \\ & 1 \end{pmatrix} \begin{pmatrix} \gamma & \\ & 1 \end{pmatrix}$$

with $\Gamma \in \mathrm{SL}(n-1, \mathbb{Z})$ and $b \in \mathbb{Z}^{n-1}$. Moreover, every $g \in \mathfrak{h}^n$ is of the form

$$g = \begin{pmatrix} g' & u \\ & 1 \end{pmatrix} \begin{pmatrix} t^{-\frac{1}{n-1}} I_{n-1} & \\ & t \end{pmatrix} = \begin{pmatrix} I_{n-1} & u \\ & 1 \end{pmatrix} \begin{pmatrix} g' & \\ & 1 \end{pmatrix} \begin{pmatrix} t^{-\frac{1}{n-1}} I_{n-1} & \\ & t \end{pmatrix},$$

where

$$u = \begin{pmatrix} u_{1,n} \\ u_{2,n} \\ \vdots \\ u_{n-1,n} \end{pmatrix}.$$

Then
$$p\cdot g=\begin{pmatrix}I_{n-1} & b\\ & 1\end{pmatrix}\begin{pmatrix}\gamma & \\ & 1\end{pmatrix}\begin{pmatrix}I_{n-1} & u\\ & 1\end{pmatrix}\begin{pmatrix}g' & \\ & 1\end{pmatrix}\begin{pmatrix}t^{-\frac{1}{n-1}}I_{n-1} & \\ & t\end{pmatrix}.$$

Let $U_n(\mathbb{Z})$ denote matrices with 1s on the diagonal, integers in the right most column, and 0s elsewhere, and similarly for $U_n(\mathbb{R})$.

Lemma 4.4. *Fix a $\gamma\in SL(n-1,\mathbb{Z})$. We have an action of $U_n(\mathbb{Z})$ on $\mathbb{R}^{n-1}$ given by left multiplication of $U_n(\mathbb{Z})$ on $\begin{pmatrix}\gamma & \\ & 1\end{pmatrix}\cdot U_n(\mathbb{R})$, with fundamental domain given by*

$$\left\{\begin{pmatrix}\gamma & \\ & 1\end{pmatrix}\begin{pmatrix}1 & & & & u_1\\ & 1 & & & u_2\\ & & \ddots & & \vdots\\ & & & 1 & u_{n-1}\\ & & & & 1\end{pmatrix}\mid 0\le u_i<1\right\}$$

Moreover,
$$U_n(\mathbb{Z})\backslash\begin{pmatrix}\gamma & \\ & 1\end{pmatrix}U_n(\mathbb{R})\cong(\mathbb{Z}\backslash\mathbb{R})^{n-1}.$$

Proof. One can write
$$\begin{aligned}\bigcup_{m\in\mathbb{Z}^{n-1}}\begin{pmatrix}I_{n-1} & m\\ & 1\end{pmatrix}\begin{pmatrix}\gamma & \\ & 1\end{pmatrix}\begin{pmatrix}I_{n-1} & (\mathbb{Z}\backslash\mathbb{R})^{n-1}\\ & 1\end{pmatrix}&=\bigcup_{m\in\mathbb{Z}^{n-1}}\begin{pmatrix}\gamma & \\ & 1\end{pmatrix}\begin{pmatrix}I_{n-1} & \gamma^{-1}m\\ & 1\end{pmatrix}\begin{pmatrix}I_{n-1} & (\mathbb{Z}\backslash\mathbb{R})^{n-1}\\ & 1\end{pmatrix}\\ &=\begin{pmatrix}\gamma & \\ & 1\end{pmatrix}\bigcup_{m\in\mathbb{Z}^{n-1}}\begin{pmatrix}I_{n-1} & (\mathbb{Z}\backslash\mathbb{R})^{n-1}+\gamma^{-1}m\\ & 1\end{pmatrix}\\ &=\begin{pmatrix}\gamma & \\ & 1\end{pmatrix}U_n(\mathbb{R}).\end{aligned}$$

□

Hence, examining our expression $p\cdot g$ and applying the lemma, we get the decomposition
$$P_n\backslash\mathfrak{h}^n\cong(\mathrm{SL}(n-1,\mathbb{Z})\backslash\mathfrak{h}^{n-1})\times(\mathbb{Z}\backslash\mathbb{R})^{n-1}\times(0,\infty).$$

Moreover, note that
$$f(\ell e_n g)=f\left(\ell e_n\begin{pmatrix}g' & u\\ & 1\end{pmatrix}\begin{pmatrix}t^{-\frac{1}{n-1}}I_{n-1} & \\ & t\end{pmatrix}\right)=f(\ell t e_n).$$

Thus we can write
$$2\sum_{\ell=1}^{\infty}\int_{P_n\backslash\mathfrak{h}^n}f(\ell\cdot e_n\cdot g)\,\mathrm{d}g=\frac{2n}{n-1}\sum_{\ell=1}^{\infty}\left(\int_{\Gamma_{n-1}\backslash\mathfrak{h}^{n-1}}\mathrm{d}g'\right)\left(\int_{(\mathbb{Z}\backslash\mathbb{R})^{n-1}}\prod_{i=1}^{n-1}\mathrm{d}x_{i,n}\right)\left(\int_0^\infty f(\ell t e_n)t^n\frac{\mathrm{d}t}{t}\right).$$

By induction, the first integral on the RHS is the volume $\Gamma_{n-1}\backslash\mathfrak{h}^{n-1}$. The second integral is 1. Thus, it suffices to compute the third integral.
Making a transformation $t\to\frac{t}{\ell}$, we have that
$$\sum_{\ell=1}^{\infty}\int_0^\infty f(\ell t e_n)t^n\frac{\mathrm{d}t}{t}=\zeta(n)\int_0^\infty f(\ell t e_n)t^n\frac{\mathrm{d}t}{t}.$$

Lemma 4.5.
$$\int_0^\infty f(\ell t e_n)t^n\frac{\mathrm{d}t}{t}=\frac{\widehat{f}((0,\ldots,0))}{Vol(S^{n-1})}.$$

Proof. Use the n dimensional spherical coordinates

$$\begin{aligned} x_1 &= t(\sin\theta_{n-1})\cdots(\sin\theta_2)(\sin\theta_1) \\ x_2 &= t(\sin\theta_{n-1})\cdots(\sin\theta_2)(\cos\theta_1) \\ &\vdots \\ x_{n-1} &= t(\sin\theta_{n-1})(\cos\theta_{n-2}) \\ x_n &= t\cos\theta_{n-1} \end{aligned}$$

In particular, note that $x_1^2 + \cdots + x_n^2 = 1$. We have the invariant measure on S^{n-1}

$$\mathrm{d}\theta = \prod_{1 \le j < n} (\sin\theta_j)^{j-1}\,\mathrm{d}j\,,$$

so

$$\mathrm{d}x_1 \cdots \mathrm{d}x_n = t^{n-1}\,\mathrm{d}t\,\mathrm{d}\theta\,.$$

This measure is invariant under rotations, so

$$f((0,\ldots,0,t)) = \frac{1}{\mathrm{Vol}(S^{n-1})}\int_{S^{n-1}} f(x_1,\ldots,x_n))\,\mathrm{d}\theta\,,$$

and thus

$$\int_0^\infty f((0,\ldots,0,t))t^n\frac{\mathrm{d}t}{t} = \frac{1}{\mathrm{Vol}(S^{n-1})}\int_{\mathbb{R}^n} f(x_1,\ldots,x_n)\,\mathrm{d}x_1\ldots\mathrm{d}x_n = \widehat{f}((0,\ldots,0)),$$

where we apply polar coordinates.

□

Now, we repeat the same process replacing $\widehat{f}$ and f, using the Poisson summation formula. Subtracting the two formulas gives an inductive formula for the volume.

□

Next time, we start the theory of automorphic forms.