(unstable) Motivic Homotopy Stuff

inverting $\mathbb{A}^1 \times X \to X$

examples of invariants which are $\mathbb{A}^1$-homotopy:
when X is smooth over a field k

- $K_0(X) \to K_0(X \times \mathbb{A}^1)$ iso ✓
- $CH^\cdot(X) \to CH(X \times \mathbb{A}^1)$ ——
- $H_{et}(X, \mu_\ell) \to H_{et}(X \times \mathbb{A}^1, \mu_\ell)$ for ℓ invertible in k

Rmk: if $\underset{\sim}{I} \times X \to X$ is invertible,
then for any $t \in I$, the map $X \simeq X \times \{t\} \to X \times \underset{\sim}{I}$ is invertible
so if $h : X \times I \to Y$, then any $h_t : X \to Y$

Work over a base scheme S, probably · Noetherian
· finite Krull dimension

Nisnevich topology:

almost a correct description of Nisnevich descent:

Say $P \in PSh(Sm_S)$ has Nis excision if for

$Z \hookrightarrow X$ a closed immersion, and $\begin{matrix} X' \\ \downarrow \\ X \end{matrix}$ an étale nbhd of Z,

the induced square
$$\begin{tikzcd} P(X) \arrow[r] \arrow[d] & P(X \setminus Z) \arrow[d] \\ P(X') \arrow[r] & P(X' \setminus Z) \end{tikzcd}$$
is Cartesian

why we use it:

- fine enough to prove "purity"
- coarse enough for K-theory to be representable

Def $\mathcal{E}_S = Shv_{Nis}(Sm_S)$

$H(S) = $ localisation of $\mathcal{E}_S$ along maps of the form $X \times \mathbb{A}^1 \xrightarrow{\text{projection}} X$ (∞-categorical)

$H(S)$ is a reflective subcategory of $\mathcal{E}_S$

also $L : \mathcal{E}_S \to H(S)$ preserves colimits, also preserves finite products

Remark: if $X \in \mathcal{E}_S$ which is $\mathbb{A}^1$-htpy invariant, so in $H(S)$,

then for any $T \in \mathcal{E}_S$, the map $\mathcal{E}_S(T, X) \to H(S)(LT, LX)$ is an equiv

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homotopy presheaves: if $X \in H(S)_*$, define $\pi_n^{\mathbb{A}^1}(X) = \pi_n \circ H(S)(-, X)$

alternative: $S^n_{mot} := \mathbb{A}^n / (\mathbb{A}^n \setminus 0)$ $\quad \pi_n^{mot}(X) = \pi_0 H(S)_*(S^n_{mot}, X)$

$S^n_{mot} \wedge S^m_{mot} = S^{m+n}_{mot}$ $\quad \pi_n^{\mathbb{A}^1}(X)(T) = \pi_0(H(S)_*(T_+ \wedge S^n, X))$

Let $i: Z \to X$ be a closed embedding of smooth mflds

$H^{\cdot}_Z(X) := H^{\cdot}(X, X|Z)$

Rmk LES $\cdots \to H^n_Z(X) \to H^n(X) \to H^n(X|Z) \to \cdots$

Intuitively $H^{\cdot}_Z(X)$ is $\tilde{H}^{\cdot}(\mathrm{hocofib}(X|Z \to X))$

Excision says that if U is a nbhd of Z in X,

then $H^{\cdot}_Z(U) \to H^{\cdot}_Z(X)$ is invertible

i.e. $U/_{U|Z} \longrightarrow X/_{X|Z}$ is w.equiv

Tubular nbhd thm $\Rightarrow$ there is a nbhd U of Z in X

such that $Z \to U$ is equivalent to $Z \to N_i$

So by excision get an equivalence $Th(N_i) = N_i/_{N_i|Z} = U/_{U|Z} \xrightarrow{\sim} X/_{X|Z}$

So $H^{\cdot}_Z(X) = \tilde{H}^{\cdot}(Th(N_i)) = H^{\cdot - \mathrm{codim}(i)}(Z)$

So get $H^{n-c}(Z) \to H^n(X) \to H^n(X|Z)$

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instead of tubular neighborhood of $Z \to X$, use def to normal cone

instead of excision along a map $(N_i, Z) \to (X, Z)$,

use excision along $(D_Z(X,Z), Z) \to (D(X,Z), Z \times \mathbb{A}^1)$

$$\begin{tikzcd} Z \times \mathbb{A}^1 \arrow[r] \arrow[dr] & D(X,Z) \arrow[d] \\ & \mathbb{A}^1 \end{tikzcd}$$

over 0 this is $Z \to N_i$

elsewhere is $Z \to X$

Thm if $Z \xrightarrow{i} X$ is a closed immersion of smooth schemes over S, then there is an equivalence

$$Th(N_i) \simeq X/X\setminus Z \quad \text{in } H(S)$$

Def a map $(X,Z) \to (X',Z')$ is w.exc if the induced map $X/X\setminus Z \to X'/X'\setminus Z'$ is an $\mathbb{A}^1$-htpy equiv

Rmk if $X \to X'$ is étale, and $Z \to Z'$ is iso

then in the Nis topology,

$$\begin{tikzcd} X\setminus Z \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & X \arrow[d] \\ X'\setminus Z' \arrow[r] & X' \end{tikzcd}$$

, so $X/X\setminus Z \to X'/X'\setminus Z'$ is iso

def Such a map is a Nisnevich map $(X,Z) \to (X',Z')$

Smooth pairs (X,Z) are pairs such that there is a Zar cov $\{U_i \subseteq X\}_i$ and étale maps $(U_i, U_i \cap Z) \to \underbrace{(\mathbb{A}^{n+c}, \mathbb{A}^n)}_{\text{inclusion of affine spaces}}$

why is $(D_t(X,Z), Z) \to (D(X,Z), Z\times\mathbb{A}^1)$ w.exc when (X,Z) is $(\mathbb{A}^n, 0)$

in this case $D(X,Z)$ is a trivial $\mathbb{A}^1$-bundle on $\mathbb{A}^n$

so

$$(D_t, Z) \to (D, Z\times\mathbb{A}^1) \to (\mathbb{A}^n, 0)$$

Lemma if $E \to X$ is a trivial $\mathbb{A}^1$-bundle, then for $p \in X$,

$(E, E_p) \to (X,p)$ is w.exc

Proof

$$\begin{tikzcd} E\setminus E_p \arrow[r] \arrow[d, "s"'] \arrow[dr, phantom, "\ulcorner"] & E \arrow[r] \arrow[d, "s"] & E/E\setminus E_p \arrow[d, "s"] \\ X\setminus p \arrow[r] & X \arrow[r] & X/X\setminus p \end{tikzcd}$$

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Motivic infinite loop spaces

the localization Thm for framed motivic spaces (generalizes this one)