Infinity Categories Talks

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These are my lecture notes for Columbia's Spring 2022 Seminar on Infinity Categories (website here). Use with caution as they may contain mistakes!

Contents

1 January 31

These are the notes from last semester's introductory talk on infinity categories. We will give a quick summary.

1.1 Introduction

There are three big things we want to understand about infinity categories:

1. Why they are a good formalism for homotopy theory?
2. How can we construct them to still be category-like?
3. What are the properties of the category of infinity categories (especially in connection to dg categories)?

We unfortunately will not get to the third goal today, but we will try to elucidate the first two.

Motivation 1: if you squint at a category, it looks like a graph. If there is any justice in this world, we can generalize to simplicial complexes.

Motivation 2: we want to think about morphisms up to homotopy, but the hom sets in a category do not record this information. So we ask for additional structure of 2-morphisms between 1-morphisms, 3-morphisms between 2-morphisms, etc. to capture the full homotopy theory. We also want higher morphisms to be invertible (like homotopies being invertible) — but only up to higher higher morphisms.

We will develop the theory of $(\infty, 1)$-categories, and henceforth they shall be known as *quasicategories*. There are several ways of formalizing them (and understanding the relationship between these formalisms is part of third goal). Today we will use simplicial sets.

1.2 Simplicial Sets

Definition 1.1. Δ is the *simplex category* with objects

$$[n] = (0, 1, ..., n), \quad n \geq 0$$

and morphisms are order preserving maps.

Example. The maps of Δ are generated by

Coface maps

$$d^k : [n-1] \to [n], \quad (0, 1, ..., n-1) \mapsto (0, 1, ..., \hat{k}, ..., n)$$

(unique injective map that misses k.)

Codegeneracy maps

$$s^k : [n+1] \to [n], \quad (0, 1, ..., n+1) \mapsto (0, 1, ..., k, k, ...n)$$

(unique surjective map that hits k twice).

We can think of Δ^n as the topological simplex, d^k includes Δ^{n-1} as the face of Δ^n opposite vertex k, and s^k crushes Δ^{n+1} onto Δ^n by pressing vertex $(n+1)$ onto vertex k.

Definition 1.2. The category **sSet** of simplicial sets is the category of functors $\Delta^{\text{op}} \to \textbf{Set}$.

The data of a simplicial set X:

- Sets X_n for $n \geq 0$
- Face maps $d_k = X(d^k) : X_n \to X_{n-1}$
- Degeneracy maps $s_k : X(s^k) : X_n \to X_{n+1}$

(satisfying identities, which we omit). X can be thought of as not just one simplex, but a simplicial complex. To see more precisely why, we introduce two new functors.

Definition 1.3. Let S be a topological space, and let $\Delta^n_{\mathbf{Top}}$ be the topological n-simplex. The *singular complex functor* $\mathrm{Sing} : \mathbf{Top} \to \mathbf{sSet}$ takes

$$\mathrm{Sing}_n(S) := \mathrm{Hom}_{\mathbf{Top}}(\Delta^n_{\mathbf{Top}}, S)$$

The *geometric realization functor* is the left adjoint $|-| : \mathbf{sSet} \to \mathbf{Top}$.

For intuition purposes,

$$X = \left(\coprod_{n=0}^{\infty} X_n \times \Delta^n_{\mathbf{Top}} \right) / \sim$$

(one n-simplex for each element of X_n, glued together according to face/degeneracy maps).

The following notion will be our segue into categories. The *k-th n-horn* Λ^n_k is the coequalizer

$$\coprod_{0 \le i < j \le n} \Delta^{n-2} \rightrightarrows \coprod_{i \ne k} \Delta^{n-1} \to \Lambda^n_k$$

Intuitively, it is $\partial \Delta^n \setminus \partial_k \Delta^n$.

Definition 1.4. A simplicial set X is a *Kan complex* (resp. *weak Kan complex*) if every horn $\Delta^n_k \to X$ for $0 \le k \le n$ (resp. $0 < k < n$) can be extended to an n-simplex $\Delta^n \to X$.

We refer to Δ^n_k for $0 \le k \le n$ as *inner horns* and Δ^n_0 and Δ^n_n as *outer horns.*

We will see examples of these and related phenomena in the next section.

1.3 Towards Quasicategories

Here are some concepts that motivate and set up our definition of quasicategories.

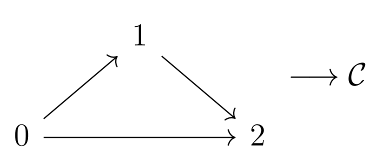
Nerves The following construction connects simplicial sets to categories.

Definition 1.5. The *nerve functor* $N : \mathbf{Cat} \to \mathbf{sSet}$ takes

$$N(\mathcal{C})_n = \mathrm{Fun}([n], \mathcal{C})$$

where $\mathrm{Fun}(-,-)$, and morphisms (functors) induce maps on $\mathrm{Fun}(-,-)$ by composition.

Example. Consider $N(\mathcal{C})_2$. This is the set of functors



Given such a functor, note that $d_1 : N(\mathcal{C})_2 \to N(\mathcal{C})_1$ is composition and $s_1 : N(\mathcal{C})_1 \to N(\mathcal{C})_2$ is inserting an identity map.

Intuitively,

- $N(\mathcal{C})_n$ is the set of strings of n composable morphisms in $\mathcal{C}$

$$C_0 \xrightarrow{f_1} C_1 \xrightarrow{f_2} \dots \xrightarrow{f_{n-1}} C_{n-1} \xrightarrow{f_n} C_n$$

- d_i *composes*

$$C_0 \xrightarrow{f_1} \dots \xrightarrow{f_{i-1}} C_{i-1} \xrightarrow{f_{i+1} \circ f_i} C_{i+1} \xrightarrow{f_{i+2}} \dots \xrightarrow{f_n} C_n$$

- s_i *inserts identity*

$$C_0 \xrightarrow{f_1} \dots \xrightarrow{f_i} C_i \xrightarrow{\mathrm{id}_{C_i}} C_i \xrightarrow{f_{i+1}} \dots \xrightarrow{f_n} C_n$$

Proposition 1.6. *A category $\mathcal{C}$ can be recovered up to isomorphism from its nerve $N(\mathcal{C})$.*

Proof. The objects of $\mathcal{C}$ are $N(\mathcal{C})_0$.

Morphisms $C_0 \to C_1$ are given by $\phi \in N(\mathcal{C})_1$ with $d_1(\phi) = C_0$ and $d_0(\phi) = C_1$.

$\mathrm{id}_C = s_0(C)$.

Given $C_0 \xrightarrow{\phi} C_1 \xrightarrow{\psi} C_2$, there is a 2-simplex $\sigma \in N(\mathcal{C})_2$ with $d_2(\sigma) = \phi$ and $d_1(\sigma) = \psi$. Then $\psi \circ \phi = d_1(\sigma)$. ☐

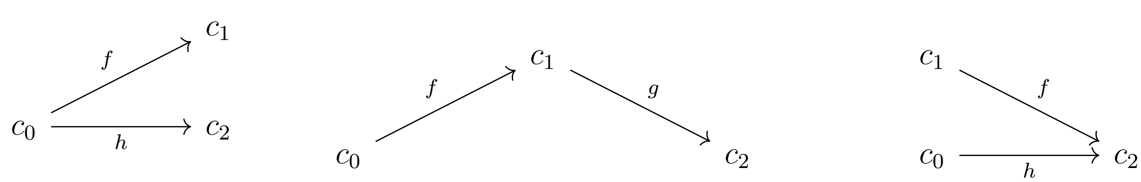
Now that we have embedded categories into simplicial sets, we can generalize them... to INFINITY (and beyond?).

The generalization will come from *horn extension properties.*

Horn Extensions

Observation.

$\Lambda^2_0 \to N(\mathcal{C})$ $\qquad$ $\Lambda^2_1 \to N(\mathcal{C})$ $\qquad$ $\Lambda^2_2 \to N(\mathcal{C})$



An extension of the inner horn to $\Delta^2 \to N(\mathcal{C})$ corresponds to a composition $g \circ f$.

(For $h = id$), extensions of the outer horns to $\Delta^2 \to N(\mathcal{C})$ correspond to an f^{-1}.

Proposition 1.7. *A simplicial set X is isomorphic to the nerve of a category iff for all $0 < k < n$ the inclusion $\Lambda^n_k \to \Delta^n$ induces a bijection*

$$X_n = Hom(\Delta^n, X) \xrightarrow{\simeq} Hom(\Lambda^n_k, X)$$

X is the nerve of a groupoid iff the above holds for all $0 \leq k \leq n$.

Proof idea. See [5] Proposition 1.1.2.2.

Given the nerve of a category $N(\mathcal{C})$, we want to show that we can uniquely extend horns. A horn $f_0 : \Delta^n_k \to \mathcal{C}$ contains the 1-skeleton

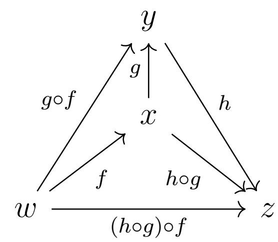
$$X_0 \xrightarrow{g_1} X_1 \xrightarrow{g_2} \dots \xrightarrow{g_n} X_n$$

which uniquely defines $f : \Delta^n \to N(\mathcal{C})$ if it exists. To see existence, check that for $j \neq i$,

$$f|_{\Delta^{0,\dots,j-1,j+1,\dots,n}} = f|_{\Delta^{0,\dots,j-1,j+1,\dots,n}}$$

(it follows from the 1-skeleton).

Conversely, given such a simplicial set, we can use the method of proof of Proposition1.6 to construct a category. However, we also need to check that the identity is actually the identity, and that composition is associative. Extending the horn $\Lambda^3_1 \to \mathcal{C}$ uniquely to $\Delta^3 \to \mathcal{C}$



shows that $h \circ (g \circ f) = (h \circ g) \circ f$.

To show that $X \cong N(\mathcal{C})$, induct on n to show

$$\mathrm{Hom}_{\mathbf{sSet}}(\Delta^n, X) \to \mathrm{Hom}_{\mathbf{sSet}}(\Delta^n, N(\mathcal{C}))$$

is a bijection. □

The fact that nerves work so nicely might hint that a generalization is in order.

Recalling the definition of a Kan complex, we have the following inclusions.

$$\begin{array}{ccc} \mathbf{Grpd} & \longrightarrow & \mathbf{Cat} \\ {\scriptstyle N}\downarrow & & \downarrow{\scriptstyle N} \\ \mathbf{Kan} & \longrightarrow & \mathbf{sSet} \end{array}$$

Quasicategories Our recipe for quasicategories: full subcategory of **sSet**, like **Cat** only need to extend inner horns, like **Kan** the extensions do not need to be unique.

Definition 1.8. A simplicial set X is a *quasicategory* iff for all $0 < k < n$ the inclusion $\Lambda^n_k \to \Delta^n$ induces a surjection

$$X_n = \mathrm{Hom}(\Delta^n, X) \twoheadrightarrow \mathrm{Hom}(\Lambda^n_k, X)$$

Why is this the correct notion? We only need compositions to be unique up to homotopy — think composition of paths.

Now, we introduce some basic terminology. This will feel familiar, as it generalizes our recovery of $\mathcal{C}$ from $N(\mathcal{C})$.

Definition 1.9. Let $\mathcal{C}$ be a quasi category.

Objects are the vertices $x \in \mathcal{C}_0$.

Morphisms are the 1-simplices $f \in \mathcal{C}_1$.

The *source map* is $s = d_1 : \mathcal{C}_1 \to \mathcal{C}_0$ and the *target map* is $t = d_0 : \mathcal{C}_1 \to \mathcal{C}_0$.

The *identity map* is $\mathrm{id} = s_0 : \mathcal{C}_0 \to \mathcal{C}_1$

The set of morphisms from x to y is

$$\begin{array}{ccc} \hom_{\mathcal{C}}(x,y) & \longrightarrow & \mathcal{C}_1 \\ \downarrow & & \downarrow {\scriptstyle (s,t)} \\ * & \underset{(x,y)}{\longrightarrow} & \mathcal{C}_0 \times \mathcal{C}_0 \end{array}$$

Note that the simplicial identities imply $d_0 s_0 = d_1 s_0 = \mathrm{id}_{\mathcal{C}_0}$, so $\mathrm{id}_x = s_0 x$ is indeed a morphism $x \to x$.

Later $\hom_{\mathcal{C}}(x,y)$ will be given additional structure as a space,and will be important for homotopy theory.

Definition 1.10. In a quasicategory $\mathcal{C}$, consider morphisms $f : x \to y$ and $g : y \to z$. These morphisms define an inner horn $\lambda : \Lambda^2_1 \to \mathcal{C}$ by $x \xrightarrow{f} y \xrightarrow{g} z$. Let $\sigma : \Delta^2 \to \mathcal{C}$ be an extension of λ. Then $d_1(\sigma)$ is a *candidate composition* for f and g.

Again, the choice of composition is not unique. But we want the space of choices to be contractible.

Examples of Quasicategories

Example. $N(\mathcal{C})$ is a quasicategory for any category $\mathcal{C}$ (bijective implies surjective).

Example. $\mathrm{Sing}(S)$ is a quasicategory for any topological space S. Since Λ^n_k is a strong deformation retract of Δ^n, any continuous map on the former extends to the latter; this works for $k = 0, n$ as well, so $\mathrm{Sing}(S)$ is in fact a Kan complex.

Lemma 1.11. *The product of quasicategories is a quasicategory.*

Proof sketch. As simplicial sets, $(X \times Y)_n = X_n \times Y_n$, with $d_k(x,y) = (d_k(x), d_k(y))$ and $s_k(x,y) = (s_k(x), s_k(y))$ (special case of limits of presheaves are computed pointwise).

Horn extensions on each coordinate together give horn extensions on the product. □

Definition 1.12. Given a simplicial set K and a quasicategory $\mathcal{C}$, the *space of functors* $\mathrm{Fun}(K, \mathcal{C})$ is the simplicial set

$$\mathrm{Fun}(K,\mathcal{C})_n = \mathrm{Map}(K,\mathcal{C})_n = \hom_{\mathbf{sSet}}(\Delta^n \times K, \mathcal{C})$$

Note that $\mathrm{Fun}(K,\mathcal{C})_0 = \hom_{\mathbf{sSet}}(K,\mathcal{C})$ recovers the hom set.

Proposition 1.13. *If K is a simplicial set and $\mathcal{C}$ is a quasicategory, then Fun$(K,\mathcal{C})$ is a quasicategory.*

Lemma 1.14. *For simplicial sets K, X, Y, there is a natural (in K, X, Y) bijection*

$$\hom_{sSet}(K, \mathrm{Fun}(X, Y)) \to \hom_{sSet}(K \times X, Y)$$

Proof sketch. There is an evaluation map $ev : \mathrm{Fun}(X, Y) \times X \to Y$ defined on $f \in \mathrm{Fun}(X, Y)_n, x \in X_n$ by

$$ev_n : (f, x) \mapsto f(\iota_n, x)$$

(should check it commutes with the face/degeneracy maps).

The forward map of our bijection takes $g : K \to \mathrm{Fun}(X, Y)$ to

$$K \times X \xrightarrow{g \times 1} \mathrm{Fun}(X, Y) \times X \xrightarrow{ev} Y$$

The inverse applied to $g : K \times X \to Y$ applied to $x \in K_n$ yields the simplicial map

$$\Delta^n \times X \xrightarrow{\iota_x \times 1} K \times X \xrightarrow{g} Y$$

□

Lemma 1.15. *If K is a simplicial set, and Y is a quasicategory, then a map of simplicial sets $K \times \Lambda^n_k \to Y$ can be extended to $K \times \Delta^n \to Y$ for all $0 < k < n$.*

Proof idea. One proof involves the concepts of anodyne extensions and saturated classes of morphisms, which we will not discuss.

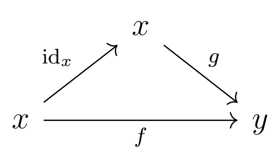
Another proof uses induction on m and repeated application of the horn extension property to fill in occurrences of Λ^m_l in $X \times \Lambda^n_k \subset X \times \Delta^n$. □

Proof of Proposition 1.13. The previous two lemmas show that $\mathrm{Fun}(K, \mathcal{C})$ satisfies the inner horn extension condition. □

Example. Consider $\mathrm{Fun}(\Delta^1, \mathcal{C})$. Note that $\mathrm{Fun}(\Delta^1, \mathcal{C})_0 = \hom_{\mathbf{sSet}}(\Delta^1, \mathcal{C})$ yields the 1-morphisms of $\mathcal{C}$.

1.4 The Homotopy Category

Definition 1.16. Given a quasicategory $\mathcal{C}$, two morphisms $f, g : x \to y$ are *homotopic* ($f \simeq g$) if there is a 2-simplex $\sigma : \Delta^2 \to \mathcal{C}$ with boundary $\partial\sigma = (g, f, \mathrm{id}_x)$



and σ is a *homotopy* from f to g ($\sigma : f \to g$).

An alternative definition could have placed id_x on edge 0, but this is equivalent by the inner horn extension property.

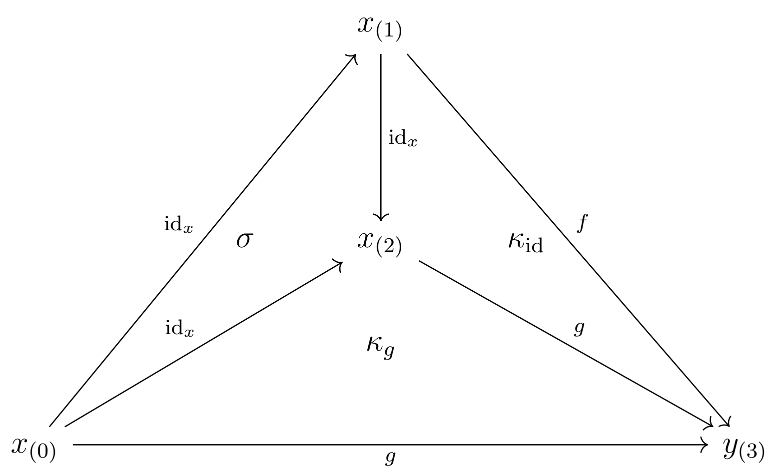
Proposition 1.17. *Let $\mathcal{C}$ be a quasicategory, and $x, y \in \mathcal{C}$. The homotopy relation is an equivalence relation on $\hom_{\mathcal{C}}(x, y)$, with the homotopy class of $f : x \to y$ denoted $[f]$.*

Proof sketch. For identity, we want a constant homotopy.

$$\kappa_f = s_0 f : \Delta^2 \to \mathcal{C}$$

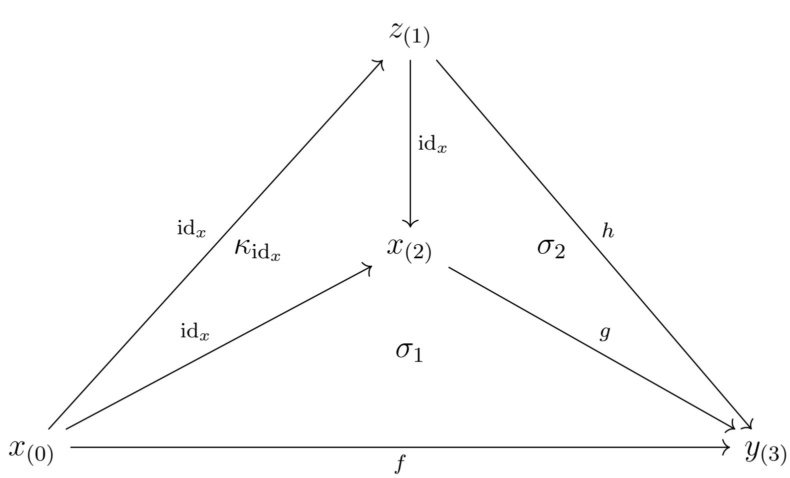
By the simplicial identities, $d_0\kappa_f = d_0 s_0 f = f$ and likewise $d_1\kappa_f = f$, and $d_2\kappa_f = d_2 s_0 f = s_0 d_1 f = \mathrm{id}_x$. So $\partial\kappa_f = (f, f, \mathrm{id}_x)$.

For symmetry, we want an inverse homotopy. Given $\sigma : f \to g$, form the inner horn $(\sigma, \kappa_g, \bullet, \kappa_{\mathrm{id}_x}) : \Lambda^3_2 \to \mathcal{C}$



Extend to $\tau : \Delta^3 \to \mathcal{C}$, and note that $\tilde{\sigma} = d_2\tau$ defines a homotopy $\tilde{\sigma} : g \to f$.

For transitivity, let $f, g, h : x \to y$ with $\sigma_1 : f \to g$ and $\sigma_2 : g \to h$. Form the inner horn $(\sigma_2, \sigma_1, \bullet, \kappa_{\mathrm{id}}) : \Lambda^3_2 \to \mathcal{C}$

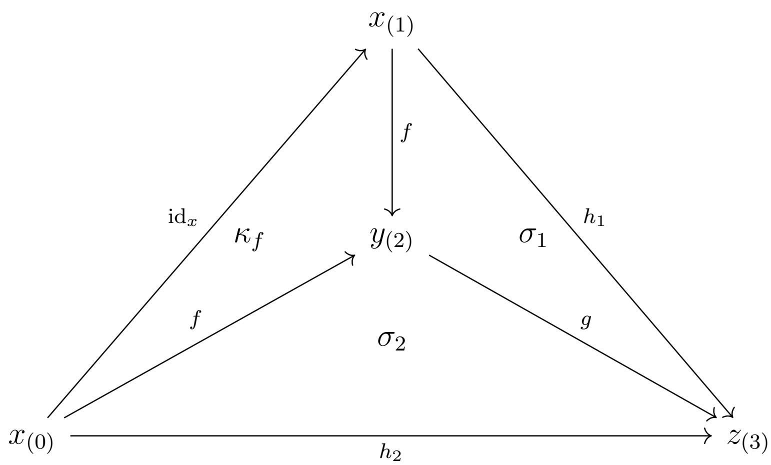


Extend to $\tau : \Delta^3 \to \mathcal{C}$ and note that $\tilde{\sigma} = d_2\tau$ gives us the desired homotopy $f \sim h$. $\square$

Now, we have our key construction.

Proposition 1.18. *Let $\mathcal{C}$ be a quasicategory. The* homotopy category $\mathrm{Ho}(\mathcal{C})$ *is an ordinary category with the same objects as $\mathcal{C}$ and morphisms the homotopy classes of morphisms in $\mathcal{C}$. Composition is $[g] \circ [f] := [g \circ f]$ for any candidate composition $g \circ f$, and $\mathrm{id}_x := [\mathrm{id}_x] = [s_0 x]$.*

Proof. Let $x \xrightarrow{f} y \xrightarrow{g} z$, and let h_1, h_2 be candidate compositions of g and f, with $\sigma_1, \sigma_2 : \Delta^2 \to \mathcal{C}$ giving $h_1 = d_1(\sigma_1)$ and $h_2 = d_2(\sigma_2)$. Form the inner horn $(\sigma_1, \sigma_2, \bullet, \kappa_f) : \Lambda^3_2 \to \mathcal{C}$



Extend to $\tau : \Delta^3 \to \mathcal{C}$, and the new face $d_2\tau : \Delta^2 \to \mathcal{C}$ gives us the desired homotopy $h_2 \to h_1$. □

Remark. There is an alternative, more abstract construction of $\mathrm{Ho}(\mathcal{C})$ using the left Kan extension along the Yoneda embedding, which shows $\mathrm{Ho}(\mathcal{C}) \cong \tau_1(\mathcal{C})$, where τ_1 is the *categorical realization functor* which is the left adjoint to N the nerve functor.

Recalling that we wanted to generalize homotopies to higher morphisms between morphisms, we have the following.

Definition 1.19. In a quasicategory $\mathcal{C}$, an *n-morphism* from $x \to y$ is a map of simplicial sets $\tau : \Delta^{n+1} \to \mathcal{C}$ such that $\tau|_{\Delta^{0,\dots,n}} = x$ and $\tau|_{\Delta^{n+1}} = y$.

There are dual, weakly equivalent notions of the *space of morphisms* $\mathrm{Map}^R_{\mathcal{C}}(x, y)$ and $\mathrm{Map}^L_{\mathcal{C}}(x, y)$ which are simplicial sets (in fact Kan complexes) of the sets of n-morphisms.

Remark. When we proved symmetry of the homotopy relation, we showed that homotopies (2-morphisms) had inverses up to a 3-morphism. In fact, this argument generalizes: n-morphisms have inverses up to $(n+1)$-morphisms as we wanted!

Finally, we have a beautiful result that characterizes quasicategories in terms of a key, long-promised property.

Theorem 1.20. *A simplicial set X is a quasicategory if and only if the restriction map*

$$\mathrm{Fun}(\Delta^2, X) \to \mathrm{Fun}(\Lambda^2_1, X)$$

is an acyclic Kan fibration.

Omitting proofs, as well as the definition of "acyclic Kan fibration" we instead explain the content of the theorem.

If we want to compose $x \xrightarrow{f} y \xrightarrow{g} z$, we consider $\lambda = (g, \bullet, f) : \Lambda^2_1 \to X$, which is the data of a vertex in $\mathrm{Fun}(\Lambda^2_1, X)$. A choice of composition $g \circ f$ is the data of on element in

$\mathrm{Fun}(\Delta^2, X)$ that restricts under i^* to the image of λ. Thus, the space of possible compositions F_λ is the pullback

$$\begin{array}{ccc} F_\lambda & \longrightarrow & \mathrm{Fun}(\Delta^2, X) \\ \downarrow & & \downarrow \\ \Delta^0 & \xrightarrow[\lambda]{} & \mathrm{Fun}(\Lambda^2_1, X) \end{array}$$

The theorem says that for quasicategories, F_λ is contractible. This means that any two choices are homotopic (which we showed), but also that all homotopies and higher homotopies comparing the two are equivalent.

The theorem further says that this property is the defining characteristic of quasicategories.

It is remarkable that we only needed to examine $\Lambda^2_1 \to \Delta^2$ to obtain this result about all higher homotopies!

2 February 7

Slightly more specific seminar outline:

- I will give the first ≈ 6 talks
  - $\approx 2-4$ on infinity categories
  - $\approx 2-4$ building up Derived Algebraic Geometry
- Last ≈ 5 talks (April/May), you are all welcome to give talks
  - Any applications (or vaguely related topics)
  - Let me know if there is background I should cover in advance

Today we will talk about model categories. Although their technical usefulness has perhaps been diminished by the advent of other infinity categorical theories, I have been advised that they are helpful for understanding why *morally speaking* things work.

The main goals for today are to build some comfort and intuition for the flavor of model categorical arguments, and to see some important foundational results of the theory. Most of this exposition is lifted from [2] and [3].

2.1 Definition and Basics of Model Categories

Definition 2.1. $f : A \to B$ is a *retract* of $g : C \to D$ if $\exists$ a diagram:

$$\begin{array}{ccccc} A & \longrightarrow & C & \longrightarrow & A \\ {\scriptstyle f}\downarrow & & {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f} \\ B & \longrightarrow & D & \longrightarrow & B \end{array}$$

with $A \to A$ and $B \to B$ the identity.

This is a retract of objects in $\mathrm{Map}(\mathcal{C})$.

Definition 2.2. A *factorization* is a pair of map $(\alpha, \beta) : \mathrm{Map}(\mathcal{C}) \to \mathrm{Map}(\mathcal{C})$ such that $f = \beta(f) \circ \alpha(f), \forall f \in \mathrm{Map}(\mathcal{C})$.

If α and β are functors, this is a *functorial factorization.*

Definition 2.3. Let $i : A \to B$ and $p : X \to Y$. Then i has the *left lifting property with respect to* p (LLP(p)) and p has the *right lifting property with respect to* i (RLP(i)) if for every diagram

$$\begin{array}{ccc} A & \xrightarrow{f} & X \\ {\scriptstyle i}\downarrow & \overset{\exists h}{\nearrow} & \downarrow{\scriptstyle p} \\ B & \xrightarrow[g]{} & Y \end{array}$$

there exists a lift $h : B \to X$.

Lemma 2.4. *LLP(p) is closed under pushouts and RLP(i) is closed under pullbacks.*

Proof. Adjoin the pullback/pushout square to the lifting problem square, and it is clear that the desired lift follows from the given lift. ☐

Definition 2.5. A *model category* is a co/complete category with the data of a *model structure*:

- Three subcategories: weak equivalences (W), cofibrations (C), and fibrations (F)
- Two (functorial) factorizations (α, β) and (γ, δ)

satisfying

1. (2-out-of-3) If f and g are composable and two out of f, g, $g \circ f$ are in W, then so is the third.
2. (Retracts) If f is a retract of g and $g \in W$ (resp. C or F) then $f \in W$ (resp. C or F).
3. (Lifting) $C \cap W \subseteq \mathrm{LLP}(F)$ and $C \subseteq LLP(F \cap W)$.
   Equivalently, $F \subseteq \mathrm{RLP}(C \cap W)$ and $F \cap W \subseteq C$.
4. (Factorization) $\alpha(f) \in C$, $\beta(f) \in F \cap W$ and $\gamma(f) \in C \cap W$ and $\delta(f) \in F$.

$$\xrightarrow{C}\xrightarrow{F\cap W}, \qquad \xrightarrow{C\cap W}\xrightarrow{F}$$

Remark. Notice the symmetry between C and F. Precisely, $\mathcal{C}^{\mathrm{opp}}$ has a model structure with C and F reversed.

Remark. Some authors (especially Quillen originally) use weaker conditions.

- Only finite co/complete (less technically convenient)
- Non-functorial factorization (in all relevant examples, can be made functorial)

To get a feel for these definitions and the relationship between the conditions, let us prove some lemmas.

Lemma 2.6. *If f is a retract of g and $g \in LLP(\varphi)$ (resp. RLP), then $f \in LLP(\varphi)$ (resp. RLP).*

Proof.

$$\begin{tikzcd} A \arrow[r] \arrow[d, "f"'] & R \arrow[r] \arrow[d, "g"'] & A \arrow[r] \arrow[d, "f"'] & X \arrow[d, "\varphi"] \\ B \arrow[r] \arrow[rrru, dashed] & S \arrow[r] \arrow[rru, dashed] & B \arrow[r] & Y \end{tikzcd}$$

The right-most square is the lifting problem we want to solve, and the left two squares give the retraction. The right two squares are a lifting problem in g which we can solve, so composing with the bottom-left-most map solves the lifting problem for the whole rectangle. But since the horizontal maps of a retract are the identity, the whole rectangle has the same maps as the right-most square!

The case of RLP is solved similarly. □

Lemma 2.7. *In a model category $\mathcal{C}$,*

$$C = LLP(F \cap W), \quad C \cap W = LLP(F)$$

and dually

$$F = RLP(C \cap W), \quad F \cap W = RLP(C)$$

Proof. By the axioms, $C \subseteq \mathrm{LLP}(F \cap W)$.

Let $f : R \to T$ with $f \in \mathrm{LLP}(F \cap W)$. Factor f as $R \xrightarrow{i} S \xrightarrow{p} T$ with $i \in C, p \in F \cap W$. Since $f \in \mathrm{LLP}(p)$:

$$\begin{tikzcd} R \arrow[r, "i"] \arrow[d, "f"'] & S \arrow[d, "p"] \\ T \arrow[r, "\mathrm{id}_T"'] \arrow[ru, dashed, "h"] & T \end{tikzcd}$$

we have h such that $i = h \circ f$ and $\mathrm{id}_T = p \circ h$. Thus,

$$\begin{tikzcd} R \arrow[r, "\mathrm{id}_R"] \arrow[d, "f"'] & R \arrow[r, "\mathrm{id}_R"] \arrow[d, "i"'] & R \arrow[d, "f"] \\ T \arrow[r, "h"'] & S \arrow[r, "p"'] & T \end{tikzcd}$$

shows that f is a retract of i, proving $f \in C$, so $C = \mathrm{LLP}(F \cap W)$.

Similarly, $C \cap W \subseteq \mathrm{LLP}(F)$ by the axioms, and factoring $f \in \mathrm{LLP}(F)$ as $f = p \circ i$ with $i \in C \cap W$ and $p \in F$ we can use the same argument to show f is a retract of i so $f \in C \cap W$ and $C \cap W = \mathrm{LLP}(F)$.

Finally, the dual statements follow immediately from applying what we have shown to $\mathcal{C}^{\mathrm{op}}$. □

Definition 2.8. In a model category $\mathcal{C}$, let $\emptyset$ be the initial object and $*$ be the terminal object. If $\emptyset \to X$ is a cofibration, then X is *cofibrant*. If $Y \to *$ is a fibration, then Y is *fibrant*.

Let $\mathcal{C}^c$ and $\mathcal{C}^f$ denote the full subcategory of cofibrant and fibrant objects respectively, and let $\mathcal{C}^{cf} = \mathcal{C}^c \cap \mathcal{C}^f$.

For any X, we can factor $\emptyset \to X$ through QX such that QX is cofibrant and $q_X : QX \to X$ is a natural trivial fibration (naturality follows from functoriality of the factorization). Similarly, we can find RX such that RX is fibrant and there is $r_X : X \to RX$ is a natural trivial cofibration.

Definition 2.9. The functor $\mathcal{C} \to \mathcal{C}^c$ mapping $X \mapsto QX$ is the *cofibrant replacement functor*, and the functor $\mathcal{C} \to \mathcal{C}^f$ mapping $X \mapsto RX$ is the *fibrant replacement functor*.

2.2 Cofibrantly Generated Model Categories

By the previous lemma, to define a model structure it suffices to define W and C (resp. F), and F (resp. C) is defined as $\text{LLP}(F \cap W)$ (resp. $\text{RLP}(C \cap W)$).

It turns out we can (sometimes) define a model category with even less data. This requires a bit of set up.

Definition 2.10. Let $I \subset \text{Mor}(\mathcal{C})$. An *$I$-diagram* is

$$Z_0 \to Z_1 \to \ldots$$

such that each $Z_k \to Z_{k+1}$ is a pushout of elements of I.

An object $X \in \mathcal{C}$ is *I-small* if for any I-diagram there is a bijection

$$\text{colim}_k \text{Hom}(X, Z_k) \xrightarrow{\simeq} \text{Hom}(X, \text{colim}_k Z)$$

Theorem 2.11. *Let $\mathcal{C}$ be a co/complete category, let $W \subset \mathcal{C}$, and let $I, J \subset Mor(\mathcal{C})$ satisfying*

- *W contains all isomorphisms, satisfies two-out-of-three, and is closed under retracts*
- *$RLP(I) = RLP(J) \cap W$*
- *Domains of I (resp. J) are I-small (resp. J-small)*
- *Colimits of J-diagrams are in $W \cap LLP(RLP(I))$*

Then $\mathcal{C}$ has a model structure with $W = W$, $F = RLP(J)$, and $C = LLP(RLP(I))$.

In such a case, we call $\mathcal{C}$ *cofibrantly generated.*

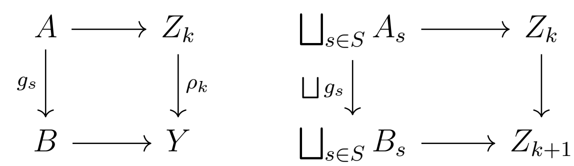
Proof sketch. The retract axiom holds by our lemma that retracts preserve lifting properties. We are given two-out of three. We are given $C \subset \text{LLP}(F \cap W)$, and $C \cap W \subset \text{LLP}(F)$ will follow from the other axioms once we have factorization (see [3] Theorem 2.1.19 for details).

The proof of factorization will follow from the *small object argument*, which we now explain. □

Theorem 2.12 (Small Object Argument)**.** *Suppose $\mathcal{C}$ is cocomplete and $I \subset Mor(\mathcal{C})$ such that domains of maps in I are I-small. Then there is a functorial factorization (γ, δ) on all $f \in \mathcal{C}$ such that $\gamma(f)$ is the colimit of an I-diagram, and $\delta(f) \in RLP(I)$.*

Proof sketch. The proof uses transfinite induction, but we will gloss over the set-theoretical technicalities.

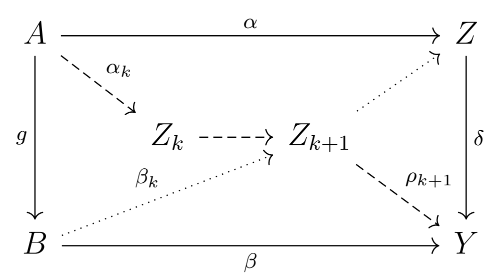
Fix $f : X \to Y$. Let $Z_0 = X$ and $\rho_0 = f$. Assume Z_k and $\rho_k : Z_k \to Y$ are defined. Let S be the set of diagrams of the form below with $g_s \in I$.



Then define Z_{k+1} by the pushout on the right above, and let $\rho_{k+1} : Z_{k+1} \to Y$ be induced by ρ_k and $\coprod_{s\in S} B_s \to Y$.

Let $Z = \text{colim} Z_k$, and let $\gamma : X \to Z$ be the composition of all $Z_k \to Z_{k+1}$. By construction, γ is the colimit of an I-diagram.

Let $\delta : Z \to Y$ be induced as $\text{colim}_k \rho_k$. It remains to show that $\delta \in \text{RLP}(I)$. Let the outer diagram be the lifting problem, with $g \in I$.



By hypothesis, A is I-small, so $\alpha \in \text{Hom}(A, Z) = \text{colim}_k \text{Hom}(A, Z_k)$ corresponds to some $\alpha_k : A \to Z_k$. By construction we have a map $\beta_k : B \to Z_{k+1}$ that commutes with the diagram. Thus, the dotted line solves the lifting problem. □

"Small object" refers to A since maps $A \to Z$ must factor through some Z_k.

Theorem 2.13. **sSet** *has a cofibrantly generated model structure, with* $f : X \to Y$ *in* W *iff* $|f|$ *is a weak homotopy equivalence,* $I = \{\partial\Delta^n \to \Delta^n\}_{n\geq 0}$ *and* $J = \{\Lambda^n_k \to \Delta^n\}_{n\geq 0}$.

Proof Sketch. It suffices to check the four conditions of the Theorem on cofibrantly generated categories.

The conditions on W follow from functoriality of π_n and the adjunction.

- Domains of I and J are I-small and J-small, because $|\partial\Delta^n| \cong S^n$ and $|\Lambda^n_k| \cong D^n$ are compact and therefore factor through a finite filtration, which applies thanks to adjunction.

- Since $||$ is a left adjoint, it preserves colimits. And $|\Lambda^n_k \to \Delta^n| \cong D^n \to D^n \times I$. Given $D^n \to X$ we have $X \to D^n \times [0, 1] \sqcup_{D^n} X$ is the inclusion into the mapping cylinder, so the colimit of a J-diagram is the inclusion into the mapping telescope which is a homotopy equivalence and so in W.

- Tautologically $I \subset \mathrm{LLP}(\mathrm{RLP}(I))$, so pushouts of I are in $\mathrm{LLP}(\mathrm{RLP}(I))$. And since each element of J is the pushout of elements of I, we have $J \subset \mathrm{LLP}(\mathrm{RLP}(I))$. And thus colimits of J-diagrams are in $\mathrm{LLP}(\mathrm{RLP}(I))$.

- It just remains to show that $\mathrm{RLP}(I) = F \cap W$. Again since elements of J are pushouts of elements of I, we have $\mathrm{RLP}(I) \subset \mathrm{RLP}(J) = F$.

  For weak equivalence, it is a fact ([2] Lemma 4.3.2) that $|f|$ is a fibration for $f \in \mathrm{RLP}(I)$, and by the LES on homotopy groups it suffices to show $|f|$ has contractible fibers. It is also a fact ([2] Lemma 4.3.1) that the fiber of $|f|$ is $|F|$, where F is the fiber of f. Since fibers pullback, $F \to \Delta^0$ is a fibration. Since products preserve lifting properties, can solve the following lifting problem:

$$\begin{array}{ccc} F \times \partial\Delta^1 & \xrightarrow{(\mathrm{id}_F, *)} & F \\ \downarrow & \nearrow & \downarrow \\ F \times \Delta^1 & \longrightarrow & \Delta^0 \end{array}$$

  which gives a homotopy between id_F and the constant map as desired. Thus, $\mathrm{RLP}(I) \subseteq F \cap W$.

  It turns out that the reverse inclusion $\mathrm{RLP}(I) \supseteq F \cap W$ is much harder to show, and builds on *Quillen's Theory of Minimal Fibrations*. We forgo an explanation, citing [3] Sections 3.3-3.6.

□

Remark. The fibrant objects of **sSet** are the Kan complexes.

Furthermore,

Proposition 2.14. *A map $f : K \to L$ in* **sSet** *is a cofibration iff it is level-wise injective. Thus, every simplicial set is cofibrant.*

Proof. $\mathrm{LLP}(\mathrm{RLP})(I)$ is precisely generated by I via pushouts, transfinite composition, and retracts (see Proposition 2.3; the proof is a straightforward application of the small object argument/retract argument).

The maps of I are injective, and injections of simplicial sets are closed under pushouts/transfinite composition/retracts, so cofibrations are injective.

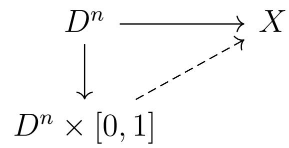
Conversely, if $f : K \to L$ is injective, it is a countable composition of pushouts of coproducts of maps in I. Since these preserve lifting properties, f is a cofibration. □

Theorem 2.15. **Top** *has a cofibrantly generated model structure with W as weak homotopy equivalences, $I = \{S^{n-1} \to D^n\}_{n \geq 0}$ and $J = \{D^n \to (D^n \times [0,1])\}_{n \geq 0}$. Then $F = \{$Serre fibrations$\}$ and $C = LLP(F \cap W)$.*

We omit the proof since it is similar to the case of **sSet**.

Corollary 2.16. *Every topological space is fibrant.*

Proof. Since $D^n \to D^n \times [0,1]$ has a section (projection) the lifting problem is trivial.



□

3 February 14

3.1 Model Structures on Chain Complexes of Modules

As a refresher on what model structures are, we define a couple of them on the category of chain complexes of modules. This is a particularly important example, as our later work today will define the derived category of modules, and really of any abelian category. I won't prove much, but there are some important high-level ideas here.

Proposition 3.1. *Let $\mathcal{C}$ be the category of complexes of R-modules in non-negative degree. Then there is a model structure on $\mathcal{C}$ defined by*

- *W: quasi-isomorphisms*
- *F: surjective maps (degree-wise)*
- *C: LLP($F \cap W$) (injective maps with projective cokernel, degreewise)*

Proof reference. See [2] 3.2.4 for details. □

Corollary 3.2. *In this model structure, all objects are fibrant. And cofibrant objects are precisely the complexes consisting of projective modules.*

Proof idea. Clearly all complexes surject onto the terminal module.

Recall that an R-module P is projective iff it $\emptyset \to P$ is in LLP(surjections). Using boundedness and quasi-isomorphism, we can lift degree-wise (might be good to do this in detail at some point). □

Remark. For unbounded complexes, cofibrant implies each degree is a projective R-module. However, the converse does not hold in general.

Remark. Recall that cofibrant replacement gave a weak-equivalence with a cofibrant object. In this case, a cofibrant replacement is a *projective resolution*.

Note that for topological spaces, CW complexes are cofibrant objects,and every topological space is weakly equivalent to a CW complex.

Recall that Blumberg said "projective resolutions are secretly CW complexes". Maybe this is what he had in mind.

Dually, there is another model structure

Proposition 3.3. *Let $\mathcal{C}$ be the category of complexes of R-modules in non-negative degree. Then there is a model structure on $\mathcal{C}$ defined by*

- *W: quasi-isomorphisms*
- *C: injective maps (degree-wise)*
- *F: RLP($C \cap W$) (surjective maps with injective kernel, degreewise)*

Corollary 3.4. *In this model structure, all objects are cofibrant. And fibrant objects are precisely the complexes consisting of injective modules.*

Remark. Fibrant replacement is an *injective resolution.*

Remark. We could also have defined the projective model structure as cofibrantly generated with W quasi-isomorphisms, $I = \{S^{n-1} \to D^n\}$ and $J = \{0 \to D^n\}$. Here S^n as the complex with R in degree n and 0 elsewhere, and D^n is $R \xrightarrow{\text{id}} R$ in degrees n and $n-1$ and 0 elsewhere.

In fact, this model structure is inherited directly from the Joyal model structure on **sSet** via the Dold-Kan correspondence.

3.2 The Homotopy Category of a Model Category

Definition 3.5. For $X \in \mathcal{C}$, a *cylinder* is a diagram

$$X \sqcup X \xrightarrow{\in C} C_X \xrightarrow{\in F \cap W} X$$

If $f_0, f_1 : X \to Y$ induce $X \sqcup X \to Y$ that factors through C_X, then f_0 and f_1 and are *left homotopic.*

Example. If $\mathcal{C} = \textbf{Top}$ with the usual model structure, then we can take cylinder object

$$X \sqcup X \xrightarrow{(i_0,i_1)} [0,1] \times X \xrightarrow{\pi_X} X$$

and the factor map $[0,1] \times X \to Y$ is our usual notion of homotopy.

Of course, there is a dual notion.

Definition 3.6. For $Y \in \mathcal{C}$, a *path object* is a diagram

$$Y \xrightarrow{\in C \cap W} P_Y \xrightarrow{\in F} Y \times Y$$

If $f_0, f_1 : X \to Y$ induce $X \to Y \times Y$ that factors through C_X, then f_0 and f_1 are *right homotopic.*

Example. If $\mathcal{C} = \textbf{Top}$ (convenient category) with the usual model structure, then we can take path object

$$Y \xrightarrow{y \mapsto \{[0,1] \mapsto y\}} Y^{[0,1]} \xrightarrow{p \mapsto (p(0),p(1))} Y \times Y$$

and the factor map $X \to Y^{[0,1]}$ is the adjoint of the standard notion of homotopy.

Lemma 3.7. *If X is cofibrant, left homotopy is an equivalence relation on maps $X \to Y$. If Y is fibrant, right homotopy is an equivalence relation for $X \to Y$.*

Proof sketch. Reflexivity and symmetry hold in general, need the cofibrant condition for transitivity. See [3] Proposition 1.2.5 for details. □

Remark. By duality, if X is cofibrant and Y is fibrant, left and right homotopy coincide.

Definition 3.8. Let $\mathcal{C}$ be a category, and $W \subset \mathrm{Mor}(\mathcal{C})$. A localization of $\mathcal{C}$ with respect to W is a functor $F : \mathcal{C} \to \mathcal{D}$ carrying all arrows of W to isomorphisms, such that

- If $f' : \mathcal{C} \to \mathcal{D}'$ carries W to isomorphisms, then $\exists(g, \theta)$ with $g : \mathcal{D} \to \mathcal{D}'$ and natural isomorphism $\theta : f' \xrightarrow{\simeq} g \circ f$.
- If (g', θ') is another such pair, $\exists!$ natural isomorphism $\alpha : g \to g'$ commuting with θ and θ'.

Remark. This exists, as we can form the free category $F(C, \mathcal{W}^{-1})$ and then impose the relations

$$\mathrm{id}_A = (1_A), \quad (f, g) = (g \circ f), \quad \mathrm{id}_{\mathrm{dom}(w)} = (w, w^{-1}), \quad \mathrm{id}_{\mathrm{codom}(w)} = (w^{-1}, w)$$

See [3] Lemma 1.2.2 to check that this satisfies the universal property.

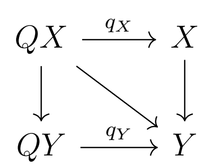
Definition 3.9. The homotopy category $\mathrm{Ho}(\mathcal{C})$ of a model category $\mathcal{C}$ is the localization at W.

Proposition 3.10. *The inclusion functors induce equivalences of categories*

$$Ho(\mathcal{C}^{cf}) \to Ho(\mathcal{C}^*) \to Ho(\mathcal{C})$$

(for $* = c, f$).

Proof. We will show that Q and R induce inverse functors. If $X \to Y$ is a weak equivalence, then by two-out-of-three, $QX \to Y$ is a weak equivalence and $QX \to QY$ is a weak equivalence.



Thus, Q induces functors $\mathrm{Ho}(\mathcal{C}^c) \to \mathrm{Ho}(\mathcal{C})$ and likewise for R.

And q_X gives natural weak equivalences $q \circ i \to 1_{\mathcal{C}^c}$ and $i \circ Q \to 1_{\mathcal{C}}$, which descend to natural isomorphisms on the homotopy categories. Thus, $\mathrm{Ho}(Q)$ and $\mathrm{Ho}(i)$ are inverse equivalences of categories, and likewise for $\mathrm{Ho}(R)$. □

The great thing (we will see later why this is so useful) is that the fibrant/cofibrant homotopy categories have a description in terms of actual homotopy.

Theorem 3.11 (Whithead). *In $\mathcal{C}^{cf}$, weak equivalences are precisely the homotopy equivalences.*

Proof idea. Entirely formal, using axioms and definitions. See Theorem 3.2. □

This is easy to show, but much harder to prove in the case of topological spaces. Reason: all the heavy lifting went into proving the model structure for **Top**.

Proposition 3.12. *There is a unique isomorphism of categories $\mathcal{C}^{cf}/\sim \to Ho(\mathcal{C}^{cf})$, and it is the identity on objects.*

Proof idea. Go through and check that $\mathcal{C}^{cf}$ satisfies the universal property of $\mathrm{Ho}(\mathcal{C}^{cf})$.

See [3] Corollary 1.2.9 for details. □

Remark. For bounded chain complexes with the projective/injective model structure, $\mathcal{C}^{cf}$ are projective/injective complexes.

3.3 Quillen Adjunction and Derived Functors

Definition 3.13. $F : \mathcal{C} \leftrightarrow \mathcal{D} : G$ adjoint functors between model categories form a *Quillen adjunction* if

- F preserves cofibrations and trivial cofibrations
- G preserves fibrations and trivial fibrations

Remark. The two bullet points above are equivalent, because $F(f) \in \mathrm{LLP}(g) \iff f \in \mathrm{LLP}(G(g))$ by naturality of the adjunction.

$$\begin{array}{ccc} F(A) & \longrightarrow & X \\ {\scriptstyle F(f)}\downarrow & \nearrow & \downarrow{\scriptstyle g} \\ F(B) & \longrightarrow & Y \end{array} \qquad \begin{array}{ccc} A & \longrightarrow & G(X) \\ {\scriptstyle f}\downarrow & \nearrow & \downarrow{\scriptstyle G(g)} \\ B & \longrightarrow & G(Y) \end{array}$$

Theorem 3.14. *A Quillen adjunction $F : \mathcal{C} \leftrightarrow \mathcal{D} : G$ induces an adjunction*

$$LF : Ho(\mathcal{C}) \leftrightarrow Ho(\mathcal{D}) : RG$$

Remark. The functors LF and RG satisfy a universal property that makes them derived functors. Given a functor $F : \mathcal{C} \to \mathcal{C}'$, the left derived functor $LF : \mathcal{C}[W^{-1}] \to \mathcal{C}'[W^{-1}]$ is part of the universal pair (LF, α) such that in the following diagram

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{F} & \mathcal{C}' \\ {\scriptstyle Q}\downarrow & & \downarrow{\scriptstyle Q} \\ \mathcal{C}[W^{-1}] & \xrightarrow[LF]{} & \mathcal{C}'[W'^{-1}] \end{array}$$

there is a natural transformation $\alpha : LF \circ Q \to Q \circ f$. For the right derived functor, there is a natural transformation $Q \circ F \to RF \circ Q$. See [2] 3.4.1 for more.

Lemma 3.15. *F preserves weak equivalence of cofibrant objects, G preserves weak equivalence of fibrant objects.*

Proof. Let $A, B \in \mathcal{C}^c$ and $f : A \to B$ with $f \in W_{\mathcal{C}}$. Factor $A \sqcup B \xrightarrow{(f,\mathrm{id})}$ as $A \sqcup B \xrightarrow{p} C \xrightarrow{q} B$ with $p \in C$, $q \in F \cap W$.

Since $\{\} \to A$ and $\{\} \to B$ are in C, their pushouts $i : A \to A \cup B$ and $j : B \to A \cup B$ are in C; and $p \in C$ so $p \circ i, p \circ j \in C$.

Since $q \circ p \circ i = f$ and $q \circ p \circ j = \mathrm{id}_B$ are in W, and $q \in W$, so $p \circ i, p \circ j \in C \cap W$ by two-out-of-three.

By Quillen condition, $F(p \circ i), F(p \circ j) \in C \cap W$.

Since $F(q) \circ F(p \circ j) = F(q \circ p \circ j) = F(\mathrm{id}_B) = \mathrm{id}_{F(B)} \in W$, so $F(q) \in W$.

Thus, $F(f) = F(q \circ p \circ i) = F(q) \circ F(p \circ i) \in W$.

And G preserving weak equivalences of fibrant objects follows dually. □

Proof of Theorem. By the lemma, F descends to a well-defined functor $\mathrm{Ho}(\mathcal{C}^c) \to \mathrm{Ho}(\mathcal{D})$ and G descends to a well-defined functor $\mathrm{Ho}(\mathcal{D}^f) \to \mathrm{Ho}(\mathcal{C})$. But by the equivalence of homotopy categories, we have well-defined functors F, G between $\mathrm{Ho}(\mathcal{C})$ and $\mathrm{Ho}(\mathcal{D})$.

For $X \in \mathcal{C}^c$ and $Y \in \mathcal{D}^f$ we want a natural isomorphism

$$\varphi : \mathrm{Hom}_{\mathrm{Ho}(\mathcal{D})}(FX, Y) \leftrightarrow \mathrm{Hom}_{\mathrm{Ho}(\mathcal{C})}(X, GY)$$

But we have natural isomorphisms

$$\mathrm{Hom}_{\mathrm{Ho}(\mathcal{D})}(FX, Y) \leftrightarrow \mathrm{Hom}_{\mathcal{D}}(FX, Y)/\sim, \quad \mathrm{Hom}_{\mathrm{Ho}(\mathcal{C})}(X, GY) \leftrightarrow \mathrm{Hom}_{\mathcal{C}}(X, GY)/\sim$$

where $\sim$ is left/right homotopy.

Thus, we just need to verify that φ respects homotopy relations. Suppose $f, g : FX \to Y$ are homotopic, with path object Y' and right homotopy $H : FX \to Y'$. Since G preserves products/fibrations/$W_{\mathcal{D}^f}$ we have $G(Y')$ is a path object for $G(Y)$. Thus, $\varphi(H) : X \to G(Y')$ is a right homotopy from φf to φg.

Conversely, let φf and φg be homotopy, with cylinder object X' for X and left homotopy $H : X' \to G(Y)$. Then since F preserves coproducts/cofibrations/$W_{\mathcal{C}^c}$ we have $F(X')$ a path object for $F(X)$, so $\varphi^{-1}(H) : FX' \to Y$ a left homotopy from f to g. □

3.4 Homotopy (co)limits

This section is a bit of a tangent, but is an important application of Quillen adjunctions. We will sketch out the main ideas.

Definition 3.16. Let $\mathcal{C}$ a model category and D a small category (think diagram). The *projective model structure* on the functor category $\mathcal{C}^D$ (if it exists) has

- W is object-wise weak equivalences in $\mathcal{C}$
- F is object-wise fibrations in $\mathcal{C}$

and the *injective model structure* on $\mathcal{C}^D$ (if it exists) has

- W is object-wise weak equivalences in $\mathcal{C}$
- C is object-wise cofibrations in $\mathcal{C}$

Lemma 3.17. *For $\mathcal{C}$ a cofibrantly generated model category and D a small category, $\mathcal{C}^D$ admits a projective model structure.*

Proposition 3.18. *Let $\mathcal{C}^D$ have a well-defined projective model structure. Then the colimit functor $(\mathcal{C}^D)_{proj} \to \mathcal{C}$ is a left Quillen functor. Dually, if $\mathcal{C}^D$ has a well-defined injective model structure, then the limit functor $(\mathcal{C}^D)_{inj} \to \mathcal{C}$ is a right Quillen functor.*

Proof. Recall that the colimit functor is left adjoint to the diagonal functor $\Delta : \mathcal{C} \to \mathcal{C}^D$. Thus, it suffices to show that Δ is a right Quillen functor; this is immediate since both F and W are define point-wise.

The dual statement follows dually. □

Definition 3.19. If $\mathcal{C}^D$ has the projective model structure, then the *homotopy colimit* functor is the total left derived functor of colimit. Dually, if $\mathcal{C}^D$ has the injective model structure, the *homotopy limit* is the total right derived functor of limit.

Example. Constructions such as homotopy fiber and homotopy cofiber.

Theorem 3.20. *Homotopy (co)limits exist for all model categories.*

Proof reference. Uses additional theory of homotopical categories, as well as Reedy model structures on functor categories.

See here for details. □

3.5 Quillen Equivalence

Definition 3.21. A Quillen adjunction $F : \mathcal{C} \leftrightarrow \mathcal{D} : G$ is a *Quillen equivalence* if for any $X \in \mathcal{C}^c$ and $Y \in \mathcal{D}^f$,

$$\{a : X \to G(Y)\} \in W \iff \{a' : F(x) \to Y\} \in W$$

A Quillen equivalence is NOT an equivalence of categories. However:

Proposition 3.22. *A Quillen adjunction $F : \mathcal{C} \leftrightarrow \mathcal{D} : G$ is a Quillen equivalence if and only if*

$$LF : Ho(\mathcal{C}) \leftrightarrow Ho(\mathcal{D}) : RG$$

is an equivalence.

Proof. The latter is equivalent to the unit and counit being isomorphisms in the homotopy category, which is equivalent to their being weak equivalences.

Consider the unit $u : \mathrm{id}_{\mathcal{C}} \to RG \circ LF$. For any $C \in \mathcal{C}^f$, we can explicitly write $(RG \circ LF)(C) = G(D)$ for $D \in \mathcal{D}^f$ and $\{F(C) \to D\} \in W_{\mathcal{D}}$.

Thus, u_C is a weak equivalence iff $\{F(C) \to D\} \in W_{\mathcal{D}} \implies \{C \to G(D)\} \in W_{\mathcal{C}}$. Likewise, for v the counit, v_D is a weak equivalence iff the converse holds. These two conditions are the definition of a Quillen equivalence. □

Proposition 3.23. *The identity functor induces a Quillen equivalence between the projective and injective model structures on $Ch_+(R)$.*

Proof. Obviously the identity is an adjunction. Since cofibrations in the projective model structure and a subset of injections, which are cofibrations in the injective model structure, this is a Quillen adjunction. And it preserves weak equivalences, since the are defined the same way in both structures, so it is a Quillen equivalence. □

Remark. Any map $f : R \to R'$ induces a Quillen adjunction between $\mathrm{Ch}_+(R)$ and $\mathrm{Ch}_+(R')$ via induction and restriction, and this is a Quillen equivalence iff f is an isomorphism.

See [3] end of Section 2.3 for more.

Proposition 3.24. *Suppose $F : \mathcal{C} \leftrightarrow \mathcal{D} : G$ is an adjunction between model categories, with $\mathcal{C}$ cofibrantly generated by I, J. Then F, G form a Quillen adjunction iff $F(I)$ are cofibrations and $F(J)$ are trivial cofibrations.*

Proof reference. See [3] Lemma 2.1.20; straightforward application of the axioms and adjunction. □

Theorem 3.25. *There is a Quillen equivalence*

$$|| : \mathbf{sSet} \leftrightarrow \mathbf{Top} : Sing$$

Proof. To see that we have a Quillen adjunction, it suffices to see that

$$|I| = \{|\partial\Delta^n \to \Delta^n|\} = \{S^n \to D^n\}$$

is a cofibration, and

$$|J| = \{|\Lambda^n_k \to \Delta^n|\} = \{D^n \to D^n \times [0, 1]\}$$

is a trivial cofibration.

To prove Quillen equivalence, let S be a simplicial set and X a topological space. Then S is automatically cofibrant and X is automatically fibrant. Thus, we just need to prove

$$\{|S| \to X\} \in W \iff \{S \to \mathrm{Sing}(X)\} \in W$$

The latter is equivalent to $\{|S| \to |\mathrm{Sing}(X)|\} \in W$, so it suffices to show that $|\mathrm{Sing}(X)| \to |X|$ is a weak homotopy equivalence, and a homotopy equivalence can be checked explicitly, such as by the first answer here. □

4 February 28

So far, quasicategories are our only model for infinity categories. Today we will see other structures that manage higher homotopical data and realize the concept of infinity categories. These will tie in with our work on model theory, and will set up the Grothendieck construction and Yoneda lemma for next time — which enable all manner of categorical constructions.

We will omit most proofs. Anyway, they are formal consequences of model structures and/or simplicial technology, and resemble proofs we've seen before.

4.1 Simplicial Categories

Recall, given a category $\mathcal{C}$, the nerve $N(\mathcal{C}) \in \mathbf{sSet}$ was defined by $N(\mathcal{C})_n = \{[n] \to \mathcal{C}\}$ (or, equivalently, n-simplices are n-compositions).

And recall the homotopy category of a quasi-category X was defined by

$$\mathrm{Hom}_{\mathrm{Ho}(X)}(x, y) = \pi_0(\mathrm{Hom}^R_X(x, y))$$

Now, we show an important result which we mentioned but brushed over last semester.

Theorem 4.1. *Let h be the left adjoint of the nerve functor:*

$$h : \mathbf{sSet} \leftrightarrow \mathbf{Cat} : N$$

Then for any quasi-category X, canonically $h(X) \cong Ho(X)$.

Proof idea. If X is a quasi-category, a map $X \to N(\mathrm{Ho}(X))$ consists of

- a map $f : X_0 \to \mathrm{Ob}(\mathrm{Ho}(X))$
- for $a \in X_1$ an assignment $f(a) \in \mathrm{Hom}_C(f(d_1 a), f(d_0 a))$
- for $\alpha \in X_2$ a composition whose face maps give relations between $f(a)$, $f(a')$

and these relations are precisely homotopy equivalences. □

We will see a beefed up version of this adjunction when it comes to simplicial categories.

Definition 4.2. Let **sCat** be the category of simplicially enriched categories.

Note that given $\mathcal{C} \in \mathbf{sCat}$, we can define $\pi_0(\mathbf{sCat})$ as the conventional category with $\mathrm{Ob}(\pi_0(\mathcal{C})) = \mathrm{Ob}(\mathcal{C})$ and $\mathrm{Hom}_{\pi_0(\mathcal{C})}(X, Y) = \pi_0(\mathrm{Map}_{\mathcal{C}}(X, Y))$.

Whereas our earlier adjunction lost information between **sSet** and **Cat**, our next adjunction will preserve information between **sSet** and **sCat**.

Definition 4.3. Let $\mathfrak{C}^{[n]}$ be the simplicial category with objects $[n]$ and mapping spaces $\mathrm{Map}_{\mathfrak{C}^{[n]}}(i, j) := N(P_{i,j})$ where

$$P_{i,j} = \begin{cases} \emptyset & i > j \\ \text{Poset of subsets of } \{i, i+1, ..., j\} \text{ that contain } \{i, j\} & i \leq j \end{cases}$$

with composition

$$\mathrm{Map}_{\mathfrak{C}^{[n]}}(i, j) \times \mathrm{Map}_{\mathfrak{C}^{[n]}}(j, k) \to \mathrm{Map}_{\mathfrak{C}^{[n]}}(j, k)$$

given by taking the union.

The point is that we are just repackaging simplicial sets as simplicial categories. This gives us functors.

Definition 4.4. Let $\mathfrak{C} : \mathbf{sSet} \to \mathbf{sCat}$ be defined by $\mathfrak{C}(\Delta^n) = \mathfrak{C}^n$, and extended by colimits.

The *homotopy coherent nerve* functor $\mathfrak{N} : \mathbf{sCat} \to \mathbf{sSet}$ is defined by $\mathfrak{N}(\mathcal{C})_n = \mathrm{Hom}_{\mathbf{sCat}}(\mathfrak{C}^n, \mathcal{C})$.

Proposition 4.5. *There is an adjunction*

$$\mathfrak{C} : \mathbf{sSet} \leftrightarrow \mathbf{sCat} : \mathfrak{N}$$

We will see later that this is more than just an adjunction, wink wink.

Proposition 4.6. **sCat** *has a model structure defined as follows. A map (simplicial functor) $f : \mathcal{C} \to \mathcal{D}$ is*

- *a weak equivalence if*
  - *f induces a weak homotopy equivalence $Map_{\mathcal{C}}(x, y) \to Map_{\mathcal{D}}(fx, fy)$ for all $x, y \in \mathcal{C}$. This is a* Dwyer-Kan equivalence*.*
  - *f is essentially surjective*
- *a fibration if*
  - *f induces a Kan fibration (i.e., $RLP(\Lambda^n_k \to \Delta_n)$) $Map_{\mathcal{C}}(x, y) \to Map_{\mathcal{D}}(fx, fy)$ for all $x, y \in \mathcal{C}$*
  - *if $\alpha : f(c) \to d$ is an equivalence in $\mathcal{D}$ (i.e., an isomorphism in $Ho(\mathcal{D})$), then there is a lift $a : c \to c'$ with $c = f(c')$, $\alpha = f(a)$*

Remark. The main content is in the first conditions. In particularly, if we fix a set of objects $\mathcal{O}$ and have f be the identity on objects, the latter conditions follow the former. This generality is enough for DK localization.

This model structure is cofibrantly generated. I won't bother giving or proving the details of I and J.

With this model structure, we have the following result

Theorem 4.7. *There exists a model structure on* **sSet** *(the* Joyal model structure*) with*

- *Cofibrations are injective maps*
- *Weak equivalences are maps carried by $\mathfrak{C}$ to DK equivalences of simplicial categories*
- *Fibrant objects are precisely the quasi-categories (RLP of inner horns)*

And the adjoint pair $(\mathfrak{C}, \mathfrak{N})$ is a Quillen equivalence.

This is a difficult theorem! But it is valuable: recall that this gives an equivalence of homotopy categories.

4.2 DK Localization

Now we will discuss derived functors in the setting of simplicial categories.

As before, we start with a naive notion of localization: for simplicial category $\mathcal{C}$ with subcategory W, we can define $\mathcal{C}[W^{-1}]$ as the simplicial category with n-cimplicies $\mathcal{C}_n[W_n^{-1}]$. DK localization will be the left derived version of this.

Recall that this satisfies a universal property, and can be computed as follows

$$\begin{array}{ccccc} \tilde{W} & \xrightarrow{\tilde{f}} & \tilde{\mathcal{C}} & \xrightarrow{\text{localization}} & L(\mathcal{C}, W) \\ \downarrow p & & \downarrow q & & \\ W & \xrightarrow{f} & C & & \end{array}$$

Let $\tilde{W}$ be cofibrant, p a trivial fibration, $\tilde{f}$ a cofibration and q a trivial fibration. Then let $L(\mathcal{C}, W) = \tilde{\mathcal{C}}[W^{-1}]$.

There is also an explicit construction $L^H(\mathcal{C}, W)$ called the *Hammock version* of DK localization. This is in [2] 5.4.4, and was covered by Emily last semester. I don't expect you to remember it from then, but I also don't expect it to make any sense now, so I'll omit the details. Here are some key facts about DK localization.

Proposition 4.8.

- *Any arrow in W becomes an equivalence in $L^H(\mathcal{C}, W)$*
- $\pi_0 L^H(\mathcal{C}, W) = \mathcal{C}[W^{-1}]$ *(as usual, the homotopy category is naive localization)*
- *If $\mathcal{C}$ is a simplicial category with model structure, then the homotopy category of $Ho(L^H(\mathcal{C}, W)) = Ho(\mathcal{C})$ (with the RHS taken as a model category).*
- *Quillen equivalence of model categories gives rise to an equivalence of DK localizations.*

Thus, DK localization gives the "infinity category underlying a model category."

That is, assuming we think of simplicial categories as infinity categories. If we want to stick with quasi-categories, then the underlying infinity category is

$$\mathfrak{N}(L^H(\mathcal{C}, W)^f)$$

4.3 Segal Spaces

Let's motivate Segal spaces. Recall:

Proposition 4.9. $N : \mathbf{Cat} \to \mathbf{sSet}$ *is fully faithful, with X in the essential image iff either of the following two equivalent properties hold*

- *Each map*
$$X_n \to X_1 \times_{X_0} X_1 \times_{X_0} \ldots \times_{X_0} X_1$$
(induced by $Sp(n) \to \Delta^n$) is bijective.
- *Each map $X_n \to Hom(\Lambda_i^n, X)$ for $1 \leq i \leq n$ (induced by $\Lambda_i^n \to \Delta^n$) is bijective.*

The first one follows since the nerve is defined by 1-morphisms, and the second because composition is unique.

To define quasi-categories, we took the latter definition and replaced "bijective" with "surjective"; to define segal spaces, we will take the former definition and replace "bijective" with "weak-equivalence".

We need some setup to make this work.

Definition 4.10. Let **ssSet** be the category of bisimplicial sets $X_{\bullet\bullet} \in \mathrm{Fun}(\Delta^{\mathrm{opp}} \times \Delta^{\mathrm{opp}}, \mathbf{Set})$.

This definition seems symmetric in the two coordinates, but our interpretation will break this symmetry. In particular, we will think of $X \in \mathbf{ssSet}$, $X_n = X_{n,\bullet}$ as a simplicial object in **sSet** (i.e., a simplicial object in spaces).

Example. Consider the two projections $\Delta \times \Delta \to \Delta$, and let $c, d : \mathbf{sSet} \to \mathbf{ssSet}$ be the induced maps. For $X \in \mathbf{sSet}$, note that $c(X)_n = X$ ("constant") while $d(X)_n = X_n$ ("discrete").

Example. Denote $\Delta^{m,n} = d(\Delta^m) \times c(\Delta^n)$ and note that

$$X_{m,n} = \mathrm{Hom}(\Delta^{m,n})$$

Thus, $\Delta^{m,n}$ is a presheaf on $\Delta \times \Delta$ represented by the pair $([m], [n])$.

Since **ssSet** is a presheaf category, it has internal hom.

Lemma 4.11. *The direct product in* **ssSet** *has a right adjoint*

$$Fun(X, Y)_{m,n} = Hom(X \times \Delta^{m,n}, Y)$$

Lemma 4.12. *The category* **ssSet** *is simplicially enriched via*

$$Map(X, Y)_n = Fun(X, Y)_{0,n} = Hom(X \times c(\Delta^n), Y)$$

Again, formally, there is another enrichment symmetrically, but this is the one of interest to us.

Note that $X_n = \mathrm{Map}(d(\Delta^n), X)$ and for any $S \in \mathbf{sSet}$ we can write $X(S) := \mathrm{Map}(d(S), X)$.

To get a model structure, let's talk about about model structures generally on functor categories.

Recall that we earlier defined "projective" and "injective" model structures on $\mathrm{Fun}(I, M)$, with weak equivalences and fibrations/cofibrations determined point-wise. But recall that these only worked for certain M. Now we define another model structure that works for all M but only some I.

Theorem 4.13 (Reedy Model Structure)**.** *The category* $\mathbf{ssSet} = Fun(\Delta^{op}, \mathbf{Set})$ *has a model structure defined as follows. For a map* $f : X \to Y$*,*

- $f \in W$ *iff* $f_n : X_n \to Y_n$ *is a weak equivalence of* **sSet**
- $f \in C$ *iff* $f_n : X_n \to Y_n$ *is a cofibration in* **sSet** *(i.e, an injection)*

- *$f \in F$ is a (resp. trivial) fibration iff $X_0 \to Y_0$ is a (resp. trivial) Kan fibration, and for each $n > 0$*
$$X_n \to Y_n \times_{Y(\partial\Delta^n)} X(\partial\Delta^n)$$
is a (resp. trivial) Kan fibration.

This model structure has an important nice formal property.

Definition 4.14. A model category M is *left proper* if all pushouts of weak equivalences along cofibrations are weak equivalences, is *right proper* if all pullbacks of weak equivalences along fibrations are weak equivalences, and is *proper* if it is both left and right proper.

Note: if all objects of M are cofibrant, then M is left proper. If all objects of M are fibrant, then M is right proper.

Lemma 4.15. *The Reedy model structure on* **ssSet** *is proper.*

Now we can express the intuition we gave earlier.

Definition 4.16. $X \in$ **ssSet** is a *Segal space* if

- X is Reedy fibrant
- $X_n \to X_1 \times_{X_0} X_1 \times_{X_0} ... \times_{X_0} X_1$ is a weak equivalence

In fact, the first condition implies that the latter map is a (trivial) Kan fibration.

Now we see how we can (almost) interpret Segal spaces as infinity categories. For a Segal space X, let

- The set of *objects* of X be $X_{0,0}$.
- For $x, y \in X_{00}$, let $\mathrm{Map}(x, y)$ be the fiber at (x, y) of
$$X_1 \to X_0 \times X_0$$
(so it is an element of X_{10}). Since this map is a fibration by definition, and fibrations pull back, the fiber is a Kan simplicial set.
- Define composition by
$$X_1 \times_{X_0} X_1 \to X_1 \xrightarrow{d_1} X_1$$
where the first map is a section of $X_2 \to X_1 \times_{X_0} X_1$, which exists since it is a trivial fibration. This is not unique.
- The homotopy category of X is defined by
$$\mathrm{Hom}_{\mathrm{Ho}(X)}(x, y) = \pi_0(\mathrm{Map}_X(x, y))$$
Check that composition is unique and associative.

Now we have yet another related notion of DK equivalence:

Definition 4.17. A map $f : X \to Y$ of Segal spaces is a *DK equivalence* if

- $\mathrm{Map}_X(x, y) \to \mathrm{Map}_Y(fx, fy)$ is an equivalence $\forall x, y \in X$
- $\forall y \in Y \exists x \in X$ and $\exists$ an equivalence $f(x) \to y$.

Proposition 4.18. *Reedy equivalence (i.e., isomorphism in the homotopy category under Reedy model structure) implies DK equivalence.*

However, the converse does not hold!

Definition 4.19. Let $\mathcal{C}$ be a category. Construct a simplicial object in simplicial sets (bisimplicial set)

$$n \mapsto N((\mathcal{C}^{[n]})^{\mathrm{iso}})$$

where $\mathcal{C}^{[0]}$ is the maximal subgroupoid of $\mathcal{C}$ and $\mathcal{C}^{[n]}$ is the arrow category of $\mathcal{C}^{[n-1]}$. This is the *Rezk nerve* denoted $B(\mathcal{C})$.

Example. Let $\mathcal{C}$ be a category. Then the natural map $d(N(C)) \to B(C)$ is a DK equivalence but not a Reedy equivalence.

But we would like it to.

To this end, we build up the definition of a complete Segal space.

Pushing simplices around yields

Lemma 4.20. *If f, g belong to the same connected component of X_1 and f is an equivalence, then g is an equivalence.*

Definition 4.21. Given a Segal space X, the *space of equivalences X^{eq}* is the subspace of X_1 spanned by the equivalences.

Note that the degeneracy $s_0 : X_0 \to X_1$ carries every object to an equivalence, so it factors through $s : X_0 \to X^{eq}$.

Definition 4.22. A Segal space X is complete if s is an equivalence.

Now we see how restricting to complete simplicial spaces solves the issue we encountered above.

Proposition 4.23. *For complete Siegel spaces, Reedy equivalence and DK equivalence coincide.*

5 March 7

Since we may have guests, here is some quick context. In topology, we care about spaces up to homotopy equivalence, but just quotienting by homotopies throws away too much structure, so we don't have things like (co)limits. One approach is to record the homotopies giving equivalences, record the homotopy between homotopies, etc. Infinity categories encapsulate the notion of higher n-arrows. There are many models of this principle: quasi-categories, simplicially enriched categories, and complete segal spaces. We took a detour through model category theory, which encapsulates homotopical data and gave us derived categories/functors. We also saw DK localization, the underlying infinity categorical notion. Today, we put all this setup to work.

5.1 Motivating the ∞-Categorical Yoneda Embedding

The key result for today is the infinity categorical Yoneda lemma. Once we have it, we can define various key universal constructions: limits, colimits, adjoints.

Recall the conventional (full and faithful) Yoneda embedding:

$$\mathcal{C} \to \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathbf{Set}), \quad B \mapsto \mathrm{Hom}_{\mathcal{C}}(-, B)$$

or equivalently,

$$\mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathbf{Set}, \quad (A, B) \mapsto \mathrm{Hom}_{\mathcal{C}}(A, B)$$

To adapt this construction to infinity categories, we need a CSS that will play the part of **Set**, which we cal $\mathcal{S}$ (will also need an analogue of op).

Constructing a functor

$$\mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{S}$$

will require a lot of coherence data, so we seek to do it non-explicitly. In particular, we will construct $\mathcal{S}$ as a classifying space for a certain kind of functor called a *left fibration*:

$$\{X \xrightarrow{\text{left fibration}} \mathcal{C}^{\mathrm{op}} \times \mathcal{C}\} \leftrightarrow \{\mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{S}\}$$

Note: from now on $\mathcal{C}$ will refer to a CSS, and $x \in \mathcal{C}$ is an object (element of $\mathcal{C}_{0,0}$) unless otherwise specified.

5.2 Left Fibrations and the Grothendieck Construction

Recall that we faced some 2-categorical issues when dealing with **Cat**: composition of functors is generally defined up to natural equivalence, so functors to **Cat** aren't actually functors. A resolution is to instead consider functors to **Grp** as follows. We state the conventional category version of left fibration first:

A functor $f : \mathcal{C} \to \mathcal{D}$ is a left fibration if

- Any $f(x) \to d$ in $\mathcal{D}$ lifts to $x \to y$ in $\mathcal{C}$
- Given $x \xrightarrow{a} y$ and $x \xrightarrow{b} z$ in $\mathcal{C}$ there is a bijection between c and $\overline{c}$ satisfying

$$\begin{array}{ccc} x & \xrightarrow{a} & y \\ {\scriptstyle b}\downarrow & \swarrow {\scriptstyle c} & \\ z & & \end{array} \qquad \begin{array}{ccc} f(x) & \xrightarrow{f(a)} & f(y) \\ {\scriptstyle f(b)}\downarrow & \swarrow {\scriptstyle \overline{c}} & \\ f(z) & & \end{array}$$

Equivalently, $f : N(\mathcal{C}) \to N(\mathcal{D})$ is in $\mathrm{RLP}(\{0\} \to \Delta^1)$ and $\mathrm{RLP}(\Lambda^2_0 \to \Delta^2)$.

The Grothendieck construction then gives a bijection between such left fibrations and functors $\mathcal{D} \to \mathbf{Grp}$. We will skip to the ∞-categorical definitions for CSS.

Definition 5.1. A Reedy fibration $f : X \to Y$ in **ssSet** is a *left fibration* if the map

$$\mathrm{Fun}(d(\Delta^1), X) \to X \times_Y \mathrm{Fun}(d(\Delta^1), Y)$$

(induced by $\{0\} \to [1]$) is a trivial fibration.

Let's pause to understand what this means. We have $d(\Delta^1)_0 = \{0,1\}$ and $d(\Delta^1)_1 = \{[0,1]\}$, so a functor $d(\Delta^1) \to \mathcal{C}$ is specified by the image of $[0,1]$ in $\mathcal{C}_1$, a space of arrows. And the image of the projections $\mathrm{Fun}(d(\Delta^1), \mathcal{C}) \to \mathcal{C}$ correspond to the source of the arrows.

So we can think of the domain as arrows of X and the codomain as arrows of Y with source in $f(X)$. And "trivial fibration" corresponds to bijection, in clear analogy with the conventional case!

Remark. I think we omit the horn filler condition, because those always hold in infinity categories.

Now here are some basic facts about left fibrations.

Lemma 5.2. *Left fibrations are preserved under base change and exponential, and left fibrations preserve CSS.*

Our next task is to construct $\mathcal{S}$, the "infinity category of spaces" which will be a classifying space for left fibrations. We will give a VERY sketchy idea of the construction, so see [2] 7.3.1 for details:

- $\mathcal{S}_{m,n}$ is the set of left fibrations $Z \to \Delta^{m,n}$ up to isomorphism (set-theoretic modifications, check functoriality)
- Gluing together these left fibrations yields a *universal left fibration* $\mathcal{E} \to \mathcal{S}$
- For any $B \in \mathbf{ssSet}$, there is a bijection

  $$\mathrm{Left}(B) \leftrightarrow \mathrm{Hom}(B, \mathcal{S})$$

  given by base change.
- This bijection of sets can be upgraded to an equivalence of CSS, giving both sides the appropriate structure.

Of course we need to check various properties, etc.

5.3 The ∞-Categorical Yoneda Embedding

Given a CSS, we need to define its opposite.

Definition 5.3. Given $\mathcal{C} \in \mathbf{ssSet}$ mapping $\Delta^{\mathrm{op}} \times \Delta^{\mathrm{op}} \to \mathbf{Set}$, let $\mathbf{C}^{\mathrm{op}} \in \mathbf{ssSet}$ be

$$\Delta^{\mathrm{op}} \times \Delta^{\mathrm{op}} \xrightarrow{\mathrm{op}\times\mathrm{id}} \Delta^{\mathrm{op}} \times \Delta^{\mathrm{op}} \xrightarrow{\mathbf{C}} \mathbf{Set}$$

Thus, note that $\mathbf{C}_n^{\mathrm{op}}$ and $\mathbf{C}_n$, but the faces and degeneracies are swapped.

Lemma 5.4. *If* $\mathbf{C}$ *is a CSS, then* $\mathbf{C}^{op}$ *is too.*

Next, our goal is to define a left fibration to $\mathbf{C}^{\mathrm{op}} \times \mathbf{C}$ that will correspond to the Yoneda map.

Definition 5.5. Let $\tau : \Delta \to \Delta$ map $[n] \mapsto [n]^{\text{op}} \star [n]$. For any $\mathcal{C} \in \mathbf{ssSet}$ let $\text{Tw}(\mathcal{C}) \in \mathbf{ssSet}$ map

$$\Delta^{\text{op}} \xrightarrow{\tau} \Delta^{\text{op}} \xrightarrow{\mathcal{C}} \mathbf{sSet}$$

Let $p : \text{Tw}(\mathcal{C}) \to \mathcal{C} \times \mathcal{C}^{\text{op}}$ be induced by the natural transformations induced by $[n] \mapsto [n]^{\text{op}} \star [n]$ and $[n]^{\text{op}} \to [n]^{\text{op}} \star [n]$.

Proposition 5.6. *$p : Tw(\mathcal{C}) \to \mathcal{C} \times \mathcal{C}^{op}$ is a left fibration.*

The left fibration $\text{Tw}(\mathcal{C}) \to \mathcal{C} \times \mathcal{C}^{\text{op}}$ induces a map

$$\tilde{Y} : \mathcal{C} \times \mathcal{C}^{\text{op}} \to \mathcal{S}$$

as desired. In particular, we can rewrite this as the *Yoneda embedding*

$$Y : \mathcal{C} \to P(\mathcal{C})$$

where

$$P(\mathcal{C}) := \text{Fun}(\mathcal{C}^{\text{op}}, \mathcal{S})$$

And the corresponding $Y(x) \to \mathcal{C}^{\text{op}}$ is the left fibration that is the base change

$$\begin{array}{ccc} Y(x) & \longrightarrow & \mathcal{C}^{\text{op}} \\ \downarrow & & \downarrow {\scriptstyle \{x\} \times \text{id}} \\ \text{Tw}(\mathcal{C}) & \xrightarrow[p]{} & \mathcal{C} \times \mathcal{C}^{\text{op}} \end{array}$$

The notion of an over category will be useful for proving the analogue of the Yoneda lemma, and will also be used for defining limits and colimits.

Definition 5.7. Given Reedy fibrant $\mathcal{C} \in \mathbf{ssSet}$ and $x \in \mathcal{C}$, define $\mathcal{C}_{/x}$ as the fiber

$$\begin{array}{ccc} \mathcal{C}_{/x} & \longrightarrow & x \\ \downarrow & & \downarrow \\ \text{Fun}(d(\Delta^1), \mathcal{C}) & \xrightarrow[\{1\} \to [1]]{} & \mathcal{C} \end{array}$$

Recall by our earlier examination of the projection map, $\mathcal{C}_{/x}$ is the CSS of arrows with target x, which is what we want!

Observation.

- *The map*

$$\mathcal{C}_{/x} \to Fun(d(\Delta^1), \mathcal{C}) \xrightarrow{\{0\} \to [1]} \mathcal{C}$$

is a right fibration (this is taking the source).

- *So the corresponding $(\mathcal{C}_{/x})^{op} \to \mathcal{C}^{op}$ is a left fibration.*
- *$\mathcal{C} \in CSS \implies \mathcal{C}_{/x} \in CSS$ (since changing base by left fibration preserves CSS).*

Lemma 5.8. *The left fibrations $Y(x) \to \mathcal{C}^{op}$ and $(\mathcal{C}_{/x})^{op} \to \mathcal{C}^{op}$ and naturally homotopy equivalent via*

$$Y(x) \to (\mathcal{C}_{/x})^{op}, \quad id_x \mapsto id_x$$

Recall that the category of left fibrations was endowed with CSS structure, so we can make sense of "homotopy equivalent".

Now we get the analogue of the Yoneda lemma.

Proposition 5.9. *Let $F \in P(\mathcal{C})$ and $x \in \mathcal{C}$. The natural evaluation map*

$$Map_{P(\mathcal{C})}(Y(x), F) \to F(x)$$

is an equivalence.

5.4 Limits, Colimits, Adjoints

Now we put the Yoneda lemma (and philosophy) to use. Let's define colimits, and limits will be dual.

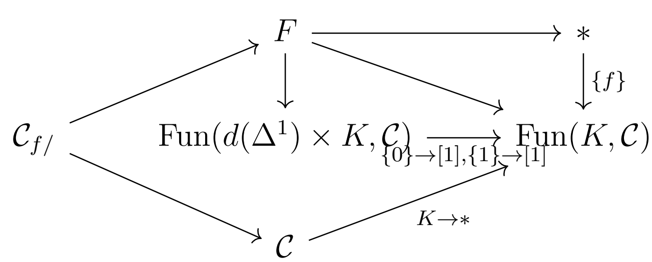
Definition 5.10. An object $x \in \mathcal{C}$ is *initial* if $\mathrm{Map}(x, y)$ is contractible for all $y \in \mathcal{C}$.

We want initial objects to be unique, but of course we can only expect uniqueness up to homotopy.

Proposition 5.11. *The full subcategory $\mathcal{C}_{init}$ of initial objects in $\mathcal{C}$ is a contractible space (or empty).*

To define more general limits, we define an undercategory.

Definition 5.12. Let $f : K \to \mathcal{C}$ be a functor between **ssSet**. Then the *undercategory* $\mathcal{C}_{f/}$ is the double fiber product



Lemma 5.13. *If $\mathcal{C}$ is a Segal space, then $\mathcal{C}_{f/} \to \mathcal{C}$ is a left fibration. So $\mathcal{C} \in CSS \implies \mathcal{C}_{f/} \in CSS$.*

Proof. The map $\mathcal{C}_{f/} \to \mathcal{C}$ is the base change of

$$\mathrm{Fun}(K, \mathcal{C})_{/f} \to \mathrm{Fun}(K, \mathcal{C})$$

which is a left fibration by our observation about undercategories earlier. $\square$

Definition 5.14. Given a functor $f : K \to \mathcal{C}$, its *colimit* is an initial object of the category $\mathcal{C}_{f/}$.

Finally, let's talk about adjunctions. For conventional categories, an adjunction is the data of

$$\mathcal{C}^{\text{op}} \times \mathcal{D} \to \mathbf{Set}$$

and $L : \mathcal{C} \to \mathcal{D}$ admits a right adjoint iff for any $y \in \mathcal{D}$ the functor $x \mapsto \text{Hom}_{\mathcal{D}}(L(x), y)$ is representable.

So for infinity categories (CSS), we expect to want a left fibration to $\mathcal{C}^{\text{op}} \times \mathcal{D}$.

Definition 5.15. For $\mathcal{C}, \mathcal{D} \in \text{CSS}$, a *correspondence* from $\mathcal{C}$ to $\mathcal{D}$ is a left fibration

$$p : \mathcal{E} \to \mathcal{C}^{\text{op}} \times \mathcal{D}$$

A correspondence is *left-presentable* if the base change by each $x \in \mathcal{C}$ defines a representable presheaf on $\mathcal{D}^{\text{op}}$, and it is *right*-presentable if the base change by each $y \in \mathcal{D}$ defines a representable presheaf on $\mathcal{C}$

$$\begin{array}{ccc} \mathcal{E}_x & \longrightarrow & \mathcal{D} \\ \downarrow & & \downarrow{\scriptstyle x \times \text{id}} \\ \mathcal{E} & \xrightarrow[p]{} & \mathcal{C}^{\text{op}} \times \mathcal{D} \end{array} \qquad \begin{array}{ccc} \mathcal{E}_y & \longrightarrow & \mathcal{C}^{\text{op}} \\ \downarrow & & \downarrow{\scriptstyle \text{id} \times y} \\ \mathcal{E} & \xrightarrow[p]{} & \mathcal{C}^{\text{op}} \times \mathcal{D} \end{array}$$

(i.e., the maps $\mathcal{E}_x \to \mathcal{S}$ or $\mathcal{E}_y \to \mathcal{S}$).

A left fibration $p\mathcal{E} \to \mathcal{C}^{\text{op}} \times \mathcal{D}$ determines an adjoint pair between $\mathcal{C}$ and $\mathcal{D}$ if p is both left and right representable.

And of course we can interpret p as a functor $\mathcal{C}^{\text{op}} \times \mathcal{D} \to \mathcal{S}$ and then as $p_C : \mathcal{C}^{\text{op}} \to P(\mathcal{D}^{\text{op}})$ or $p_D : \mathcal{D} \to P(\mathcal{C})$.

Then p is left representable if p_C factors through $\mathcal{D}^{\text{op}}$, an d1it is right representable if p_C factors through $\mathcal{C}$.

6 March 21

Unfortunately due to Covid, we are on Zoom today and I am not as well prepared as I'd hoped. Apologies in advance!

Let's start by tying together some important concepts that we rushed over or skipped.

6.1 DK localization and Classification Diagrams

Recall that DK localization took a model category $(\mathcal{C}, W)$ and gave a simplicial category $L^H(\mathcal{C}, W)$ (most explicitly via the hammock construction) satisfying

- Arrows of W become equivalences in $L^H(\mathcal{C}, W)$
- $\pi_0(L^H(C, W)) = C[W^{-1}]$

- A Quillen equivalence of model categories yields an equivalence of DK localizations.

Recall also that $\mathcal{C} : \mathbf{sSet} \leftrightarrow \mathbf{sCat} : \mathcal{N}$ was a Quillen equivalence ($\mathcal{N}$ is the homotopy coherent nerve).

We can use this to construct the *infinity category of spaces*, which we mentioned last time.

Example. Let $(\mathcal{C}, W)$ be **sSet** with weak homotopy equivalences. Then $L^H(\mathcal{C}, W)$ is $\mathcal{S}$, the simplicial category of Kan simplicial sets with inner Hom for morphism spaces.

Note that $\mathcal{S}$ is also $\mathcal{N}$ of the simplicial category of Kan simplicial sets.

Recall that we had a Quillen equivalence between **sSet** and **Top** (and also that all objects of **Top** are fibrant), justifying the name "infinity category of spaces".

Let us also justify using this term last time:

Proposition 6.1. *$\mathcal{S}$ classifies left-fibrations (i.e., is equivalent to $\mathcal{S}$ from last time).*

Proof sketch. The CSS $\mathcal{S}$ from last time has a weak equivalence

$$\mathrm{Left}(B) \to \mathrm{Fun}(B, \mathcal{S})$$

Letting $B = *$, the right hand side is $\mathcal{S}$, and the left hand side is $\mathrm{Left}(*)$ which is equivalent to the category of Kan simplicial sets (same lifting condition). □

In addition to DK localization, let's look at another bridge to a model of infinity categories. In particular, the following *classification diagram* construction takes a category with weak equivalences to a CSS. This modifies the Rezk Nerve construction

$$\mathcal{C} \mapsto B\mathcal{C} \in \mathbf{ssSet}, \quad B\mathcal{C}_n = N(\mathrm{Fun}([n], \mathcal{C})^{\mathrm{iso}})$$

Definition 6.2. Given $(\mathcal{C}, W)$, define $B(C, W) \in \mathbf{ssSet}$ by

$$B(\mathcal{C}, W)_n = N(\mathrm{Fun}([n], (\mathcal{C}, W)))$$

where $\mathrm{Fun}([n], (\mathcal{C}, W))$ has objects as functors $[n] \to \mathcal{C}$ and whose arrows are pointwise weak equivalences of functors.

Note that $B(\mathcal{C}, W)_{n,m}$ consists of commutative $n \times m$ rectangles whose vertical arrows belong to W (this is reminiscent of the $L^H(\mathcal{C}, W)$ construction!).

This construction by itself is not a CSS, but:

Theorem 6.3. *$B(\mathcal{C}, W)^f$ (Reedy fibrant replacement) is a CSS.*

6.2 ∞-localization

Now we will discuss localization of infinity categories via adjoint functors. We will see that DK localization calculates it.

Let $\mathcal{S} \to \mathbf{Cat}_\infty$ be the embedding of the infinity category of spaces into the infinity category of infinity categories (this can be done in any model, e.g., including $\mathcal{N}(\mathrm{Kan}) \to \mathbf{sCat}$ or essentially constant CSS into CSS). We want adjoints to this functor.

Proposition 6.4. *There is a right adjoint to this embedding,*

$$\mathbf{Cat}_\infty \to \mathcal{S}, \quad \mathcal{C} \mapsto \mathcal{C}^{eq}$$

where $\mathcal{C}^{eq}$ is the maximal subspace*, obtained by discarding non-equivalences.*

Proof idea. Obtained by a Quillen adjunction $\mathbf{sSet} \leftrightarrow (\mathbf{ssSet}, CSS)$. See [2] 8.5.1. □

On the other hand, we have:

Proposition 6.5. *There is a left adjoint to this embedding,*

$$\mathbf{Cat}_\infty \to \mathcal{S}, \quad \mathcal{C} \mapsto \mathcal{C}^f$$

replacing a quasi-category with its Kan fibrant replacement.

Proof idea. This again comes from a Quillen adjunction, this time on the standard vs. Joyal model structures on **sSet** (an example of Bousfield localization). The adjoint is thus the left derived functor of the identity. Again see [2] 8.5.1. □

Now we come to general localization. Consider the functor

$$K : \mathbf{Cat}_\infty \to \mathrm{Fun}(\Delta^1, \mathbf{Cat}_\infty), \mathcal{C} \mapsto \mathcal{C}^{\mathrm{eq}}$$

(recall that $\mathcal{C}^{\mathrm{eq}}$ has discarded all non-equivalences, so we can embed it into the arrow category).

Definition 6.6. General localization is the functor

$$L : \mathrm{Fun}(\Delta^1, \mathbf{Cat}_\infty) \to \mathbf{Cat}_\infty$$

left adjoint to K.

We can check that DK localization is a special case of this construction.

6.3 General Infinity Categories

Now that we have explicitly constructed many of the main tools of infinity categories, we will employ a general language independent of model. Here are the basic points (which should be familiar by now):

- Let **Cat** denote the infinity category of infinity categories.
- There is a subcategory $\mathcal{S} \to \mathbf{Cat}$ of spaces. This inclusion has a right adjoint (maximal subspace) and left adjoint (total localization).
- For $\mathcal{C} \in \mathbf{Cat}$ and any $x, y \in \mathcal{C}$, there is a space $\mathrm{Map}_{\mathcal{C}}(x, y)$ (defined "up to a contractible space of choices").
- There is a full subcategory $\mathbf{Cat}^{\mathrm{conv}} \to \mathbf{Cat}$ of conventional categories.

- There is a functor Ho : $\mathbf{Cat} \to \mathbf{Cat}^{\text{conv}}$ defined by

$$\mathrm{Hom}_{\mathrm{Ho}(\mathcal{C})}(x, y) = \pi_0(\mathrm{Map}_{\mathcal{C}}(x, y))$$

- An arrow in $\mathcal{C}$ is an equivalence if its image in $\mathrm{Ho}(\mathcal{C})$ is an isomorphism. A functor (arrow in $\mathcal{C}\dashv\sqcup$) is an equivalence if it induces equivalences mapping spaces and an equivalence of homotopy categories.

Next, an example to motivate our next topic, Cocartesian fibrations (as well as left fibrations from last time). See here and here for details.

Recall from topology that given a (nice) topological space X, there is an equivalence of categories

$$[\text{covering spaces } Y \to X] \to \mathrm{Fun}(\Pi_1(X) \to \mathrm{Set})$$

(where $\Pi_1(X)$ is the fundamental groupoid, with objects points of X and morphisms homotopy classes of paths). This equivalence is given by taking fibers, and it follows by the path lifting property of covering spaces.

To upgrade this result, instead let $X \in \mathbf{Kan}$ be an ∞-groupoid. Instead of $\pi_1(X)$, we look at $\mathrm{Ho}(X)$. The result will now say that we have an equivalence

$$[\text{Kan fibrations } Y \to X] \to \mathrm{Fun}(X, \mathcal{S})$$

with the map again given by taking fibers. This was our result last time! (i.e., functors to $\mathcal{S}$ correspond to left fibrations). The next upgrade will be the following equivalence:

$$[\text{cocartesian fibrations } Y \to X] \to \mathrm{Fun}(X, \mathbf{Cat}_\infty)$$

We will give this equivalence in a few steps.

6.4 Complete Segal Objects

The first link in our chain of equivalences will involve Complete Segal objects, which we motivate as follows.

Recall that we have the Yoneda embedding $\mathbf{Cat} \to \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S})$. We can compose this with the map $\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}) \to \mathrm{Fun}(\Delta^{\mathrm{op}}, \mathcal{S})$ defined by restriction along $\Delta \to \mathcal{C}$, and we have the following:

Proposition 6.7. *The map* $\mathbf{Cat} \to \mathit{Fun}(\Delta^{op}, \mathcal{S})$ *is a fully faithful embedding.*

Proof idea. Morphism spaces are built out of simplices, so any presheaf can be recovered from values on Δ^{op}. □

The image of this embedding will have the additional properties of being *complete* and *Segal*.

Definition 6.8. Let $\mathcal{C}$ be an infinity category with finite limits. A simplicial object $X \in \mathrm{Fun}(\Delta^{\mathrm{op}}, \mathcal{C})$ is *Segal* if $X(\Delta^n) \to X(\mathbf{Sp}(n))$ is an equivalence in $\mathcal{C}$ for all n.

Let $\mathrm{Seg}(\mathcal{S})$ be the full subcategory spanned by Segal objects.

This should of course be familiar from our definition of Segal spaces.

Definition 6.9. A simplicial object $X \in \mathrm{Fun}(\Delta^{\mathrm{op}}, \mathcal{C})$ is a *groupoid object* if for any $[n] = S \cup T$ with $S \cap T = \{s\}$,

$$X_n \cong X(S) \times_{X(\{s\})} X(T)$$

Let $\mathbf{Grp}(\mathcal{C})$ denote the full subcategory spanned by groupoid objects.

Note that $\mathbf{Grp}(\mathcal{S})$ consists of the images of groupoids under the Yoneda embedding, motivating the definition. Furthermore,

Lemma 6.10. $\mathbf{Grp}(\mathcal{C}) \subset$ *Seg*$(\mathcal{C})$.

Proof idea. Since $\mathbf{Sp}(n)$ consists of gluing $[i-1, i] \sqcup_{\{i\}} [i, i+1]$, the map $X([n]) \to X(\mathbf{Sp}(n))$ is in fact an isomorphism by inducting on the groupoid condition. □

Proposition 6.11. *Let $\mathcal{C}$ be an infinity category with finite limits. The embedding $\mathbf{Grp}(\mathcal{C}) \to$ Seg$(\mathcal{C})$ has a right adjoint functor, $X \mapsto X^{eq}$.*

Proof sketch. In the case $\mathcal{C} = \mathcal{S}$, we can construct this adjoint explicitly: let X_n^{eq} be the subspace of $X_n \sim X(\mathbf{Sp}(n))$ whose image in the homotopy category is a sequence of isomorphisms.

From that, we can construct an adjoint for the case of $\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S})$. And then we can use the Yoneda embedding (and the fact that it preserves limits). □

Definition 6.12. A Segal object $X \in \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S})$ is complete if X^{eq} is essentially constant.

Let $\mathbf{CS}(\mathcal{C})$ denote the full category of complete Segal objects.

Now, we have the promised:

Lemma 6.13. *The Yoneda embedding*

$$\mathbf{Cat} \to \mathit{Fun}(\Delta^{op}, \mathcal{S})$$

identifies $\mathbf{Cat}$ *with* $\mathbf{CS}(\mathcal{S})$.

Proof idea. The idea is to realize $\mathrm{Fun}(\Delta^{\mathrm{op}}, \mathcal{S})$ as Reedy fibrant bisimplicial sets and **Cat** as the full subcategory of complete segal spaces, which can be shown to be equivalent to the Yoneda embedding. See [2] 9.4.4 for details. □

The following relative version will be our key link.

Lemma 6.14. *The equivalence*

$$\mathit{Fun}(B, \mathit{Fun}(\Delta^{op}, \mathcal{C})) = \mathit{Fun}(\Delta^{op}, \mathit{Fun}(B, \mathcal{C}))$$

identifies $\mathit{Fun}(B, \mathbf{CS}(\mathcal{C}))$ *with* $\mathbf{CS}(\mathit{Fun}(B, \mathcal{C}))$.

Proof idea. Check componentwise. □

Corollary 6.15. *We have the following chain of equivalences:*

$$\mathit{Fun}(B, \mathbf{Cat}) \cong \mathit{Fun}(B, \mathbf{CS}(\mathcal{S})) \cong \mathbf{CS}(\mathit{Fun}(B, \mathcal{S})) \cong \mathbf{CS}(\mathit{Left}(B))$$

So our next goal is to classify $\mathbf{CS}(\mathrm{Left}(B))$.

6.5 Cocartesian fibrations

There is a rich theory of cocartesian fibrations, of which we unfortunately present less than the bare minimum (my apologies for being underprepared today).

In what follows, let $f : X \to B$ be a morphism in **Cat**.

Definition 6.16. An arrow $a : x \to y$ in X is called *f-cocartesian* if the following diagram is cartesian:

$$\begin{array}{ccc} X_{y/} & \longrightarrow & X_{x/} \\ \downarrow & & \downarrow \\ X \times_B B_{f(y)/} & \longrightarrow & X \times_B B_{f(x)/} \end{array}$$

Recall our definition of over-categories in the setting of CSS.

Definition 6.17. An arrow $\overline{a} : b \to c$ in B *admits a cocartesian lifting* if for any $x \in X$ with $f(x) = b$, there exists a cocartesian arrow $a : x \to y$ in X with $f(a) = \overline{a}$.

Definition 6.18. A map $f : X \to B$ is a *cocartesian fibration* if for any $x \in X$ and any $a : f(x) \to b'$ it admits a cocartesian lifting of a.

Proposition 6.19 (Grothendieck construction)**.** *There is an equivalence G : $\mathbf{Coc}(B) \to \mathbf{CS}(Left(B))$ defined by*

$$G(X)_n = Fun_B([n], X)^{coc}$$

i.e., the left fibration over B defined by the cocartesian arrows in X.

7 March 28

Today we dive into derived algebraic geometry. This talk should be a somewhat fresh start, so it's alright if you are not up to speed on the previous material.

7.1 Stable Infinity Categories and Ring Spectra

These ideas came up last semester in the seminar and in Andrew Blumberg's class, but let's refresh our memories.

Definition 7.1. An ∞-category $\mathcal{C}$ is *stable* if

- It has a zero object
- Every morphism has a kernel and a cokernel
- A triangle is exact $\iff$ it is coexact

Example. A *spectrum* is a sequence of pointed spaces $\{X_i\}$ (in $\mathcal{S}$) together with equivalences $X_i \simeq \Omega X_{i+1}$ where Ω is the loop space functor $X \mapsto 0 \times_X 0$. The infinity category of spectra is stable.

There are various important results to prove, e.g., that the homotopy category of a stable infinity category is triangulated, with shift given by suspension. But we are interested in algebraic structures.

Definition 7.2. An associative monoid in an ∞-category $\mathcal{C}$ with products is a "special Δ-space", i.e., a functor $A : \Delta^{\text{op}} \to \mathcal{C}$ satisfying

- A_0 is a terminal object
- $p_n : A_n \to (A_1)^n$ induced by $\text{Sp}(n) \to \Delta^n$ is an equivalence

Of course, this is reminiscent of Segal spaces!

Remark. The way to think about this definition is that A_1 is an algebra object and we have multiplication

$$A_1 \times A_1 \xrightarrow{p_2^{-1}} A_2 \xrightarrow{d_1} A_1$$

We can check associativity up to homotopy using the face/degeneracy maps identities.

Definition 7.3. A left-module over a monoid in $\mathcal{C}$ with products is a functor

$$F : \Delta^{\text{op}} \times [1] \to \mathcal{C}$$

such that

- The simplicial object $F|_{\Delta^{\text{op}} \times \{1\}}$ is an algebra object
- F carries $([n], 0) \to ([n], 1)$ and $(\{n\}, 0) \to ([n], 0)$ to a product diagram

$$F([n], 0) \cong F([0], 0) \times F([n], 1)$$

(where $[0] \xrightarrow{\{n\}} [n]$).

Remark. Again, there is a way to interpret this definition more concretely. The idea is that $F|_{\Delta^{\text{op}} \times \{1\}}$ is A and $F|_{\Delta^{\text{op}} \times \{0\}}$ is $A \times M$.

More precisely, let $M = F([0], 0)$ and A be the simplicial (algebra) object $A_n = F([n], 1)$. Thus, we are given $F([n], 0) \cong A_n \times M \cong A_1^n \times M$.

As before, multiplication $A \times A \to A$ is $F([2], 1) \to F([1], 1)$ induced by $[1] \xrightarrow{d_1} [2]$. And now the action $A \times M \to M$ is $F([1], 0) \to F([0], 0)$ induced by $[0] \xrightarrow{d_1} [1]$.

There is one piece of data we have not used: the map $0 \to 1$ in $[1]$. It's image under F gives a map between simplicial objects $F|_{\Delta^{\text{op}} \times \{0\}} \to F|_{\Delta^{\text{op}} \times \{1\}}$, or $A \times M \to A$. This is precisely the coherence data making multiplication in A compatible with its action on M.

A naive definition of A_∞ ring spectra is to take monoid objects in spectra. Lurie modifies this to give stronger associativity than just up-to-homotopy.

An E_∞-ring spectrum has the additional requirement of homotopy-coherent commutativity.

One important technical definition:

Definition 7.4. The ∞-category of connective E_∞-ring spectra, $E_\infty\text{-alg}^{\text{cn}}$, is the full subcategory of E_∞-monoids in the ∞-category of spectra such that $\pi_i(R) = 0$ for $i < 0$.

7.2 Simplicial Commutative Rings

Now we get into our first main construction.

For motivation, consider the following equivalent description of the category **CRing** of (ordinary) commutative rings. For $R \in \mathbf{CRing}$, consider $\mathrm{Hom}(-, R) \in P_{\mathbf{Set}}(\mathbf{Poly})$, where $P_{\mathbf{Set}}(\mathbf{Poly}) = \mathrm{Fun}(\mathbf{Poly}^{\mathrm{op}}, \mathbf{Set})$ and $\mathbf{Poly} \subset \mathbf{CRing}$ is the full subcategory spanned by $\mathbf{Z}[T_1, ..., T_n]$.

Fact: $R \mapsto \mathrm{Hom}(-, R)$ is an equivalence between **CRing** and the subcategory of $P_{\mathbf{Set}}(\mathbf{Poly})$ of presheaves sending finite coproducts to products.

For an ∞-categorical version, we of course replace **Set** with $\mathcal{S}$.

Definition 7.5. Let **SCR** denote the ∞-category of *simplicial commutative rings*, i.e., the subcategory of $P_{\mathcal{S}}(\mathbf{Poly})$ of presheaves sending finite coproducts to products.

Since the Yoneda embedding respects coproducts, we have $\mathbf{Poly} \hookrightarrow \mathbf{SCR}$. As before, we would like a result guaranteeing that we lose nothing by working with this subcategory.

To make this formal, we use the idea of sifted (homotopy) colimits.

Recall that for conventional categories, a category (diagram) D is sifted if colimits of shape D commute with finite products in **Set**. That is for any S a finite discrete category, $F : D \times S \to \mathbf{Set}$ induces a canonical isomorphism

$$\mathrm{colim}_{d \in D} \prod_{s \in S} F(d, s) = \prod_{s \in S} \mathrm{colim}_{d \in D} F(d, s)$$

And a sifted colimit is a colimit over a sifted diagram.

A diagram is sifted iff $D \xrightarrow{\Delta} D \times D$ is a cofinal functor, meaning precomposition preserves colimits.

This latter condition naturally adapts to infinity categories:

Definition 7.6. An ∞-category D is *sifted* if for any $F : D \times D \to \mathcal{C}$,

$$\mathrm{colim}(D \times D \xrightarrow{F} \mathcal{C}) \xrightarrow{\sim} \mathrm{colim}(D \xrightarrow{\Delta} D \times D \xrightarrow{F} \mathcal{C})$$

is a weak equivalence.

A *sifted colimit* is the colimit of a sifted diagram.

Now we get our desired proposition.

Proposition 7.7. *Let $\mathcal{C}$ be an ∞ category which admits sifted colimits. Then the Yoneda embedding* $\mathbf{Poly} \hookrightarrow \mathbf{SCR}$ *induces an equivalence of ∞-categories*

$$\mathit{Fun}_{\mathit{sift}}(\mathbf{SCR}, \mathcal{C}) \xrightarrow{\sim} \mathit{Fun}(\mathbf{Poly}, \mathcal{C})$$

where $\mathit{Fun}_{\mathit{sift}} \subset \mathit{Fun}$ is the full subcategory spanned by functors that commute with sifted colimits.

This amounts to: "**SCR** is freely generated by **Poly**" under sifted colimits." It follows from the fact that "**SCR** is the non-abelian derived category of **Poly**" and general theory.

Unpacking what this means: for the conventional category case, we restricted to presheaves that sent finite coproducts to products. Preserving sifted colimits is a strengthening of that condition (since products are sifted). It makes sense that the infinity category case condition would be stricter since it must respect homotopy coherence data.

Now let's see what a simplicial ring encodes.

Definition 7.8. Give $R \in \mathbf{SCR}$, the *underlying space* is

$$R_{\mathcal{S}} := R(\mathbf{Z}[T]) \in \mathcal{S}$$

But there is further structure encoded by **Poly**. Since R respects products and $\mathbf{Z}[T_1, ..., T_n] \cong \bigotimes_{i=1}^{n} \mathbf{Z}[T]$ (this is the product in **Poly**) we have a map

$$\text{mult} : R_{\mathcal{S}}^{\times n} \to R_{\mathcal{S}}$$

induced by

$$R_{\mathcal{S}}^{\times n} = R(\mathbf{Z}[T])^n \cong R\left(\bigotimes_{i=1}^{n} \mathbf{Z}[T]\right) = R(\mathbf{Z}[T_1, ..., T_n]) \xrightarrow{T \mapsto T_1 ... T_n} R(\mathbf{Z}[T]) = R_{\mathcal{S}}$$

Likewise, we have a map induced by addition $R(\mathbf{Z}[T_1, ..., T_n]) \xrightarrow{T \mapsto T_1 + ... + T_n} R(\mathbf{Z}[T])$

$$\text{add} : R_{\mathcal{S}}^{\times n} \to R_{\mathcal{S}}$$

And since $R(\mathbf{Z}) = *$ (since $\mathbf{Z} \otimes \mathbf{Z} \cong \mathbf{Z}$) the maps $\mathbf{Z}[T] \to \mathbf{Z}$, $T \mapsto 0, 1$ induce

$$0, 1 : * \to R_{\mathcal{S}}$$

Call $R \in \mathbf{SCR}$ *discrete* if the presheaf has values in $\text{Set} \subset \mathcal{S}$. The inclusion $\mathbf{CRing} \hookrightarrow \mathbf{SCR}$ is an equivalence onto discrete subrings. It has a left adjoint, denoted

$$R \mapsto \pi_0(R)$$

7.3 Algebras and Modules over a Simplicial Commutative Ring

Algebras are straightforward:

Definition 7.9. Let $\mathbf{SCR}_R$ be the ∞-category of R-algebras, i.e., $A \in \mathbf{SCR}$ equipped with $R \to A$.

For modules, we will use an embedding into ring spectra as follows. Recall that there is an Eilenberg Maclane functor

$$\mathbf{CRing} \to E_\infty\text{-alg}^{\text{cn}}, \quad R \mapsto \{K(R, n)\}_n$$

this identifies ordinary commutative rings with *discrete* E_∞-ring spectra.

By our equivalence proposition, we obtain a functor of ∞-categories (unique up to contractible choice) that commutes with sifted colimits:

$$\mathbf{SCR} \to E_\infty\text{-alg}^{\text{cn}}, \quad R \mapsto R_{\text{Spt}}$$

Remark. $\Omega^\infty(R_{\mathrm{Spt}}) = R_{\mathcal{S}}$.

Definition 7.10. An *R-module* is a module over the E_∞-ring spectrum R_{Spt}.

Mod_R is the stable symmetric monoidal ∞-category of R-modules, and $\mathrm{Mod}_R^{\mathrm{cn}}$ is the full subcategory spanned by connective R-modules.

Example. If R is discrete, Mod_R is canonically equivalent to the derived category of the abelian category of conventional R-modules.

Definition 7.11. A connective R-module M is *flat* if $M \otimes_R N$ is discrete for any discrete R-module N.

It is *flat* if it is flat if $M \otimes_R N$ is zero iff N is zero for any connective R-module N.

Now let's go on a slight tangent to see the relationship between strict and non-strict models for rings. For any $R \in \mathbf{SCR}$, we have an induced functor

$$\mathbf{SCR}_R \to E_\infty\text{-alg}^{\mathrm{cn}}_{R_{Spt}}$$

But this functor is neither fully faithful nor essentially surjective, since our strict definitions of commutativity and associativity makes $\mathbf{SCR}$ a "stricter" category than E_∞-alg.

Finally, we note that there is a forgetful functor

$$\mathbf{SCR}_R \to E_\infty\text{-alg}^{\mathrm{cn}}_{R_{Spt}} \to \mathrm{Mod}^{\mathrm{cn}}_{R_{Spt}} = \mathrm{Mod}^{\mathrm{cn}}_R$$

7.4 Derived Schemes

Once again, we recall conventional algebraic geometry in order to motivate the derived case. And again, it serves well to recast definitions into presheaf categories which we can adapt to the infinity categorical setting.

For any scheme $S \in \mathbf{Sch}$, we have

$$\mathrm{Maps}_{\mathbf{Sch}}(-, S) \in P_{\mathbf{Set}}(\mathbf{Sch})$$

Grothendieck proved that this presheaf satisfies fpqc descent. The condition that every scheme has an affine Zariski cover implies that $\mathbf{CRing}^{\mathrm{op}} \hookrightarrow \mathbf{Sch}$ induces an equivalences

$$\mathrm{Sh}^{\mathrm{fpqc}}(\mathbf{CRing}^{\mathrm{op}}) \xrightarrow{\simeq} \mathrm{Sh}^{\mathrm{fpqc}}(\mathbf{Scheme})$$

Thus, the Yoneda embedding of $\mathbf{Sch}$ induces

$$\mathbf{Sch} \hookrightarrow \mathrm{Sh}^{\mathrm{fpqc}}(\mathbf{CRing}^{\mathrm{op}})$$

and we can take this essential immage as an alternative definition of the category of schemes.

Let's build up these ideas for $\mathbf{SCR}$.

Let's start with the category we want to refine.

Definition 7.12. A *derived prestack* is an element of $P_{\mathcal{S}}(\mathbf{SCR}^{\mathrm{op}}) = \mathrm{Fun}(\mathbf{SCR}, \mathcal{S})$.

Now we want to set up the descent condition. We can define the fpqc pretopology on $\mathbf{SCR}^{\mathrm{op}}$.

Definition 7.13. A family of homomorphisms $(R \to R_\alpha)_{\alpha \in \Lambda}$ is *fpqc covering* if:

- Λ is finite
- Each $R \to R_\alpha$ is flat (i.e., the underlying R-module of R_α is flat)
- $R \to \prod_\alpha R_\alpha$ is faithfully flat

Let $(R \to R_\alpha)_\alpha$ be an fpqc covering family and $\tilde{R} = \prod_\alpha R_\alpha$. Let $\check{C}(R/\tilde{R}) \in \mathrm{Fun}(\Delta, \mathbf{SCR})$ be the cosimplicial object defined by

$$\check{C}(R/\tilde{R})^n = \tilde{R} \otimes_R \ldots \otimes_R \tilde{R}, \quad n+1 \text{ times}$$

This is the *Čech nerve of* $R \to \tilde{R}$.

Definition 7.14. A derived prestack $\mathcal{X}$ satisfies *fpqc descent* if for all fpqc covering families $(R \to R_\alpha)_\alpha$, the canonical morphism

$$\mathcal{X}(R) \to \lim_{n \in \Delta} \mathcal{X}(\check{C}(R/\tilde{R})^n)$$

is an isomorphism.

A *derived stack* is a derived prestack satisfying fpqc descent.

Now we can define derived affine schemes. For $R \in \mathbf{SCR}$, define the derived prestack $\mathrm{Spec}(R) = \mathrm{Map}_{\mathbf{SCR}}(R, -) \in P_{\mathcal{S}}(\mathbf{SCR}^{\mathrm{op}})$.

Proposition 7.15. $\mathbf{SCR}$ *is a derived stack for any* $R \in \mathbf{SCR}$.

Definition 7.16. An *affine derived scheme* is a derived stack isomorphic to $\mathrm{Spec}(R)$ for some $R \in \mathbf{SCR}$.

To define schemes in general, we need an idea of *open immersions*. We need an idea of what it means to be a covering. This will generalize the descent condition earlier. If $X \to Y$ is a morphism of sheaves, define $\check{C}(X/Y)_\bullet \in \mathrm{Fun}(\Delta^{\mathrm{op}}, \mathbf{Sh})$

$$\check{C}(X/Y)_n = X \times_Y \ldots \times_Y X \quad n+1 \text{ times}$$

We say that $X \to Y$ is an *effective epimorphism* if the canonical morphism of sheaves

$$\mathrm{colim}_{n \in \Delta^{\mathrm{op}}} \check{C}(X/Y)_n \to Y$$

is an isomorphism.

Definition 7.17. For $R, R' \in \mathbf{SCR}$, $R \to R'$ is *locally of finite presentation* if the functor

$$\mathbf{SCR} \to \mathcal{S}, \quad A, \mapsto \mathrm{Maps}_{\mathbf{SCR}_R}(R', A)$$

commutes with filtered colimits.

Definition 7.18. Let $j : U \to \mathcal{X}$ be a morphism of derived stacks.

- If $\mathcal{X} = \mathrm{Spec}(R)$ and $U = \mathrm{Spec}(A)$, then j is an *open immersion* if the corresponding $R \to A$ is
  - locally of finite presentation
  - flat
  - epimorphism
- If $\mathcal{X} = \mathrm{Spec}(R)$ and U is general, j is an open immersion if it is a monomorphism and there exists a family $(U_\alpha \to U)_\alpha$ with
  - U_α affine
  - $\coprod_\alpha U_\alpha \to U$ is an effective epimorphism
  - Each $U_\alpha \to U \to \mathcal{X}$ is an open immersion in the previous sense
- If $\mathcal{X}$ and U are general, then j is an open immersion if for any $\mathrm{Spec}(A) \to \mathcal{X}$,

  $$U \times_{\mathcal{X}} \mathrm{Spec}(A) \to \mathrm{Spec}(A)$$

  is an open immersion in the above sense.

Of course it would be good to check consistency etc. Here's one handy fact

Lemma 7.19. *$R \to A$ induces an open immersion iff $A \otimes_R A \to A$ is invertible.*

Note that this is on the nose; recall that we chose a stricter version of infinity-rings.
Now we can get a familiar looking definition of derived schemes.

Definition 7.20. A *Zariski cover* of a derived stack $\mathcal{X}$ is a family of open immersions $(j_\alpha : U_\alpha \hookrightarrow \mathcal{X})_\alpha$ and

$$\bigsqcup_\alpha U_\alpha \to \mathcal{X}$$

is an effective epimorphism.
A Zariski cover is *affine* if each U_α is an affine derived scheme, and $\mathcal{X}$ is a *derived scheme* if it admits an affine Zariski cover.

Next time we will see derived quasi-coherent sheaves and descent.

8 April 4

8.1 More on Modules over Simplicial Rings

Let's talk more about modules and clear up some things skipped last time.
First, we provide intuition for modules over a monoid (this will come up again, so might be worth revisiting). Recall:

Definition 8.1. A left-module over a monoid in $\mathcal{C}$ with products is a functor

$$F : \Delta^{\mathrm{op}} \times [1] \to \mathcal{C}$$

such that

- The simplicial object $F|_{\Delta^{\mathrm{op}} \times \{1\}}$ is an algebra object
- F carries $([n], 0) \to ([n], 1)$ and $(\{n\}, 0) \to ([n], 0)$ to a product diagram

$$F([n], 0) \cong F([0], 0) \times F([n], 1)$$

(where $[0] \xrightarrow{\{n\}} [n]$).

Here is the interpretation:

Remark. Again, there is a way to interpret this definition more concretely. The idea is that $F|_{\Delta^{\mathrm{op}} \times \{1\}}$ is A and $F|_{\Delta^{\mathrm{op}} \times \{0\}}$ is $A \times M$.

More precisely, let $M = F([0], 0)$ and A be the simplicial (algebra) object $A_n = F([n], 1)$. Thus, we are given $F([n], 0) \cong A_n \times M \cong A_1^n \times M$.

As before, multiplication $A \times A \to A$ is $F([2], 1) \to F([1], 1)$ induced by $[1] \xrightarrow{d_1} [2]$. And now the action $A \times M \to M$ is $F([1], 0) \to F([0], 0)$ induced by $[0] \xrightarrow{d_1} [1]$.

There is one piece of data we have not used: the map $0 \to 1$ in $[1]$. It's image under F gives a map between simplicial objects $F|_{\Delta^{\mathrm{op}} \times \{0\}} \to F|_{\Delta^{\mathrm{op}} \times \{1\}}$, or $A \times M \to A$. This is precisely the coherence data making multiplication in A compatible with its action on M.

Next, we give more detail about flatness of modules over simplicial rings. The following is (part of) the "Derived Lazard Theorem". We state it in the generality of A_∞-ring spectra:

Proposition 8.2. *Let R be a connective A_∞-ring spectrum, and let M be a connective left R-module. The following are equivalent:*

- *If N is a discrete right R-module, then $N \otimes_R M$ is discrete*
- *The $\pi_0 R$-module $\pi_0 R \otimes_R M$ is discrete and flat (as a conventional module)*

Proof sketch. The idea is to show by induction that for any connective R-module N, there are isomorphisms

$$\mathrm{Tor}_0^{\pi_0 R}(\pi_i N, \pi_0 M) \to \pi_i(N \otimes_R M)$$

Then we get

$$\pi_i(N \otimes_R M) = \pi_i(N \otimes_{\pi_0 R} \pi_0 R \otimes_R M) = \mathrm{Tor}_i^{\pi_0 R}(N, \pi_0 R \otimes_R M)$$

□

There are other equivalent conditions; see [4] Theorem 2.5.2 for the full result.

8.2 Derived Quasi-coherent Sheaves

Let $S = \mathrm{Spec}(R)$ be an affine derived scheme. Define the stable ∞-category of quasi-coherent sheaves on S by

$$\mathrm{Qcoh}(S) := \mathrm{Mod}_R$$

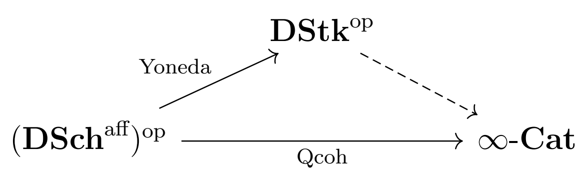
This defines a presheaf of ∞-categories

$$(\mathbf{DSch}^{\mathrm{aff}})^{\mathrm{op}} \to \infty{-}\mathbf{Cat}$$

where $f : \mathrm{Spec}(R') \to \mathrm{Spec}(R)$ induces

$$f^* : \mathrm{Mod}_R \to \mathrm{Mod}_{R'}, \quad M \mapsto M \otimes_R R'$$

To define derived quasi-coherent sheaves in general, we want to take the right Kan extension along the Yoneda embedding:



This gives,

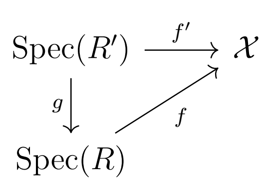
Definition 8.3. Let $\mathcal{X}$ be a derived stack. The stable ∞-category of quasi-coherent shaves on $\mathcal{X}$ is

$$\mathrm{Qcoh}(\mathcal{X}) := \lim_{S \to \mathcal{X}} \mathrm{Qcoh}(S)$$

where the limit is taken over all morphisms from $S \in \mathbf{DSch}^{\mathrm{aff}}$.

To unpack this data, note that $\mathcal{F} \in \mathrm{Qcoh}(X)$ determines:

1. An R-module $f^*(\mathcal{F})$ for each $f : \mathrm{Spec}(R) \to \mathcal{X}$
2. For every commutative triangle



   an isomorphism $g^*(f^*\mathcal{F}) \to (f')^*\mathcal{F}$ in $\mathrm{Qcoh}(\mathrm{Spec}(R'))$.

3. A homotopy coherent system of compatibilities between these isomorphisms

Remark. Actually, the presheaf on $\mathbf{DSch}^{\mathrm{aff}}$ is valued in symmetric monoidal ∞-categories, and you can check that the right Kan extension provides a lift (since the forgetful functor from symmetric monoidal ∞-categories to ∞-categories preserves and detects limits). Thus, $\mathrm{Qcoh}(\mathcal{X})$ is always canonically symmetric monoidal.

In the case that X is in particular a derived scheme, we have a slightly nicer description of quasi-coherent sheaves.

Proposition 8.4. *If $X \in \mathbf{DSch}$, then*

$$Qcoh(X) \xrightarrow{\sim} \lim_{U \hookrightarrow X} Qcoh(U)$$

where the limit is over all open immersions $U \to X$ from $U \in \mathbf{DSch}^{aff}$.

Proof. The goal is to reduce to the case that X is affine, which follows directly from properties of modules.

The key data we are given is an affine Zariski cover $\{X_\alpha \hookrightarrow X\}_\alpha$. Since Qcoh was constructed as a right Kan extension, it sends colimits of derived stacks to limits, so we have

$$\mathrm{Qcoh}(X) = \lim_{n \in \Delta} \mathrm{Qcoh}(\check{C}(\tilde{X}/X)_n) = \lim_{n \in \Delta} \mathrm{Qcoh}(\tilde{X} \times_X ... \times_X \tilde{X})$$

(we have switched Qcoh with $\lim_{n \in \Delta}$ in the effective epimorphism condition of the Zariski cover).

Letting $U \to X$ be any open immersion from an affine, we have a pull-back Zariski cover $\{U_\alpha \hookrightarrow U\}_\alpha$ for $U_\alpha = X_\alpha \times_X U$. Thus, we similarly have

$$\mathrm{Qcoh}(U) = \lim_{n \in \Delta} \mathrm{Qcoh}(\check{C}(\tilde{U}/U)_n) = \lim_{n \in \Delta} \mathrm{Qcoh}(\tilde{U} \times_U ... \times_U \tilde{U})$$

Thus, it suffices to show for all n that the following is an equivalence:

$$\mathrm{Qcoh}(\check{C}(\tilde{X}/X)_n) \to \lim_{U \hookrightarrow X} \mathrm{Qcoh}(\check{C}(\tilde{X}/X)_n \times_X U) = \lim_{U \hookrightarrow X} \mathrm{Qcoh}(\tilde{X} \times_X ... \times_X \tilde{X} \times_X U)$$

(since $\tilde{X} \times_X ... \times_X \tilde{X} \times_X U = \tilde{U} \times_U ... \times_U \tilde{U}$) which is equivalent to showing that

$$\mathrm{Qcoh}(\check{C}(\tilde{X}/X)_n) \xrightarrow{\sim} \lim_{U \hookrightarrow \check{C}(\tilde{X}/X)_n} \mathrm{Qcoh}(U)$$

is an equivalence.

We want to say that $\check{C}(\tilde{X}/X)_n = \tilde{X} \times_X ... \times_X \tilde{X}$ is affine, but it is the intersection of affines. Since each X_α is affine, their pairwise intersections are open subschemes in affine derived schemes, and so are separated. Then they admit affine Zariski covers where each of the pairwise intersections are affine. This reduces to the affine case. □

8.3 Faithfully Flat Descent

Let us recall what a descent datum for classical quasi-coherent sheaves is. The idea is that we want local data of a sheaf and compatibility conditions so we can glue them together. We make this precise.

Given a scheme S and a family of morphisms $\{f_i : S_i \to S\}_i$, a descent datum is $(\mathcal{F}_i, \varphi_{ij})_{ij}$, where

- $\mathcal{F}_i$ is a quasi-coherent sheaf on S_i

- $\varphi_{ij} : \mathrm{pr}_0^*\mathcal{F}_i \to \mathrm{pr}_1^*\mathcal{F}_j$ is an isomorphism of quasi-coherent sheaves on $S_i \times_S S_j$
- For every triple i, j, k, the following *cocycle condition* diagram commutes

$$\begin{array}{ccc} \mathrm{pr}_0^*\mathcal{F}_i & \xrightarrow{\mathrm{pr}_{02}^*\varphi_{ik}} & \mathrm{pr}_2^*\mathcal{F}_k \\ & {\scriptstyle \mathrm{pr}_{01}^*\varphi_{ik}} \searrow \quad \nearrow {\scriptstyle \mathrm{pr}_{02}^*\varphi_{jk}} & \\ & \mathrm{pr}_1^*\mathcal{F}_j & \end{array}$$

There is also a natural notion of morphisms of descent data (see Stacks), making them a category. We can check that any $\mathcal{F} \in \mathrm{Qcoh}(S)$ defines a descent datum on a family via pulling back, defining a functor from $\mathrm{Qcoh}(S)$ to the category of descent data.

Fixing $F \in \mathrm{Qcoh}(S)$, we have a trivial descent datum $(\mathcal{F}, \mathrm{id})$ on $\{S \xrightarrow{id} S\}$, and a canonical descent datum on $\{S_i \to S\}$ by pulling back, called $(\mathcal{F}|_{S_i}, \mathrm{can})$.

Now, given any descent datum $(\mathcal{F}_i, \varphi_{ij})$, we say it is effective *effective* if there is some $\mathcal{F} \in \mathrm{Qcoh}(S)$ such that $(\mathcal{F}_i, \varphi_{ij}) \cong (\mathcal{F}|_{S_i}, \mathrm{can})$.

The *descent condition* is then

1. Every descent datum is effective
2. The functor from $\mathrm{Qcoh}(S)$ to the category of descent data (with respect to the covering) is fully faithful

This is nearly the same as positing an equivalence of categories! To generalize this notion to the derived setting, we make some tweaks

- The descent datum will require much higher coherency than the cocycle condition
- It is not natural to ask for isomorphism

Let us go directly to the statement of fpqc descent for derived quasi-coherent sheaves:

Theorem 8.5. *Given a derived scheme S and a fpqc covering family $\{f_\alpha : S_\alpha \to S\}_\alpha$, let $\tilde{S} = \coprod_\alpha S_\alpha$ and $f : \tilde{S} \to S$. Then the canonical functor*

$$Qcoh(S) \to \lim_{n \in \Delta} Qcoh(\check{C}(\tilde{S}/S)_n)$$

is an equivalence.

The target object is called the *totalization*. Recall that $\check{C}(\tilde{S}/S)_n = \tilde{S} \times_S \ldots \times_S \tilde{S}$. We saw this last time, but let's take a moment to get some intuition. Given $\mathcal{F} \in \mathrm{Qcoh}(S)$, the above functor defines:

- in degree 0: an object of $\mathrm{Qcoh}(\tilde{S})$, i.e., an object of $\mathrm{Qcoh}(S_\alpha)$ for each α
- in degree 1: an object of $\mathrm{Qcoh}(\tilde{S} \times_S \tilde{S})$, i.e., an object of $\mathrm{Qcoh}(S_\alpha \times_S S_\beta)$ for each α, β
- in higher degree: higher coherence data

Thus, via the coface and codegeneracy maps, the structure of the limit of this cosimplicial set encapsulates the cocycle condition along with higher compatibility results.

Before we prove this, we build a bit of general theory.

8.4 Monadicity and the Proof

We now give some background on (co)-monadicity while simultaneously sketching the proof of fpqc descent for quasi-coherent sheaves.

See HA 4.7 and here for details.

Definition 8.6. Let $\mathcal{C}$ be an ∞-category. A *monad* T on $\mathcal{C}$ is an algebra object of $\mathrm{Fun}(\mathcal{C}, \mathcal{C})$ (with respect to the composition monoidal structure). Let $\mathrm{Alg}_T(\mathcal{C})$ be the ∞-category of (left) T-algebras in $\mathcal{C}$.

Let's unpack this structure. A monad on $\mathcal{C}$ consists of a functor $T : \mathcal{C} \to \mathcal{C}$ with maps $\mathrm{id} \to T$ and $T \circ T \to T$ satisfying unit/associativity conditions up to homotopy.

A T-algebra is an algebra object $C \in \mathcal{C}$ with a structure map $T(C) \to C$ comaptible with the algebra structure on T, again up to coherent homotopy.

Example. An adjunction has a naturally associated monad (and co-monad).

There are of course dual notions of co-monads and co-algebras, which will turn out to be more relevant in our case.

Observation. *The totalization* $\lim_{n \in \Delta} Qcoh(\check{C}(\tilde{S}/S)_n)$ *can be identified with the* ∞*-category of co-algebras in* $Qcoh(\tilde{S})$ *over the co-monad associated to the adjunction*

$$f^* : Qcoh(S) \leftrightarrow Qcoh(\tilde{S}) : f_*$$

Let's spell this out a bit more. The co-monad is the functor $f^* f_* : \mathrm{Qcoh}(\tilde{S}) \to \mathrm{Qcoh}(\tilde{S})$. Given an object in the totalization, we want the degree zero piece, $\mathcal{F} \in \mathrm{Qcoh}(\tilde{S})$, to be the algebra object. The map $\mathcal{F} \to f^* f_* \mathcal{F}$ comes from the rest of the structure of the totalization (need to figure this out).

Now, given an adjunction $F : \mathcal{C} \leftrightarrow \mathcal{D} : G$ with associated monoid T, the left action of T on G yields a functor $G' : \mathcal{D} \to \mathrm{Alg}_T(\mathcal{C})$ such that G is the composition of G' with the forgetful functor $\mathrm{Alg}_T(\mathcal{C}) \to \mathcal{C}$.

Thus, the following results implies the desired equivalence.

Theorem 8.7 (Barr-Beck-Lurie)**.** *Given a left adjoint* $F : \mathcal{C} \to \mathcal{D} : G$*, the induced functor* $\mathcal{D} \to Alg(\mathcal{C})$ *is an equivalence if and only if*

- *G is conservative (reflects equivalences)*
- *If V is a G-split simplicial object of $\mathcal{D}$, then V admits a colimit in D that is preserved by G.*

Splitting is a technical condition about general simplicial objects in categories. See Stacks for the definition.

In our case, the first condition is that f^* is conservative, which follows immediately since f is faithfully flat. The second condition is much more complicated, and relies nontrivially on the homotopy theory of spectra.

9 April 11

See website for Kevin Chang's lecture notes.

10 April 18

Today we dive into the Bhatt paper, which we advertised as one terminus of this seminar. The appeal of this paper is that it uses infinity-categorical derived algebraic geometry to prove concrete results about algebraic spaces/schemes.

First we build up a bit more general theory of derived algebraic geometry and provide a brief introduction to algebraic spaces. Then we present a powerful result giving an equivalence between hom sets and various functor categories. From there we state and outline the proofs of results related to: formal points, gluing, and products.

10.1 Algebraic Spaces

We say the bare minimum about algebraic spaces. The intuition is that while schemes are affine schemes glued together in the Zariski topology, algebraic spaces are affine schemes glued together in the étale topology.

Definition 10.1. Given a scheme $S \in \mathbf{Sch}_{fppf}$, an *algebraic space* over S is a presheaf

$$F : (\mathrm{Sch}/S)^{\mathrm{op}}_{fppf} \to \mathbf{Set}$$

satisfying

1. F is a sheaf
2. $F \to F \times F$ is representable
3. There exists a scheme $U \in (\mathrm{Sch}/S)_{fppf}$ and a surjective étale map $h_U \to F$

10.2 More Derived Preliminaries

Definition 10.2. Given an algebraic space X, let $D(X)$ be the *quasi-coherent derived category of X*, i.e., the subcategory of complexes of $\mathcal{O}_X$-modules whose cohomology sheaves are quasi-coherent, viewed as a symmetric stable ∞-category.

Recall that a complex $\mathcal{E}^\bullet$ of $\mathcal{O}$-modules is strictly perfect if $\mathcal{E}^i$ is zero for all but finitely any i and each $\mathcal{E}^i$ is a direct summand of a finite free $\mathcal{O}$-module.

A complex $\mathcal{E}^\bullet$ of $\mathcal{O}_X$ modules is perfect if there exists an open covering $X = \bigcup U_i$ and strictly perfect complexes $\mathcal{E}_i^\bullet$ of $\mathcal{O}_{U_i}$ modules with $\mathcal{E}_i^\bullet \to \mathcal{E}_i^\bullet|_{U_i}$ a quasi-isomorphism.

Definition 10.3. Let $D_{\mathrm{perf}}(X) \subset D(X)$ be the full subcategory of *perfect complexes*, i.e., objects of $D(X)$ that can be represented by perfect complexes.

An object is *compact* if its representable functor commutes with colimits.

An object A is *dualizable* if it admits a dual object $A^\vee$ with evaluation and unit maps

$$\mathrm{ev}_A : A^\vee \otimes A \to 1, \quad i_A : 1 \to A \otimes A^\vee$$

satisfying the natural identities after passing to the homotopy category.

Lemma 10.4. *For an object $K \in D(X)$, the following are equivalent: $K \in D_{perf}(X)$, K is compact, and K is dualizable.*

Rough idea of proof: prove for modules over discrete ring R, and then use connectivity and gluing to generalize.

Lemma 10.5. *The $Ind(D_{perf}(X)) = D(X)$.*

Finally, some notation: $\mathrm{Fun}_\otimes(-,-)$ is the category of symmetric monoidal functors, $\mathrm{Fun}^L_\otimes(-,-)$ is the category of co-continuous symmetric monoidal functors, $\mathrm{Fun}^L_{\otimes,c}(-,-)$ is the category of co-continuous symmetric monoidal functors preserving compact objects.

10.3 The Equivalence Results

We prove a key theorem, which is the Tannaka duality of the title.

Lemma 10.6. *There are natural identifications*

$$Fun_\otimes(D_{perf}(X), D_{perf}(S)) \cong Fun^L_{\otimes,c}(D_{perf}(X), D_{perf}(S)) \cong Fun^L_\otimes(D_{perf}(X), D_{perf}(S))$$

Proof. The first follows from $D(X) = \mathrm{Ind}(D_{\mathrm{perf}}(X))$ (so every functor on left uniquely induces a co-continuous functor on the right, and every functor on the right restricts to one on the left).

The second follows because preserving compact objects $\iff$ preserving D_{perf} $\iff$ preserving dualizable objects, which is true for all symmetric monoidal functors. □

Theorem 10.7. *If X and S are qcqs algebraic spaces, then there is an isomorphism*

$$Hom(S, X) \cong Fun_\otimes(D_{perf}(X), D_{perf}(S))$$

induced by pullback.

Proof sketch of full faithfulness. Fact: $\mathrm{Hom}(-, X)$ and $\mathrm{Fun}_\otimes(D_{\mathrm{perf}}(X), -)$ are stacks for the Zariski topology, i.e., they satisfy descent. So we can reduce to the case S is affine, meaning $S \to X$ is quasi-affine.

In the affine case, X, S correspond (via Yoneda) to functors $\mathrm{CAlg}^{\mathrm{cn}} \to \hat{\mathcal{S}}$ (infinity category of not-necessarily small spaces; set theoretic concern). So we want to show full faithfulness of

$$\mathrm{Map}_{\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \hat{\mathcal{S}})}(S, X) = \mathrm{Fun}_\otimes(D(X), D(Y))$$

Lurie shows this (see Proposition 3.3.11) with the assumptions that X is quasi-geometric and $D(X)$ is presentable.

The proof is very homotopy theoretic: show that any pair of points $\eta, \eta' \in X(R)$ yield maps $f, f' : Y \to X$ for which every $\mathrm{QCoh}(X)$-linear symmetric monoidal transformation from f^* to f'^* is an equivalence, and then show homotopy equivalence of homotopy fibers in a convenient commutative diagram. □

The proof of essential surjectivity is much more involved; we give a sketchier sketch.

Proof sketch of essential surjectivity. The idea is to progressively build étale hypercovers of X and S by quasi-affine schemes (generalizations of Čech nerve), and a map between them that is carried to a given F. To show compatibility, we need to prove that $F : \mathrm{QAff}/X \to \mathrm{QAff}/S$ preserves étale morphisms, finite limits, étale surjections, etc. To do so, construct a right adjoint G to F and prove various symmetric-monoidal properties. □

In the case of schemes, we get underived results.

Theorem 10.8. *For qcqs schemes S and X,*

$$Hom(S, X) \simeq Fun^L_{\otimes}(QCoh(X), QCoh(S))$$

Proof idea. For full faithfulness, take an affine open cover and use the nice properties of j_* and j^* for open immersions j.

For essential surjectivity, the affine case is straight forward, and then check that they glue together.

(See Theorem 3.1). □

Corollary 10.9. *Let X and S be qcqs schemes. Assume X has enough vector bundles.*

$$Hom(S, X) \simeq Fun^L_{\otimes}(Vect(X), Vect(S))$$

Proof reference. (See Corollary 3.2) □

10.4 Formal Points

Using the previous theory, we obtain the following result, which is the *algebrization* of a formal point.

Theorem 10.10. *Let X be a qcqs algebraic space, and let A be a ring with an ideal I such that A is I-adically complete. Then $X(A) \cong \lim_n X(A/I^n)$ via the natural map.*

We first prove a lemma.

Lemma 10.11. *If $A = \lim A/I^n$ then $D_{Perf}(A) \cong \lim D_{Perf}(A/I^n)$.*

Proof sketch. Full faithfulness follows since $K \otimes_A -$ commutes with limits, by compactness of $K \in D_{\mathrm{perf}}(A)$ and adjunction. So $\lim K \otimes_A A/I^n \cong K \otimes_A (\lim A/I^n) \cong K \otimes_A A \cong K$.

For essential surjectivity, given $\{K_n\}_n \in \lim D_{\mathrm{perf}}(A/I^n)$ use induction to construct a representative $\{P_n\}_n$ with P_n a complex of finite projective A/I^n modules with $P_{n+1} \to P_n$ inducing $P_{n+1}/I^n \cong P_n$. □

Proof of Theorem. Let $\{\epsilon_n : \mathrm{Spec}(A/I^n) \to X\}$ be a compatible system of maps. These induce via pullback a compatible system of exact symmetric monoidal functors $\{D_{\mathrm{Perf}}(X) \to \lim D_{\mathrm{Perf}}(A/I^n)\}$, i.e.,

$$F : D_{\mathrm{Perf}}(X) \to \lim D_{\mathrm{Perf}}(A/I^n)$$

symmetric monoidal. By the lemma, we have $F \in \mathrm{Fun}_{\otimes}(D_{\mathrm{Perf}}(X), D_{\mathrm{Perf}}(A))$. By our big equivalence theorem, there is a unique map $\epsilon : \mathrm{Spec}(A) \to X$ with $F \simeq \epsilon^*$. Since F extends each ϵ_n^*, so ϵ extends each ϵ_n, providing an inverse. □

We can give a result about colimits of spaces, not just their points.

Proposition 10.12. *Let X be a qcqs algebraic space, and let $\{X_i\}$ be a diagram of qcqs X-spaces with $D(X) \cong \lim D(X_i)$. Then $D_{perf}(X) \cong \lim D_{perf}(X_i)$ and $X = colim\ X_i$.*

Proof. Duals are computed pointwise in the limit, so dualizable objects correspond to limits of dualizable objects.

This gives a natural equivalence $D_{\text{perf}}(X) \to \lim D_{\text{perf}}(X_i)$ which is symmetric monoidal and exact, so by our equivalence theorem it corresponds to an equivalence $X \to \text{colim } X_i$. □

10.5 Gluing

Now we give a formal gluing result.

Proposition 10.13. *Let X be a qcqs algebraic space with constructible closed subspace $Z \subset X$. Let $\pi : Y \to X$ be a qcqs map of algebraic spaces such that $\pi^* : D_Z(X) \simeq D_{\pi^{-1}(Z)}(Y)$ is an equivalence. Let $U = X \setminus Z$ and $V = Y \setminus \pi^{-1}(Z)$. Then*

$$\Phi : D(X) \to D(Y) \times_{D(V)} D(U)$$

is an equivalence.

Proof sketch. Objects in $D(Y) \times_{D(V)} D(U)$ correspond to (K, L, η) for $K \in D(Y)$, $L \in D(U)$ and $\eta : j^*K \simeq \pi^*L$.

To check full faithfulness, suffices to check locally on X. Applying the projection formula, suffices to check that the following is a cofiber/fiber sequence

$$\mathcal{O}_X \xrightarrow{a} \pi_*\mathcal{O}_Y \oplus j_*\mathcal{O}_U \xrightarrow{b} \pi_*j_*\mathcal{O}_V$$

where b is induced by η (this is the ∞-categorical ingredient).

To do so, apply $\underline{\Gamma}_Z(-) \oplus (- \otimes j_*\mathcal{O}_U)$, which is a conservative functor. What you get is clearly isomorphism/inclusion/projection (using base change and the assumption).

For essential surjectivity, construct inverse

$$\Psi : D(Y) \times_{D(V)} D(U) \to D(X), \quad (K, L, \eta) \mapsto \text{fib}\,(\pi_*K \oplus j_*L \to (\pi \circ j)_*j^*K)$$

again using η. Here Ψ is the right adjoint to Φ, and we can check that the unit/counit are naturally isomorphic to the identity using the given isomorphism. □

Corollary 10.14. *In the proposition, Φ induces*

$$D_{perf}(X) \to D_{perf}(Y) \times_{D_{perf}(V)} D_{perf}(U), \quad Vect(X) \to Vect(Y) \times_{Vect(V)} Vect(U)$$

and if π is flat, then Φ induces

$$QCoh(X) \to QCoh(Y) \times_{QCoh(V)} QCoh(U)$$

and

$$\begin{array}{ccc} V & \xrightarrow{j} & Y \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi} \\ U & \xrightarrow{j} & X \end{array}$$

is a pushout in qcqs algebraic spaces.

Proof sketch. The D_{Perf} statement follows from the previous proposition.

The Vect statement follows from showing that the essential image of $\text{Vect}(S) \to D(S)$ coincides with dualizable objects in $D^{\leq 0}_{\text{perf}}(S)$ for S a qcqs space. Then we show K is connective iff $\pi^* K$ and $j^* K$ is connective, which follows from the isomorphism assumption.

For the QCoh statement, use flatness and vanishing cohomology sheaves to show that the desired restriction exists and gives an equivalence.

And the pushout diagram follows from the previous proposition. □

Remark. This proposition/corollary requires ∞-categories; faithfullness fails if we work with homotopy categories. This is because the homotopy categories forget η, the data of how objects over Y and U are identified over V.

Example. Consider the following counter example in the case of classical categories.

Let p be prime, and

$$A = \text{colim}\left(\mathbf{Z}_p[x] \to \mathbf{Z}_p\left[\frac{x}{p}\right] \to \mathbf{Z}_p\left[\frac{x}{p^2}\right] \to \ldots\right)$$

the ring of germs of bounded algebraic functions at 0 on the p-adic unit disc. Equivalently, $A \subset \mathbf{Q}_p[x]$ consists of $f(x)$ with $f(0) \in \mathbf{Z}_p$. Can check that $A/p^n \cong \mathbf{Z}/p^n$ and $\hat{A} = \mathbf{Z}_p$. Let

$$X = \text{Spec}(A), \quad Y = \text{Spec}(\hat{A}), \quad Z = \text{Spec}(A/p)$$

and U, V open complements as before. We will show that

$$\text{QCoh}(X) \to \text{QCoh}(Y) \times_{\text{QCoh}(V)} \text{QCoh}(U)$$

is not faithful.

Let $M = A/(x) \in \text{QCoh}(X)$. Consider the map

$$\eta : M \to (M \otimes_A \hat{A}) \oplus M\left[\frac{1}{p}\right] \in \text{QCoh}(Y) \times_{\text{QCoh}(V)} \text{QCoh}(U)$$

Note that $M \otimes_A \hat{A} \cong \mathbf{Z}_p$ and $M\left[\frac{1}{p}\right]$ are both p-torsion free, but $0 \neq \frac{x}{p^n} \in M$ is p-torsion, so it must be in the kernel of the map. Considering this as the functor on $\text{Hom}_A(A, M)$ proves the failure of faithfulness.

10.6 Products

We state and briefly give an idea of the last main result: algebrization of products.

Theorem 10.15. *Fix a set I of rings $\{A_i\}_{i \in I}$ with product $A := \prod_i A_i$ and a qcqs algebraic space X. Then $X(A) \cong \prod_i X(A_i)$ via the natural map.*

The proof in the case of algebraic spaces is much more difficult than the earlier results, though it is still based on the equivalence theorem. The proof involves bounding Nisnevich covers (étale covers admitting sections over a constructible stratification).

The proof for schemes is more elementary, and is based on the non-derived equivalence result for schemes. It involves more classical scheme theory, but also uses techniques of bounding numbers of generators.

11 April 25

See website for notes from Roy Magen's talk.

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