§ Proper Test Functions & Microstalks

$(x,\xi)\in T^*M$, $f(x)=0$, $df_x=\xi$, $L\subseteq T^*M$ conic Lagrangian.
s.t. (x,ξ) a sm. pt of L, $ss(F)\subseteq L$ in a nbhd of (x,ξ).

· Assume L & Γ_{df} intersect transversely at (x,ξ).

Prop 2.23 pinwheels paper $\Rightarrow$ Then $\mathcal{F}_{(x,\xi),f}$ does not depend on f. (up to shifts)

i.e. if $\mathcal{F}_{(x,\xi),f}=0 \iff (x,\xi)\notin ss(F)$.

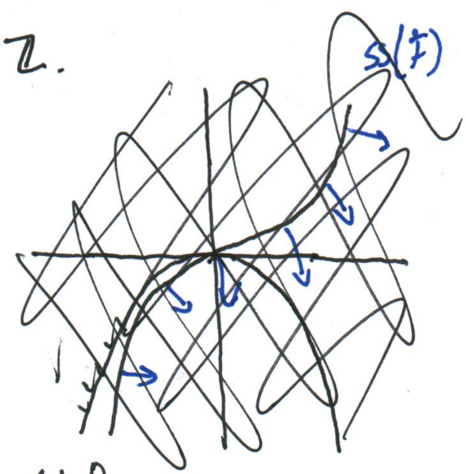
An explicit calculation can be given by:

$$\mathcal{F}_{(x,\xi),f}=\varinjlim_{x\in U} C\left(R\Gamma(U,\mathcal{F})\to R\Gamma(U\cap f^{-1}(-\infty,0),\mathcal{F})\right)[-1]$$

↳ ~~The correct category~~

$\Rightarrow$ What goes wrong if L & Γ_{df} don't intersect transversely:

$Z=\{y\le x^3\}\subseteq\mathbb{R}^2$, $i: Z\hookrightarrow\mathbb{R}^2$; $\mathcal{F}=i_*\mathbb{Z}$.



① - Test function $f(x,y)=-y$ will not detect $(0,-dy)\in ss(\mathcal{F})$ since $f|_{\partial Z}(x)=x^3$.
→ No transverse intersection!

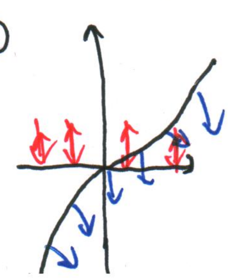
② - $f(x,y)=-x^2-y$ will be transverse to $\{y=x^3\}$ at 0.
Since its a nondegenerate critical point.

$x=0$
$\xi=-dy$

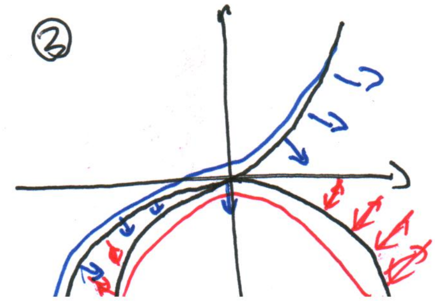
① $\mathcal{F}_{(x,\xi),f}=C(\mathbb{Z}\to\mathbb{Z})=0$.

② $\mathcal{F}_{(x,\xi),f}=C(\mathbb{Z}\to\mathbb{Z}^2)\neq 0$

①



②



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§ Co representability:

Prop 4.9 $X \subseteq T^*M$ closed & conical. $\Lambda \subseteq T^*M \setminus X$ closed conical isotropic subanalytic (generically Lagr. / co. isotropic)

Then $Sh_X(M) \subseteq Sh_{X\cup\Lambda}(M)$ consists of the kernels of all microstalk functors at Lagr. pts of Λ.

- We now would like to find co representatives of micro stalk functors.

Thm 4.10 $X \subseteq T^*M$ as before, $ss(\mathcal{F}) \subseteq X$, $\varphi: M \to \mathbb{R}$ proper s.t. on $\varphi^{-1}([a,b))$,

$T_{d\varphi} \cap ss(\mathcal{F}) = (x, \xi)$ a sm. Lagr. pt of X

$A = \varphi^{-1}(-\infty, a)$, $A' = \varphi^{-1}(a, \infty)$

$B = $ " $B' = $ "

Then, up to a shift the following functors are iso:

- $\mu_{(x,\xi)}$ the microstalk functor
- $\mathrm{Hom}(C(A_! \mathbb{Z} \to B_! \mathbb{Z}), -)$
- $\mathrm{Hom}(C(A'_* \mathbb{Z} \to B'_* \mathbb{Z}), -)$

$\to$ maps we're coning are restrictions of sections.

- Rk: $C(A_! \mathbb{Z} \to B_! \mathbb{Z})$ not necessarily in $Sh_X(M)$

So we're not quite done

Category Lemma $X \subseteq X' \subseteq T^*M$ closed, $i: Sh_X(M) \to Sh_{X'}(M)$ has both adjoints $(i^*, i, i^!)$

e.g. $X' = T^*M$. contains X. Then $i^*(C(A_! \mathbb{Z} \to B_! \mathbb{Z}))$ co represents the microstalk

Usually $(i^*, i, i^!)$ are hard to understand / describe geometrically

In the case $X' \setminus X$ is isotropic, Prop 4.9 lets us say the following: $i^*: Sh_{X'} \to Sh_{X}$ realizes the quotient

$$Sh_{X'}/D \xrightarrow{\sim} Sh_X$$

where D denotes co-representing objects for microstalks at Lagr. pts of $X' \setminus X$.

$\Rightarrow$ Seems like nobody has written down what co-representing objects are? Can't find a good example

§ Compact Objects

(For main thm, we care about $Sh_\Lambda(M)^c$, Λ subanal closed conic isotropic)

- Lemma: S a triangulation. Then $Sh_S(M)$ is compactly generated and $Sh_S^c(M)$ are sheaves w/ perfect stalks and cpt support

Pf: Recall equivalence of S-constructible sheaves & S-locally const sheaves

$$Sh_S(M) \cong \mathrm{Mod}\, S \ (:= \mathrm{Fun}(S^{op}, \mathbb{Z}\text{-mod}))$$

Compact objects in $\mathrm{Mod}\, S$ are generated by $\mathbb{Z}_S$ which will generate sheaves w/ perf stalks & cpt supp. $\square$

~~Lemma~~

Prop: S subanal whitney triangulation. $Sh_S(M)$ is compactly generated by co-representatives of microstalk functors at smooth ~~Lagr.~~ pts. of $N^* S$

~~Pf~~

Pf: Cone $(A_! \mathbb{Z} \to B_! \mathbb{Z})$ compact in a finer triangulation S' by prev. lemma
w/ $f^{-1}(a), f^{-1}(b) \subseteq S'$ $(\iota: Sh_S \to Sh_{S'}$.
Categorical reasons imply ι^* Cone$(\cdots)$ compact in Sh_S.
$\Rightarrow$ So corepresentatives of micro stalk functs compact.
$\Rightarrow$ By prop. 4.9, sm. Lagr. pts of Λ closed conical isotropic suband. generate Sh_Λ $\rightsquigarrow$ apply to N^*S. $\square$.

Cor: $\Lambda \subseteq T^*M$ cl. con. suband. isotr. $Sh_\Lambda(M)$ cptly gen by coreps of sm. Lagr. pts.

Pf: Embed $\Lambda \hookrightarrow N^*S$ for S suband whitney triang $\square$.

Cor: Previous equiv. $Sh_{X'}/_D \xrightarrow{\sim} Sh_X$ extends to compact objects in categories. $(X \subseteq X' \subseteq T^*M$, $D =$ sm Lagr pts of $X' \setminus X)$

Unsure what these mean. Skip?

Cor: Yoneda embedding ~~implies~~ induces an equivalence
$\begin{Bmatrix} \text{objects in } Sh_\Lambda(M) \\ \text{w/ perfect stalks} \end{Bmatrix} \longleftrightarrow \text{Prop } Sh_\Lambda(M)^c$

Pf: Category Theory

Cor: M compact, S triang. Then $Sh_S(M)$ sm. and proper.

Cor: M compact, Λ closed conical subanalytic sing. isotr. Then $Sh_\Lambda(M)^c$ smooth, Prop $Sh_\Lambda(M)^c \subseteq$ Perf $Sh_\Lambda(M)^c$ and Prop $Sh_\Lambda(M)^c$ proper