Algebraic Geometry I, Fall 2021

Problem Set 11

Due Monday, November 22, 2021 at 11:59 pm

1. Let $X = \operatorname{Spec}(A)$ be an affine scheme. Show that the functor $\widetilde{(-)}\colon \operatorname{Mod}_A \to \operatorname{Mod}_{\mathcal{O}_X}$ is left adjoint to the global sections functor $\Gamma\colon \operatorname{Mod}_{\mathcal{O}_X} \to \operatorname{Mod}_A$. That is, for $M \in \operatorname{Mod}_A$ and $\mathcal{F} \in \operatorname{Mod}_{\mathcal{O}_X}$, construct a natural isomorphism
$$\operatorname{Hom}_A(M, \Gamma(X, \mathcal{F})) \cong \operatorname{Hom}_{\mathcal{O}_X}(\widetilde{M}, \mathcal{F}).$$
(You don't need to write out the details of why the isomorphism is functorial in M and $\mathcal{F}$, just construct the isomorphism.)

2. Let $X = \operatorname{Spec}(A)$ for an integral domain A with fraction field K. Let $\mathcal{F} = i_{p*}\underline{K}$ where i_p is the inclusion of a point $p \in X$ and $\underline{K}$ is the constant sheaf with value K on $\{p\}$. Show that $\mathcal{F}$ has a natural $\mathcal{O}_X$-module structure, and determine for which p it is quasi-coherent.

3. Consider the following finiteness conditions on an $\mathcal{O}_X$-module $\mathcal{F}$ on a scheme X:
   (i) $\mathcal{F}$ is *of finite type* if for all $p \in X$ there exists an open neighborhood U of p, an integer $n \geq 0$ (depending on p), and an exact sequence $\mathcal{O}_U^n \to \mathcal{F}|_U \to 0$.
   (ii) $\mathcal{F}$ is *of finite presentation* if for all $p \in X$ there exists an open neighborhood U of p, integers $m, n \geq 0$ (depending on p), and an exact sequence $\mathcal{O}_U^m \to \mathcal{O}_U^n \to \mathcal{F}|_U \to 0$.
   (iii) $\mathcal{F}$ is *coherent* if it is of finite type and for every open subset $U \subset X$, every integer $n \geq 0$, and every $\mathcal{O}_U$-module map $u\colon \mathcal{O}_U^n \to \mathcal{F}|_U$, the kernel $\ker(u)$ is of finite type.

   Now let X be a locally noetherian scheme. Prove the following:
   (a) $\mathcal{F}$ is coherent if and only if $\mathcal{F}$ is of finite presentation, if and only if $\mathcal{F}$ is of finite type and quasi-coherent, if and only if for every open affine $U = \operatorname{Spec}(A) \subset X$ we have $\mathcal{F}|_U \cong \widetilde{M}$ for a finitely generated A-module M.
   (b) The full subcategory $\operatorname{Coh}(X) \subset \operatorname{QCoh}(X)$ of coherent sheaves is preserved under finite direct sums, kernels, and cokernels. Hence $\operatorname{Coh}(X)$ is an abelian category, as $\operatorname{QCoh}(X)$ is.

4. Let $f\colon X \to Y$ be a morphism of locally noetherian schemes.
   (a) Show that if $\mathcal{G} \in \operatorname{Coh}(Y)$, then $f^*\mathcal{G} \in \operatorname{Coh}(X)$.
   (b) Show that if f is a finite morphism and $\mathcal{F} \in \operatorname{Coh}(X)$, then $f_*\mathcal{F} \in \operatorname{Coh}(Y)$.
   Remark: A fundamental result that you will eventually learn about is that this is true more generally for f a proper morphism.
   (c) Give an example where X and Y are varieties over a field and $\mathcal{F} \in \operatorname{Coh}(X)$ is a coherent sheaf on X, but $f_*\mathcal{F}$ is not coherent.

5. This problem does not need to be submitted:

   Let $i\colon Z \to X$ be a closed immersion of schemes. Let $\mathcal{I} \subset \mathcal{O}_X$ be the quasi-coherent ideal sheaf corresponding to Z.

   (a) Show that the functor
   $$i_*\colon \mathrm{Mod}_{\mathcal{O}_Z} \to \mathrm{Mod}_{\mathcal{O}_X}$$
   is exact and fully faithful, with essential image consisting of $\mathcal{O}_X$-modules $\mathcal{G}$ such that $\mathcal{I} \cdot \mathcal{G} = 0$. Here, $\mathcal{I} \cdot \mathcal{G}$ is defined as the image of the multiplication map $\mathcal{I} \otimes_{\mathcal{O}_X} \mathcal{G} \to \mathcal{G}$, induced by the multiplication map $\mathcal{I}(U) \otimes_{\mathcal{O}_X(U)} \mathcal{G}(U) \to \mathcal{G}(U)$ on opens $U \subset X$.

   (b) Deduce that the analogous statement holds for the restriction of i_* to the subcategory $\mathrm{QCoh}(Z) \subset \mathrm{Mod}_{\mathcal{O}_Z}$.

6. Recall that if $\mathcal{F}$ is a sheaf of abelian groups on a topological space X, then its support $\mathrm{Supp}(\mathcal{F}) \subset X$ is the subset of $p \in X$ such that $\mathcal{F}_p \neq 0$.

   (a) Give an example of a quasi-coherent sheaf on a scheme whose support is not closed.

   (b) Show that if $\mathcal{F}$ is a finite type $\mathcal{O}_X$-module on a scheme X, then $\mathrm{Supp}(\mathcal{F}) \subset X$ is closed.

   For the rest of this problem we assume that $\mathcal{F}$ is a finite type quasi-coherent sheaf on a scheme X. Note that if $U \subset \mathcal{F}$ is an open subset and $f \in \mathcal{O}_X(U)$, then there is a map of sheaves $f\cdot\colon \mathcal{F}|_U \to \mathcal{F}|_U$ given on an open subset $V \subset U$ by multiplication by $f|_V$. We define the *annihilator* $\mathrm{Ann}(\mathcal{F})$ of $\mathcal{F}$ by the formula
   $$\mathrm{Ann}(\mathcal{F})(U) = \{f \in \mathcal{O}_X(U) \mid f\cdot\colon \mathcal{F}|_U \to \mathcal{F}|_U \text{ is } 0\}\,.$$

   (c) Show that $\mathrm{Ann}(\mathcal{F})$ is a quasi-coherent sheaf of ideals on $\mathcal{O}_X$.

   (d) In class we will see that for any quasi-coherent sheaf of ideals $\mathcal{I}$, there is an associated closed subscheme $V(\mathcal{I}) \subset X$. Show that the underlying topological space of $V(\mathrm{Ann}(\mathcal{F}))$ is equal to $\mathrm{Supp}(\mathcal{F})$ — we thus call $V(\mathrm{Ann}(\mathcal{F}))$ the *scheme theoretic support* of $\mathcal{F}$ — and give an example of two finite type quasi-coherent sheaves $\mathcal{F}_1, \mathcal{F}_2$ on a scheme X which have $\mathrm{Supp}(\mathcal{F}_1) = \mathrm{Supp}(\mathcal{F}_2)$ but different scheme theoretic support.

   (e) Let $i\colon Z = V(\mathrm{Ann}(\mathcal{F})) \to X$ be the closed immersion described above. Show that there exists $\mathcal{G} \in \mathrm{QCoh}(Z)$ such that $i_*\mathcal{G} \cong \mathcal{F}$.