Some features of derived algebraic geometry

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April 11, 2022

1 Recall

I'll be following (parts of) Sections 2 and 3 of Toën's survey "Derived Algebraic Geometry." Some of the ∞-stuff will be loose, as Toën puts it. For technical stuff, you can check the references in the survey.

I'll review Toën's definition of a derived scheme because he does it a bit differently from Amal. We start by defining the ∞-category **dRgSp** of **derived ringed spaces**. The objects of this ∞-category are pairs $(X, \mathcal{O}_X)$, where X is a space and $\mathcal{O}_X \in \mathbf{sComm}(X)$ is a stack of derived rings on X. The mapping spaces are given by

$$Map((X, \mathcal{O}_X), (Y, \mathcal{O}_Y)) = \coprod_{u: X \to Y} Map_{\mathbf{sComm}(Y)}(\mathcal{O}_Y, u_* \mathcal{O}_X).$$

Just as we can take π_0 of a simplicial ring, any derived ringed space $(X, \mathcal{O}_X)$ admits a **truncation** $t_0(X) = (X, \pi_0(\mathcal{O}_X))$, its underlying ringed space.

We can then define the sub-∞-category $\mathbf{dRgSp}^{loc}$ of **derived locally ringed spaces** whose objects are derived ringed spaces $(X, \mathcal{O}_X)$ such that $t_0(X)$ is a locally ringed space and whose mapping spaces consist of morphisms whose truncations are local morphisms. We then define the ∞-category of **derived schemes** to be the full sub-∞-category of $\mathbf{dRgSp}^{loc}$ consisting of objects $(X, \mathcal{O}_X)$ satisfying:

(i) $t_0(X)$ is a scheme.

(ii) For all i, the sheaf of $\pi_0(\mathcal{O}_X)$-modules $\pi_i(\mathcal{O}_X)$ is quasi-coherent.

We denote the ∞-category of derived schemes by **dSch**. We have an adjunction

$$t_0 : \mathbf{dSch} \rightleftharpoons \mathbf{Sch} : i,$$

where i treats a scheme as a derived scheme. Moreover, we have a map $t_0(X) \to X$ for any X; this is analogous to the closed embedding $Y_{red} \hookrightarrow Y$ for a scheme.

2 Fiber products of derived schemes

In algebraic geometry, we have a lot of fiber products. However, sometimes, they're really stupid; for example, if $X \hookrightarrow Y$ is a closed embedding, then $X \times_Y X \cong X$. One key feature of derived schemes is that their intersections retain a lot more information.

We will describe fiber products of derived schemes on the level of affines. Suppose we have a diagram of simplicial rings $A \leftarrow C \to B$. The pushout $D := A \otimes_C^{\mathbb{L}} B$ can be computed by replacing

B with a **cellular** C-algebra (i.e. a cofibrant replacement) and then taking the levelwise tensor product. In the case that C is an ordinary ring, a cellular C-algebra is a simplicial C-algebra whose components are all polynomial algebras in C. We globalize this to get fiber products of derived schemes.

When A, B, C are ordinary rings, D has homotopy $\pi_n(D) \cong \mathrm{Tor}_n^C(A, B)$. Thus, when $X \to S \leftarrow Y$ are schemes, the fiber product $Z := X \times_S Y$ has $\pi_i(\mathcal{O}_Z) \cong \mathcal{T}\wr\nabla_i^{\mathcal{O}_S}(\mathcal{O}_X, \mathcal{O}_Y)$. Thus, when $\mathcal{O}_X$ and $\mathcal{O}_Y$ are Tor-independent over $\mathcal{O}_S$, the derived fiber product agrees with the ordinary one. Otherwise, they differ. This harkens back to Serre's intersection formula: if $X \to S \leftarrow Y$ are closed subschemes intersecting properly and S is regular, then the intersection multiplicity of X and Y at a generic point s of $X \cap Y$ is

$$\sum_{i \geqslant 0} (-1)^i \mathrm{length}_{\mathcal{O}_{S,s}} \mathrm{Tor}_i^{\mathcal{O}_{S,s}}(\mathcal{O}_{X,s}, \mathcal{O}_{Y,s}).$$

According to Toën, Serre's intersection formula is the origin of derived algebraic geometry.

One case that might be interesting is that of self-intersections. Suppose $Y \hookrightarrow X$ is a closed embedding of schemes, and consider the derived scheme $Y \times_X Y$. The underlying scheme $t_0(Y \times_X Y)$ is just Y, but the derived scheme might have nontrivial higher information. For convenience, suppose that $Y \hookrightarrow X$ is a regular embedding, i.e. the conormal sheaf $\mathcal{I}/\mathcal{I}^2$ is a vector bundle, denoted $\mathcal{N}^\vee$. Shrinking X, we may consider $X = \operatorname{Spec} A$ and $Y = \operatorname{Spec} A/I$, where $I = (f_1, \ldots, f_r)$ is generated by a regular sequence. We want to compute the derived tensor product $B := A/I \otimes_A^{\mathbb{L}} A/I$. To do this, we replace A/I by a simplicial ring $K(A, f_*)$ (I think the K stands for Koszul), which is obtained by freely adding 1-simplices h_i to A such that $d_0(h_i) = 0$ and $d_1(h_i) = f_i$, i.e. homotopies between the f_i and 0. Since $f_1, \ldots, f_r$ is a regular sequence, the map $K(A, f_*) \to A/I$ can be shown to be an equivalence. Moreover, $K(A, f_*)$ is cellular because it is obtained by freely adjoining things. Now $B = A/I \otimes_A^{\mathbb{L}} A/I \simeq A/I \otimes_A K(A, f_*)$, so $\pi_*(B) \cong \bigoplus_{i \geqslant 0} \Lambda^i(I/I^2)[i]$. In characteristic 0, it can be shown that $B \simeq Sym_{A/I}(I/I^2[1])$, so that $Z \simeq \mathbf{Spec}(Sym_{A/I}(I/I^2[1]))$ is like the total space of the normal bundle, except it's shoved into higher homotopy.

3 Base change

Recall from last time that Amal defined the **quasi-coherent derived ∞-category** of a derived scheme X:

$$L_{qcoh}(X) := \varprojlim_{U \subset X \text{ affine}} L_{qcoh}(U).$$

Given a morphism $f : X \to Y$, there are functors

$$f^* : L_{qcoh}(Y) \rightleftharpoons L_{qcoh}(X) : f_*.$$

Now suppose we are given a cartesian diagram of derived schemes.

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow{g} & Y \end{array}$$

There is a natural transformation $h : g^* f_* \implies f'_* g'^*$ between functors $L_{qcoh}(X) \to L_{qcoh}(Y')$. When X, Y, Y' are sufficiently nice ordinary schemes and g is flat, X' is an ordinary scheme, and the flat base change theorem says that h is an equivalence. The base change theorem in derived algebraic geometry is an extension of flat base change: if all the derived schemes in the diagram are qcqs, then h is an equivalence. In contrast, for a cartesian diagram of ordinary schemes, base change is not an isomorphism in general. This reflects the fact that a cartesian diagram of ordinary schemes is not necessarily a cartesian diagram of derived schemes.

4 The cotangent complex

Given a morphism of schemes $f : X \to Y$, the cotangent complex $\mathbb{L}_{X/Y} \in D^-_{qcoh}(X)$ governs the deformation theory of f. We always have $H^0(\mathbb{L}_{X/Y}) \cong \Omega_{X/Y}$. Thus, the cotangent complex is a higher version of the sheaf of differentials. Some properties of the cotangent complex include the following:

(1) f is smooth of relative dimension r iff f is locally of finite presentation and $\mathbb{L}_{X/Y}$ is concentrated in degree 0 and locally free of rank r.

(2) If f is a closed embedding and I is the ideal sheaf of X, then $H^1(\mathbb{L}_{X/Y}) \cong I/I^2$. If f is a regular embedding, then $\mathbb{L}_{X/Y} \simeq I/I^2[1]$.

(3) For $X \xrightarrow{f} Y \xrightarrow{g} Z$, there is a distinguished triangle $Lf^*\mathbb{L}_{Y/Z} \to \mathbb{L}_{X/Z} \to \mathbb{L}_{X/Y}$.

(4) If $h : X \to Z$ factors as $X \xrightarrow{f} Y \xrightarrow{g} Z$ with f a closed embedding and g a smooth morphism, then $\tau_{\geqslant -1}\mathbb{L}_{X/Z} \simeq [I/I^2 \to f^*\Omega_{Y/Z}]$, where I is the ideal sheaf for f and the map is induced by $d : \mathcal{O}_Y \to \Omega_{Y/Z}$. If f is additionally a regular embedding (i.e. h is an lci morphism), then $\mathbb{L}_{X/Z} \simeq \tau_{\geqslant -1}\mathbb{L}_{X/Z} \simeq [I/I^2 \to f^*\Omega_{Y/Z}]$.

We get the following familiar exact sequences from these axioms:

- For $X \xrightarrow{f} Y \xrightarrow{g} Z$, there is an exact sequence

$$f^*\Omega_{Y/Z} \to \Omega_{X/Z} \to \Omega_{X/Y} \to 0.$$

- For $X \xrightarrow{f} Y \xrightarrow{g} Z$ with f a closed embedding with ideal sheaf I, there is an exact sequence

$$I/I^2 \to f^*\Omega_{Y/Z} \to \Omega_{X/Z} \to 0.$$

  This sequence is exact on the left if g and $g \circ f$ are smooth.

To construct the cotangent complex of a map of schemes, we first construct the cotangent complex of a map of rings. For a map $A \to B$, the cotangent complex $\mathbb{L}_{B/A} \in D^-(B)$ is defined as the left derived functor of the Kähler differentials functor. What this means in the simplicial setting is that instead of taking Kähler differentials of B, we first take a simplicial resolution (an equivalence from a cofibrant simplicial ring) $\epsilon : P_\bullet \to B$ and then apply Kähler differentials, so that we don't lose information. We then let $\mathbb{L}_{B/A}$ be the normalized complex associated to the simplicial B-module $\Omega_{P_\bullet/A} \otimes_{P_\bullet} B$, which can be shown to be independent of simplicial reoslution. In the model category of simplicial commutative A-algebras, the cofibrant objects are those whose components are polynomial algebras over A (it makes sense because polynomial algebras have a lifting property).

For a set S, let $A[S]$ denote the free polynomial algebra whose generators are elements of S. For any $A \to B$, there is a standard simplicial resolution $P_\bullet \to B$ with $P_0 = A[B]$, $P_1 = A[A[B]]$, etc. Here, all the face/degeneracy/augmentation maps collapse consecutive A's or the A and B. Since this resolution is canonical, we can globalize it to a morphism of schemes $f : X \to Y$. We take $\mathbb{L}_{X/Y}$ to be the complex of sheaves associated to the complex of presheaves $U \mapsto \mathbb{L}_{\mathcal{O}_X(U)/f^{-1}\mathcal{O}_Y(U)}$, where $\mathbb{L}$ is computed with the standard resolution.

While the cotangent complex (or at least its first few terms) is useful in ordinary algebraic geometry (e.g. deformation theory, virtual fundamental classes), it's unclear what it means geometrically, outside the 0th term $H^0(\mathbb{L}_{X/Y}) \cong \Omega_{X/Y}$. However, in derived algebraic geometry, the cotangent complex has a nice interpretation in terms of derivations.

Given a derived scheme X and an object $M \in L_{qcoh}(X)$ concentrated in nonpositive degrees, we can form a derived scheme $X[E]$, which is the trivial square-zero extension of X by M. The **space of derivations with coefficients in** M is the mapping space $Map_{X/\mathbf{dSch}}(X[M], X)$. The **absolute cotangent complex** $\mathbb{L}_X$ is an object of $L_{qcoh}(X)$ with the universal property

$$Map_{X/\mathbf{dSch}}(X[M], X) \simeq Map_{L_{qcoh}(X)}(\mathbb{L}_X, M).$$

In other words, $\mathbb{L}_X$ carries the universal derivation on X. We can also define all these things relative to $f : X \to Y$, in which case we get the **relative cotangent complex** $\mathbb{L}_Y := cofib(f^*\mathbb{L}_Y \to \mathbb{L}_X)$.

The cotangent complex lets us define smooth and étale morphisms of derived schemes. A morphism $f : X \to Y$ is **étale** (resp. **smooth**) if it is locally of finite presentation and $\mathbb{L}_{X/Y} \simeq 0$ (resp. $\mathbb{L}_{X/Y}$ is a vector bundle on X).