Wrapped microlocal sheaves on pair-of-pants

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In this note we give an account of Nadler’s computation of the dg category of wrapped microlocal sheaves on \(n\)-dimensional pair-of-pants, which in particular verifies the homological mirror symmetry conjecture in this case.

We will be working over an algebraically closed field \(K\) of characteristic 0.

1 Matrix factorizations

Consider a superpotential \(W : \mathbb{A}^n \to \mathbb{A}^1\) such that \(0 \in \mathbb{A}^1\) is the unique critical value. Denote by \(X - W^{-1}(0)\) the singular fiber.

Let \(\text{Perf}(X)\) and \(\text{Coh}(X)\) be the dg enhancements of the triangulated category of perfect complexes and the bounded derived category of coherent sheaves, respectively. Let \(D_{\text{Sing}}(X) = \text{Coh}(X)/\text{Perf}(X)\) be the 2-periodic dg quotient.

Let \(\text{MF}(\mathbb{A}^n, W)\) be the differential \(\mathbb{Z}/2\)-graded category of matrix factorizations. Denote by \(\text{MF}(\mathbb{A}^n, W)_{2\mathbb{Z}}\) the unfurling of \(\text{MF}(\mathbb{A}^n, W)\), then there is a quasi-equivalence between 2-periodic dg categories

\[
\text{MF}(\mathbb{A}^n, W)_{2\mathbb{Z}} \cong D_{\text{Sing}}(X). \tag{1}
\]

We are interested in the superpotential \(W_{n+1} : \mathbb{A}^{n+1} \to \mathbb{A}^1\) given by the product of coordinates \(z_1 \cdots z_{n+1}\), so that the central fiber \(X_n - W_{n+1}^{-1}(0)\) is the union of \(n + 1\) coordinate hyperplanes.

Consider the natural projection \(\pi : X_n \to X_{n-1}\).

**Proposition 1.1.** The pullback of coherent sheaves

\[
\pi^* : \text{Coh}(X_{n-1}) \to \text{Coh}(X_n) \tag{2}
\]

induces an equivalence of differential \(\mathbb{Z}/2\)-graded categories

\[
\text{Coh}(X_{n-1})_{\mathbb{Z}/2} \cong \text{MF}(\mathbb{A}^{n+1}, W_{n+1}). \tag{3}
\]

Let \(d\text{gst}_\mathbb{K}\) be the \(\infty\)-category of \(\mathbb{K}\)-linear small stable dg categories with exact functors. Let \(d\text{gSt}_\mathbb{K}\) be the \(\infty\)-category of \(\mathbb{K}\)-linear cocomplete dg categories with continuous functors. Let \(d\text{gSt}_\mathbb{K}^c \subset d\text{gSt}_\mathbb{K}\) be the (not full) \(\infty\)-subcategory of \(\mathbb{K}\)-linear cocomplete dg categories with functors preserving compact objects. Taking ind-categories provides an equivalence

\[
\text{Ind} : d\text{gst}_\mathbb{K} \xrightarrow{\cong} d\text{gSt}_\mathbb{K}^c. \tag{4}
\]

Taking compact objects provides an inverse equivalence

\[
\kappa : d\text{gSt}_\mathbb{K}^c \xrightarrow{\cong} d\text{gst}_\mathbb{K}. \tag{5}
\]

Let \(\mathcal{X}_\mathbb{K}\) be the category of affine locally complete intersection \(\mathbb{K}\)-schemes and closed embeddings. Passing to coherent sheaves and pushforwards yields a functor

\[
\text{Coh}_* : \mathcal{X}_\mathbb{K} \to d\text{gst}_\mathbb{K}. \tag{6}
\]

Passing to perfect complexes and \(*\)-pullbacks yields a functor

\[
\text{Perf}_* : \mathcal{X}_\mathbb{K}^{op} \to d\text{gst}_\mathbb{K}. \tag{7}
\]
and similarly for perfect complexes with proper support

\[ \text{Perf}_{\text{prop}}^\ast : \mathcal{X}_{K}^{op} \to \text{dgSt}_K. \] (8)

Passing to ind-coherent sheaves and pushforwards provides a functor

\[ \text{IndCoh} : \mathcal{X}_K \to \text{dgSt}_K^c. \] (9)

Passing to quasi-coherent sheaves and \(*\)-pullbacks yields a functor

\[ \text{QCoh}^\ast \simeq \text{IndPerf}^\ast : \mathcal{X}_{K}^{op} \to \text{dgSt}_K^c. \] (10)

Passing to ind-coherent sheaves with \(*\)-pullbacks or \(!\)-pullbacks yield functors

\[ \text{IndCoh} : \mathcal{X}_K \to \text{dgSt}_K, \text{IndCoh}^! : \mathcal{X}_K^{op} \to \text{dgSt}_K. \] (11)

Tensoring with the dualizing complex provides a natural intertwining equivalence

\[ \otimes \omega : \text{IndCoh}^\ast \Rightarrow \text{IndCoh}^!. \] (12)

Let us return to \((\mathbb{A}^n, W_n)\). Let \(\mathcal{J}_n^p\) denote the poset of subsets \(I \subset \{1, \cdots , n\}\) under inclusions. For \(I \in \mathcal{J}_n^p\), consider the corresponding coordinate subspace

\[ X_I = \text{Spec} \mathbb{K}[z_1, \cdots , z_n]/(z_a | a \notin I). \] (13)

We have a colimit diagram of closed embeddings

\[ \text{colim}_{I \in \mathcal{J}_n^p} X_I \Rightarrow X_{n-1}. \] (14)

**Proposition 1.2.** The colimit diagram (14) is taken to a colimit diagram by \(\text{Coh}_\ast\) and \(\text{IndCoh}_\ast\), and a limit diagram by \(\text{Perf}^\ast\), \(\text{Perf}_{\text{prop}}^\ast\), \(\text{IndCoh}^!\), and \(\text{IndCoh}^\ast\).

## 2 Microlocal sheaves

Let \(Z\) be a real analytic manifold. We will often work with a closed conic Lagrangian subvariety \(\Lambda \subset T^* Z\) and its Legendrian ideal boundary

\[ \Lambda^\circ = (\Lambda \cap (T^* Z \setminus Z))/\mathbb{R}_{>0} \subset S^\circ Z. \] (15)

Denote by \(Y\) the front projection \(\pi^\circ(\Lambda^\circ)\), where \(\pi^\circ : S^\circ Z \to Z\). In the generic situation, the projection \(\pi^\circ|_{\Lambda^\circ} : \Lambda^\circ \to Y\) is finite, so the front projection is a hypersurface.

We will often fix a Whitney stratification \(S = \{Z_\alpha\}_{\alpha \in A}\) of \(Z\) so that \(Y \subset Z\) is a union of strata. Hence we have inclusions

\[ \Lambda \subset T^p_{\delta} Z := \bigsqcup_{\alpha \in A} T^p_{Z_\alpha} Z, \quad \Lambda^\circ \subset S^\circ_{\delta} Z := \bigsqcup_{\alpha \in A} S^\circ_{Z_\alpha} Z. \] (16)

Given a Whitney stratification \(\delta\), by a small open ball \(B \subset Z\) around a point \(z \in Z\) we will mean an open ball \(B = B(r) \subset Z\) of some radius \(r > 0\) such that the corresponding spheres \(S(r') \subset Z\), for all \(0 < r' < r\), are transverse to the strata of \(\delta\).

Let \(\text{Sh}^\circ(Z)\) denote the dg category of complexes of sheaves of \(\mathbb{K}\)-vector spaces on \(Z\) such that the total cohomology sheaf is locally constant with respect to some Whitney stratification \(\delta\). For a fixed Whitney stratification \(\delta\), denote by \(\text{Sh}^\circ_{\delta}(Z) \subset \text{Sh}^\circ(Z)\) the full subcategory which are cohomologically locally constant with respect to this specific \(\delta\). It follows that \(\text{Sh}^\circ(Z) = \bigsqcup_{\delta} \text{Sh}^\circ_{\delta}(Z)\).

Let \(\text{Sh}(Z) \subset \text{Sh}^\circ(Z)\) be the full dg subcategory of constructible complexes of sheaves of \(\mathbb{K}\)-vector spaces on \(Z\). In other words, it consists of objects of \(\text{Sh}^\circ(Z)\) whose total cohomology sheaf, when restricted to each \(Z_\alpha\), has finite rank. We can introduce the
notation $\text{Sh}_{S}(Z)$ as above, and it follows that $\text{Sh}(Z) = \bigcup_{S} \text{Sh}_{S}(Z)$. The objects of $\text{Sh}_{S}(Z)$ will be referred to as large constructible sheaves, and the objects of $\text{Sh}(Z)$ as constructible sheaves.

All functors between dg categories of sheaves will be derived in the dg sense. When dealing with large constructible sheaves, since we are working with co-complete dg categories, the functors should also be co-continuous (preserves colimits). For example, for a closed embedding $i : Y \hookrightarrow Z$, by the $!$-restriction $i^{!} : \text{Sh}^c(Z) \to \text{Sh}^c(Y)$, we will mean the shifted cone $i^{!} \cong \text{Cone}(\mathcal{F} \to j_{!*}\mathcal{F})[-1]$, where $j : U \to Z$ is the inclusion of the open complement $U = Z \setminus Y$. For a smooth map $f : Y \to Z$, by the $!$-pullback $f^{!} : \text{Sh}^c(Z) \to \text{Sh}^c(Y)$, we will mean the twist of the $*$-pullback $f^{!}\mathcal{F} \cong f^{*}\mathcal{F} \boxtimes \omega_{f}$, where $\omega_{f} \cong \text{or}_{f}[(\dim Y / Z)]$ is the relative dualizing complex.

Fix a point $(z, \xi) \in T^*Z$. Let $B \subset Z$ be an open ball around $z \in Z$, and $f : B \to \mathbb{R}$ a smooth function such that $f(z) = 0$ and $df|_{z} = \xi$. We will refer to $f$ as a compatible test function.

Define the vanishing cycles functor

$$ \phi_{f} : \text{Sh}^c(Z) \to \text{Mod}_{\mathbb{R}}, \quad \phi_{f}(\mathcal{F}) = \Gamma_{1<0}(B, \mathcal{F}|_{B}) \cong \text{Cone}(\Gamma(B, \mathcal{F}|_{B}) \to \Gamma(\{f < 0\}, \mathcal{F}|_{\{f<0\}}))[-1], $$

(17)

where we take $B \subset Z$ to be sufficiently small.

To any object $\mathcal{F}$ of $\text{Sh}^c(Z)$, define its singular support $ss(\mathcal{F}) \subset T^*Z$ to be the largest closed subset such that $\phi_{f}(\mathcal{F}) \cong 0$ for any $(z, \xi) \in T^*Z \setminus ss(\mathcal{F})$, and any compatible test function $f$.

For a conic Lagrangian subvariety $\Lambda \subset T^*Z$, write $\text{Sh}^c_{\Lambda}(Z) \subset \text{Sh}^c(Z)$, resp. $\text{Sh}_{\Lambda}(Z) \subset \text{Sh}(Z)$ for the full dg subcategory with singular support $ss(\mathcal{F}) \subset \Lambda$.

Given a Whitney stratification $S$, an inclusion $\Lambda \subset \mathcal{T}_{S}^{0}Z$ induces the fully faithful embeddings $\text{Sh}^c_{\Lambda}(Z) \subset \text{Sh}^c_{S}(Z)$, $\text{Sh}_{\Lambda}(Z) \subset \text{Sh}_{S}(Z)$. More generally, an inclusion $\Lambda \subset \Lambda'$ induces the fully faithful embeddings $\text{Sh}^c_{\Lambda}(Z) \subset \text{Sh}^c_{\Lambda'}(Z)$, $\text{Sh}_{\Lambda}(Z) \subset \text{Sh}_{\Lambda'}(Z)$.

When $U \subset Z$ is an open subset, we will abuse notations and write $\text{Sh}^c_{\Lambda}(U) \subset \text{Sh}^c(U)$, resp. $\text{Sh}_{\Lambda}(U) \subset \text{Sh}(U)$ for the full dg subcategory with objects satisfying $ss(\mathcal{F}) \subset \Lambda \cap \pi^{-1}(U)$. $\pi : T^*Z \to Z$ is the natural projection.

**Remark 2.1.** Let $\omega_{Z} \cong \text{or}_{Z}[(\dim Z)] \cong p^{!}\mathbb{R}_{pt}$, for $p : Z \to pt$, be the Verdier dualizing complex. For a conic Lagrangian subvariety $\Lambda \subset T^*Z$ and the antipodal conic Lagrangian subvariety $-\Lambda \subset T^*Z$. Verdier duality provides an involutive equivalence

$$ D_{Z} : \text{Sh}_{\Lambda}(Z)^{op} \xrightarrow{\cong} \text{Sh}_{-\Lambda}(Z), D_{Z}(\mathcal{F}) = \mathcal{H}om(\mathcal{F}, \omega_{Z}). $$(19)

The above discussions can be generalized to the slightly more general setting. To a conic open subspace $\Omega \subset T^*Z$, we associate the dg category $\mu \text{Sh}^c_{\Lambda}(\Omega)$ of large microlocal sheaves on $\Omega$ supported along $\Lambda$. Before describing its construction, we first mention some of its formal properties.

- Given an inclusion of conic open subspaces $\Omega' \subset \Omega$, there is a natural restriction functor $\mu \text{Sh}^c_{\Lambda}(\Omega) \to \mu \text{Sh}^c_{\Lambda}(\Omega')$. These assignments assemble into a sheaf $\mu \text{Sh}^c_{\Lambda}$ of dg categories supported along $\Lambda$.

- There exists a Whitney stratification of $\Lambda$ such that the restriction of $\mu \text{Sh}^c_{\Lambda}$ to each stratum is locally constant. Thus we can reconstruct $\mu \text{Sh}^c_{\Lambda}$ from the assignments $\mu \text{Sh}^c_{\Lambda}(\Omega)$ for small conic open neighborhoods $\Omega$ of $(z, \xi) \in \Lambda$.

- Given a closed embedding of conic Lagrangian subvarieties $\Lambda' \subset \Lambda$, there is a natural full embedding $\mu \text{Sh}^c_{\Lambda'} \subset \mu \text{Sh}^c_{\Lambda}$ of sheaves of dg categories.

All of the above facts follow from the local description of $\mu \text{Sh}^c_{\Lambda}(\Omega)$ which we now recall. Note that for a point $(z, \xi) \in \Lambda$ there are two local cases to consider: either $\xi = 0$
so locally $\Omega$ is the cotangent bundle $T^*B$ of a small open ball $B \subset Z$, or $\xi \neq 0$ so that locally $\Omega$ is the cone over a small open ball $\Omega^\circ \subset S^\circ Z$.

- For $B = \pi(\Omega)$, there is always a canonical functor $\text{Sh}_A^\circ(B) \to \mu\text{Sh}_A^\circ(\Omega)$. When $\Omega = T^*B$, this functor is in fact an equivalence
  \[
  \text{Sh}_A^\circ(B) \xrightarrow{\sim} \mu\text{Sh}_A^\circ(T^*B). \tag{20}
  \]

- Suppose $\Omega$ is the cone over small open ball $\Omega^\circ \subset S^\circ Z$. Set $B = \pi(\Omega)$, and let $\text{Sh}_A^\circ(B, \Omega) \subset \text{Sh}_A^\circ(B)$ denote the full dg subcategory of objects with $ss(\mathcal{F}) \cap \Omega \subset \Lambda$. Then there is a natural equivalence
  \[
  \text{Sh}_A^\circ(B, \Omega)/K^\circ(B, \Omega) \xrightarrow{\sim} \mu\text{Sh}_A^\circ(\Omega), \tag{21}
  \]
  where $K^\circ(B, \Omega) \subset \text{Sh}_A^\circ(B, \Omega)$ denote the full dg subcategory of objects with $ss(\mathcal{F}) \cap \Omega = \emptyset$.

We similarly introduce the full dg subcategory $\mu\text{Sh}_A(\Omega) \subset \mu\text{Sh}_A^\circ(\Omega)$ of microlocal sheaves on $\Omega$ supported along $\Lambda$. It is constructed as above by working with constructible sheaves instead of large constructible sheaves.

The dg category $\mu\text{Sh}_A(\Omega)$ is the sections of a subsheaf $\mu\text{Sh}_A \subset \mu\text{Sh}_A^\circ$ of full dg categories supported along $\Lambda$. Given a Whitney stratification of $\Lambda$ such that the restriction of $\mu\text{Sh}_A^\circ$ to each stratum is locally constant, the restriction of $\mu\text{Sh}_A$ to each stratum will also be locally constant. Finally, given a closed embedding of conic Lagrangian subvarieties $N \subset \Lambda$, the full embedding $\mu\text{Sh}_A^\circ \subset \mu\text{Sh}_A$ restricts to a full embedding $\mu\text{Sh}_N \subset \mu\text{Sh}_\Lambda$.

**Remark 2.2.** For a conic Lagrangian subvariety $\Lambda \subset T^*Z$, with antipodal subvariety $-\Lambda \subset T^*Z$, and conic open subspace $\Omega \subset T^*Z$, with antipodal subspace $-\Omega \subset T^*Z$. Verdier duality induces an involutive equivalence
  \[
  D_Z : \mu\text{Sh}_A(\Omega)^{op} \xrightarrow{\sim} \mu\text{Sh}_A(-\Omega). \tag{22}
  \]

Fix a Whitney stratification $\mathcal{S} = \{Z_\alpha\}_{\alpha \in A}$ of $Z$ such that $\Lambda \subset T^*_\mathcal{S}Z := \bigsqcup_{\alpha \in A} T^*_{Z_\alpha}Z$. To each stratum $Z_\alpha \subset Z$, introduce the frontier $\partial(T^*_{Z_\alpha}Z) := \overline{T^*_{Z_\alpha}Z \setminus T^*_{Z_\alpha}Z}$ of its conormal bundle, and the dense, open, smooth locus of their complement
  \[
  (T^*_{Z_\alpha}Z)^{\circ} := T^*_{Z_\alpha}Z \setminus \bigcup_{\alpha \in A} \partial(T^*_{Z_\alpha}Z). \tag{23}
  \]

Introduce the corresponding dense, open, smooth locus
  \[
  \Lambda^\circ := \Lambda \cap (T^*_{\mathcal{S}}Z)^{\circ} \subset \Lambda. \tag{24}
  \]

Note that $\Lambda^\circ$ depends on $\mathcal{S}$, refining $\mathcal{S}$ leads to smaller $\Lambda^\circ$.

Fix a point $(z, \xi) \in \Lambda^\circ$. Let $B \subset Z$ be a small open ball around $z \in Z$, and $f : B \to \mathbb{R}$ a compatible test function. Let $L \subset T^*Z$ be the graph of $df$, and assume that $L$ intersects $\Lambda^\circ$ transversely at the single point $(z, \xi) \in \Lambda^\circ$.

**Definition 2.1.** Let $\Omega \subset T^*Z$ be a conic open subspace containing $(z, \xi) \in \Lambda^\circ$. Define the microstalk along $L \subset T^*Z$ to be the vanishing cycles
  \[
  \phi_L : \mu\text{Sh}_A^\circ(\Omega) \to \text{Mod}_{\mathcal{L}^\circ}, \quad \phi_L(\mathcal{F}) := \Gamma_{\{t \geq 0\}}(B, \tilde{\mathcal{F}}|_B), \tag{25}
  \]
  where $\tilde{\mathcal{F}} \in \text{Sh}_A^\circ(B, \Omega_B)$ represents the restriction of $\mathcal{F} \in \mu\text{Sh}_A^\circ(\Omega)$ to a small open neighborhood $\Omega_B \subset \Omega$ of the point $(z, \xi) \in T^*Z$.

**Remark 2.3.** The microstalk is well-defined since by construction it vanishes on the kernel of the localization $\text{Sh}_A^\circ(B, \Omega_B) : \to \mu\text{Sh}_A^\circ(\Omega_B)$ with respect to the singular support.

**Lemma 2.1.** An object $\mathcal{F} \in \mu\text{Sh}_A^\circ(\Omega)$ is trivial if and only if all of its microstalks are trivial. An object $\mathcal{F} \in \mu\text{Sh}_A^\circ(\Omega)$ lies in the full subcategory $\mu\text{Sh}_A(\Omega) \subset \mu\text{Sh}_A^\circ(\Omega)$ if and only if all its microstalks are perfect (i.e., proper) $\mathbb{L}$-modules.
3 Wrapped microlocal sheaves

**Definition 3.1.** Define the small dg category $\mu Sh_{\Lambda}^w(\Omega)$ of wrapped microlocal sheaves on $\Omega$ supported along $\Lambda$ to be the full dg subcategory of compact objects within the dg category $\mu Sh_{\Lambda}^c(\Omega)$ of large microlocal sheaves.

The dg categories $\mu Sh_{\Lambda}(\Omega)$ and $\mu Sh_{\Lambda}^c(\Omega)$ can now be defined for any Liouville manifold instead of conic open subsets in the cotangent bundle. See the work of Nadler-Shende.

**Remark 3.1.** Given the full dg subcategory $\mathcal{C}_c \subset \mathcal{C}$ of compact objects in a stable cocomplete dg category, the canonical functor $\text{Ind}\mathcal{C}_c \to \mathcal{C}$ is an equivalence. Thus we have

$$\text{Ind}\mu Sh_{\Lambda}^c(\Omega) \cong \mu Sh_{\Lambda}^c(\Omega). \quad (26)$$

Geometrically, the partially wrapped Fukaya category $\mathcal{W}(X,f)$ is generated by cocores and linking discs. Denote their endomorphism $A_{\mathcal{W}}$-algebra by $\mathcal{W}(X,f)$. Then the definition above just says that the derived category $D_{\text{perf}}^\mathcal{W}(X,f)$ can be defined as the category $D_{\text{perf}}(\mathcal{W}(X,f))$ of perfect modules over the $A_{\mathcal{W}}$-algebra $\mathcal{W}(X,f)$. Note that $D_{\text{perf}}(\mathcal{W}(X,f)) \subset D_{\text{mod}}(\mathcal{W}(X,f))$ is the subcategory of compact objects.

There is a more concrete geometric characterization of wrapped microlocal sheaves. Recall the microstalk functors $\phi_L : \mu Sh_{\Lambda}^c(\Omega) \to \text{Mod}_X$. Note that $\phi_L$ preserves products, hence admits a left adjoint $\phi'_L : \text{Mod}_X \to \mu Sh_{\Lambda}^c(\Omega)$, and also preserves coproducts, hence $\phi'_L$ preserves compact objects.

**Definition 3.2.** Define the microlocal skyscraper $\mathcal{F}_L = \phi'_L(\mathbb{K}) \in \mu Sh_{\Lambda}^c(\Omega)$ to be the object corepresenting the microstalk

$$\phi_L(\mathcal{F}) \cong \text{hom}(\mathcal{F}_L, \mathcal{F}), \mathcal{F} \in \mu Sh_{\Lambda}^c(\Omega). \quad (27)$$

**Lemma 3.1.** $\mu Sh_{\Lambda}^c(\Omega)$ is split-generated by the microlocal skyscrapers $\mathcal{F}_L \in \mu Sh_{\Lambda}^c(\Omega)$.

**Proof.** By Lemma 2.1, the microlocal skyscrapers $\mathcal{F}_L$ compactly generate $\mu Sh_{\Lambda}^c(\Omega) \cong \text{Ind}\mu Sh_{\Lambda}^c(\Omega)$. (For any non-trivial object $\mathcal{F}$ of $\mu Sh_{\Lambda}^c(\Omega)$, there must be some $L$ so that $\text{hom}(\mathcal{F}_L, \mathcal{F}) \neq 0$.) Thus we may invoke the general fact that if a collection of objects of a small stable dg category $\mathcal{C}_c$ generates the ind-category $\mathcal{C} \cong \text{Ind}\mathcal{C}_c$, then it split-generates $\mathcal{C}_c$. \hfill \square

**Remark 3.2.** One should think of $\mu Sh_{\Lambda}^c(\Omega)$ as the (derived) partially wrapped Fukaya category associated to the stopped Liouville manifold $(\Omega, \Lambda \cap \partial \Omega)$, and $\mu Sh_{\Lambda}(\Omega)$ the (derived) infinitesimal Fukaya category. The microlocal skyscrapers correspond to cocores and linking discs which intersect the smooth part of $\Lambda \cap \Omega$ transversely at a single point. Geometrically, they are given by the Lagrangian disc $L \subset T^*B$.

Recall that for conic open subspaces $\Omega \subset T^*Z$, the dg category $\mu Sh_{\Lambda}^c(\Omega)$ of large microlocal sheaves is the sections of a sheaf $\mu Sh_{\Lambda}^c$ of dg categories supported along $\Lambda$. For an inclusion $\Omega' \subset \Omega$ of conic open subspaces, the restriction functor $\rho : \mu Sh_{\Lambda}^c(\Omega) \to \mu Sh_{\Lambda}^c(\Omega')$ preserves products, hence admits a left adjoint $\rho' : \mu Sh_{\Lambda}^c(\Omega') \to \mu Sh_{\Lambda}^c(\Omega)$, and also preserves coproducts, hence $\rho'$ preserves compact objects. Thus its restriction to the subcategory of compact objects defines a natural corestriction functor

$$\rho^w : \mu Sh_{\Lambda}^c(\Omega') \to \mu Sh_{\Lambda}^c(\Omega). \quad (28)$$

**Proposition 3.1.** The dg categories $\mu Sh_{\Lambda}^w(\Omega)$ for conic open subspaces $\Omega \subset T^*Z$ and corestriction functors $\rho^w : \mu Sh_{\Lambda}^w(\Omega') \to \mu Sh_{\Lambda}^w(\Omega)$ for inclusions $\Omega' \subset \Omega$, assemble into a cosheaf $\mu Sh_{\Lambda}^w$ of dg categories supported along $\Lambda$. Furthermore, there exists a Whitney stratification of $\Lambda$ such that the restriction of $\mu Sh_{\Lambda}^w$ to each stratum is locally constant.
Given a closed embedding of conic Lagrangian subvarieties $\Lambda \subset \Lambda'$, there is a natural full embedding $i : \mu Sh^w_{\Lambda'} \to \mu Sh^w_{\Lambda}$ of sheaves of dg categories. Observe that $i$ preserves products, hence admits a left adjoint $i^! : \mu Sh^w_{\Lambda} \to \mu Sh^w_{\Lambda'}$, and also preserves coproducts, so $i^!$ preserves compact objects. Thus its restriction to compact objects defines an essentially surjective functor (i.e. surjective on objects up to isomorphism)

$$i^w : \mu Sh^w_{\Lambda} \to \mu Sh^w_{\Lambda'}.$$  
(29)

The evaluation of $i^w$ on a microlocal skyscraper $\mathcal{F}_L \in \mu Sh^w_{\Lambda}(\Omega)$ is straightforward. If the small Lagrangian ball $L \subset T^* Z$ is centered at a point $(z, \xi) \in \Lambda^c$ that is not contained in $\Lambda' \subset \Lambda$, then $i^w(\mathcal{F}_L) \equiv 0$. If the small Lagrangian ball $L$ is centered at a point $(z, \xi) \in \Lambda^c$ contained in $\Lambda' \subset \Lambda$, then $i^w(\mathcal{F}_L)$ simply represents the restriction of the microstalk functor to sections of $\mu Sh^w_{\Lambda} \subset \mu Sh^w_{\Lambda'}$. (Geometrically, what the functor $i^w$ does is sending linking discs of $\Lambda$ to linking discs of $\Lambda'$. Since $\Lambda$ has more linking discs, $i^w$ is essentially surjective.)

**Theorem 3.1.** The natural hom-pairing provides an equivalence

$$\mu Sh_{\Lambda}(\Omega) \cong \text{Fun}^{ex}(\mu Sh^w_{\Lambda}(\Omega)^{op}, \text{Perf}_K),$$

where $\text{Fun}^{ex}$ denotes the dg category of exact functors, and $\text{Perf}_K$ that of perfect $K$-modules.

**Remark 3.3.** While objects of $\mu Sh^w_{\Lambda}(\Omega)$ similarly give functionals on $\mu Sh_{\Lambda}(\Omega)$, it is not in general true that they produce all possible functionals. One could think about the specific example where $\Lambda = S^1 \subset T^* S^1$ is the zero section, and $\Omega = T^* S^1$ is the entire cotangent bundle. Then we have $\mu Sh_{S^1}(T^* S^1) \cong \text{Perf}_{\text{prop}}(\mathcal{G}_m)$ and $\mu Sh^w_{S^1}(T^* S^1) \cong \text{Coh}(\mathcal{G}_m)$. The hom-pairing gives an equivalence

$$\text{Perf}_{\text{prop}}(\mathcal{G}_m) \cong \text{Fun}^{ex}(\text{Coh}(\mathcal{G}_m)^{op}, \text{Perf}_K).$$

(31)

Clearly there are more functionals on $\text{Perf}_{\text{prop}}(\mathcal{G}_m)$ than those coming from $\text{Coh}(\mathcal{G}_m)$. For example, one could take the hom-pairing with a direct sum of skyscraper sheaves at infinitely many points.

**Remark 3.4.** In terms of symplectic topology, Theorem 3.1 is a version of the Eilenberg-Moore equivalence between the partially wrapped Fukaya category and the infinitesimal Fukaya category associated to the stopped Liouville manifold $(\Omega, \Lambda \cap \partial_x \Omega)$. Choosing $\Lambda$ to be the zero section, $\Lambda \cap \partial_x \Omega = \mathcal{G}$, the infinitesimal Fukaya category becomes the compact Fukaya category, and the partially wrapped Fukaya category becomes the fully wrapped Fukaya category. In general, this Eilenberg-Moore equivalence does not give the Koszul duality between Fukaya categories, a typical example is $\Omega = T^* S^1$.

On the mirror side, Koszul duality between $\text{Coh}(X)$ and $\text{Perf}_{\text{prop}}(X)$ holds for proper schemes $X$.

**Proof of Theorem 3.1.** First, let us observe that it suffices to prove the assertion locally. If one choose a cover $\{\Omega_i\}_{i \in I}$ of $\Omega$ by conic open subspaces, since $\mu Sh_\Lambda(\Omega)$ is a sheaf and $\mu Sh^w_\Lambda(\Omega)$ is a cosheaf we have

$$\mu Sh_\Lambda(\Omega) \cong \varinjlim_{i \in I} \mu Sh_\Lambda(\Omega_i), \quad \mu Sh^w_\Lambda(\Omega) \cong \varprojlim_{i \in I} \mu Sh^w_\Lambda(\Omega_i).$$

(32)

Thus if we have the assertion locally, i.e.

$$\mu Sh_\Lambda(\Omega_i) \cong \text{Fun}^{ex}(\mu Sh^w_\Lambda(\Omega_i)^{op}, \text{Perf}_K),$$

(33)
then we have it globally

\[
\mu Sh_\Lambda(\Omega) \cong \lim_{\alpha \to 1} \mu Sh_\Lambda(\Omega, \alpha) \cong \lim_{\alpha \to 1} \text{Fun}_K^\text{ex}(\mu Sh_\Lambda(\Omega, \alpha)^\text{op}, \text{Perf}_K)
\]

\[
\cong \text{Fun}_K^\text{ex}(\text{colim}_{\alpha \to 1} \mu Sh_\Lambda(\Omega, \alpha)^\text{op}, \text{Perf}_K) \cong \text{Fun}_K^\text{ex}(\mu Sh_\Lambda(\Omega)^\text{op}, \text{Perf}_K).
\]

(34)

We may assume that \( \Omega \subset T^* Z \) is the cone over a small open ball \( \Omega^c \subset S^c Z \) centered at a point of \( \Lambda^c \subset S^c Z \).

We may deform \( \Lambda^c \subset S^c Z \) to a Legendrian subvariety \( \Lambda_{arb}^c \subset S^c Z \) with arboreal singularities. Taking \( \Lambda_{arb} \) to be the cone over \( \Lambda_{arb}^c \), we have an equivalence \( \mu Sh_\Lambda(\Omega) \cong \mu Sh_\Lambda(\Omega, \alpha) \), restricting to an equivalence \( \mu Sh_\Lambda(\Omega) \cong \mu Sh_\Lambda(\Omega, \alpha) \). Passing to the compact objects in the first equivalence, we have \( \mu Sh_\Lambda(\Omega) \cong \mu Sh_\Lambda(\Omega, \alpha) \).

Thus we may assume that the conic Lagrangian subvariety \( \Lambda \subset T^* Z \) has arboreal singularities. Moreover, we may further assume that \( \Omega \subset T^* Z \) is the cone over a small open ball \( \Omega^c \subset S^c Z \) centered at a point of \( \Lambda^c \subset S^c Z \) that is an arboreal singularity. In this situation, \( \mu Sh_\Lambda(\Omega) \) is equivalent to the dg category \( \text{Mod}_K(T) \) of modules over a directed tree \( T \), and \( \mu Sh_\Lambda(\Omega) \) is equivalent to the dg category \( \text{Perf}_K(T) \) of perfect modules. Passing to compact objects under the first equivalence, we have that \( \mu Sh_\Lambda(\Omega) \) is also equivalent to \( \text{Perf}_K(T) \).

Finally, for perfect modules over a directed tree, it is straightforward to check that the hom-pairing provides an equivalence

\[
\text{Perf}_K(T) \cong \text{Fun}_K^\text{ex}(\text{Perf}_K(T)^\text{op}, \text{Perf}_K).
\]

(35)

\[\square\]

**Remark 3.5.** Locally, the dg categories \( \mu Sh_\Lambda(\Omega) \) and \( \mu Sh_\Lambda(\Omega) \), which correspond respectively to the infinitesimal Fukaya category and the partially wrapped Fukaya category, are Koszul dual. Moreover, these two categories are smooth and proper, therefore “self-dual” in the derived sense. This explains the fact that their derived categories are both equivalent to \( \text{Perf}_K(T) \) in the above argument.

### 4 Lagrangian skeleton

By the \( n \)-dimensional **pair-of-pants**, we mean the Liouville manifold \( (P_n, \alpha_{P_n}) \) given by the generic hyperplane

\[
P_n = \{1 + z_1 + \cdots + z_{n+1} = 0\} \subset TZ^{n+1},
\]

(36)
equipped with the restriction of the Liouville form \( \alpha_n \) on \( T^* T^{n+1} \). The Lagrangian skeleton of \( P_n \) can be described by the combinatorics of the permutahedron.

To get a more convenient Lagrangian skeleton, we need to break the symmetry and work with the pair-of-pants in a slightly modified form where we alter its embedding near infinity. This is called **tailored pair-of-pants**, and we denote it by \( (Q_n, \alpha_{Q_n}) \).

To provide the tailored pair-of-pants a particularly simple skeleton, it will be useful to break the symmetry and apply a natural isotopy to its Liouville structure. For \( x = (x_1, \cdots, x_{n+1}) \in \mathbb{R}^{n+1} \), consider the family of Liouville structures on \( T^* T^{n+1} \) given by

\[
\alpha_n^x = \sum_{a=1}^{n+1} (\xi_a - x_a) d\theta_a, \omega_n^x = -d\alpha_n^x - \sum_{a=1}^{n+1} (\xi_a - x_a) d\theta_a.
\]

(37)

The restriction of \( \alpha_n^x \) and \( \omega_n^x \) to the pair-of-pants \( P_n \) provide a family of Liouville structures. We may construct the tailored pair-of-pants \( Q_n \subset T Z^{n+1} \) so that the restricted Liouville form \( \alpha_{Q_n}^x = \alpha_n^x |_{Q_n} \) provide a family of Liouville structures as well. Choose
$\ell \gg 0$, and let $x = (-\ell, \cdots, -\ell) \in \mathbb{R}^{n+1}$. Let us focus on the Liouville structure on $Q_n$ given by

$$\beta_{Q_n} := \alpha_{Q_n}^{x} = \left(\sum_{a=1}^{n+1} (\xi_a + \ell) d\theta_a\right) |_{Q_n}. \quad (38)$$

Write $L_n \subset Q_n$ for the resulting skeleton, we will describe its geometry.

Let $S^1_\Delta \subset T^{n+1}$ be the diagonal circle. The translation $S^1_\Delta$-action on $T^{n+1}$ induces a Hamiltonian $S^1$-action on $T^*T^{n+1}$, with moment map

$$\mu_{\Delta} : T^*T^{n+1} \to \mathbb{R}, \quad \mu_{\Delta}(\theta_1, \xi_1, \cdots, \theta_n, \xi_n) = -\sum_{a=1}^{n+1} \xi_a. \quad (39)$$

Consider the quotient $T^n - T^{n+1}/S^1_\Delta$ consisting of $(n+1)$-tuples $[\theta_1, \cdots, \theta_n]$ taken up to simultaneous translation. If we distinguish the last entry, then we obtain an identification $T^n \cong T^n$ via the coordinates $\theta_a - \theta_n$, where $1 \leq a \leq n$.

Let $t_n^* = \left\{ \sum_{a=1}^{n+1} \xi_a = 0 \right\} \subset \mathbb{R}^{n+1}$ be the dual of the Lie algebra of $T^n$. We have the identification $T^*T^n \cong T^n \times t_n^*$. In terms of coordinates, a point of $T^*T^n$ can be represented by $[(\theta_1, \cdots, \theta_n), (\xi_1, \cdots, \xi_{n+1})]$, where $\sum_{a=1}^{n+1} \xi_a = 0$.

For $\chi \in \mathbb{R}$, we have a twisted Hamiltonian reduction correspondence

$$T^*T^{n+1} \xrightarrow{q_{\chi}} \mu_{\Delta}^{-1}(\chi) \xrightarrow{p_{\chi}} T^*T^n, \quad (40)$$

where $q_{\chi}$ is the inclusion of level set, while $p_{\chi}$ is the translation projection

$$p_{\chi}((\theta_1, \cdots, \theta_n), (\xi_1, \cdots, \xi_{n+1})) := ([\theta_1, \cdots, \theta_n], (\xi_1 - \chi, \cdots, \xi_{n+1} - \chi)), \quad (41)$$

where $\chi = \chi/(n+1)$. In particular, when $\chi = 0$, we recover the usual Hamiltonian reduction correspondence

$$T^*T^{n+1} \xrightarrow{q_0} T^*S^1 \xrightarrow{p_0} T^*T^n, \quad (42)$$

where $T^*S^1 \subset T^*T^{n+1}$ is the conormal bundle.

Introduce the conic Lagrangian subvariety

$$\Lambda_1 := \{ (\theta, 0) | \theta \in S^1 \} \cup \{ (0, \xi) | \xi \in \mathbb{R}_{\geq 0} \} \subset T^*S^1, \quad (43)$$

and the product conic Lagrangian subvariety

$$\Lambda_{n+1} := (\Lambda_1)^{n+1} \subset T^*T^{n+1}. \quad (44)$$

Note that $\Lambda_{n+1} \subset \mu_{S^1}^{-1}(\mathbb{R}_{\geq 0})$, and that $\Lambda_{n+1}$ and $\mu_{S^1}^{-1}(\chi)$ are transverse for $\chi > 0$. Fix some $\chi > 0$, define the Lagrangian subvariety

$$E_n := p_{\chi}(q_{\chi}^{-1}(\Lambda_{n+1})) \subset T^*T^n. \quad (45)$$

**Remark 4.1.** We do not include $\chi$ in the notation for $E_n$ as we will eventually specialize to the case $\chi = n + 1$.

To describe $E_n \subset T^*T^n$, consider the moment map $\mu_{n+1} : T^*T^{n+1} \to \mathbb{R}^{n+1}$ of the Hamiltonian $T^{n+1}$-action and restrict it to $\Lambda_{n+1} \subset T^*T^{n+1}$. Note that $\mu_{n+1}(\Lambda_{n+1}) = \mathbb{R}_{\geq 0}^{n+1}$. For $I \subset \{ 1, \cdots, n+1 \}$, consider the relatively open coordinate cone

$$\sigma_I = \{ \xi_a = 0, \xi_b > 0 | a \in I, b \notin I \} \subset \mathbb{R}_{>0}^{n+1}. \quad (46)$$

For $x \in \sigma_I$, $\mu_{n+1}^{-1}(x) \cap \Lambda_{n+1}$ is the orthogonal coordinate subtorus

$$T^I = \{ \theta_a - 0 | a \notin I \} \subset T^{n+1}. \quad (47)$$
Consider the closed simplex
\[ \tilde{\Xi}(\chi) = \left\{ (\xi_1, \cdots, \xi_{n+1}) | \xi_a \geq 0 \text{ for } 1 \leq a \leq n+1, \sum_{a=1}^{n+1} \xi_a = \chi \right\} \subset \mathbb{R}_{\geq 0}^{n+1}. \] (48)

Note that the projection \( p_\chi \) restricts to an isomorphism
\[ \mu_\Delta^{-1}(\chi) \cap \Lambda_{n+1} = \mu_{n+1}^{-1} \left( \tilde{\Xi}(\chi) \right) \xrightarrow{\sim} \mathcal{L}_n \] (49)
since for any point of \( \mu_\Delta^{-1}(\chi) \cap \Lambda_{n+1} \), we must have \( \xi_a > 0 \) and hence \( \theta_n = 0 \) for some \( a \in \{1, \cdots, n+1\} \), so that no points are identified by the \( S^1_\Delta \)-translations.

For a proper subset \( I \subset \{1, \cdots, n+1\} \), consider the relatively open subsimplex
\[ \tilde{\Xi}_I(\chi) = \tilde{\Xi}_n(\chi) \cap \sigma_I. \] (50)

Then \( p_\chi \) restricts to an isomorphism
\[ \bigcup_I T^I \times \tilde{\Xi}_I(\chi) \xrightarrow{\sim} \mathcal{L}_n, \] (51)
where we take the union over non-empty \( I \subset \{1, \cdots, n+1\} \). Note that when \( n = 2 \), \( \mathcal{L}_n \) is the union of two circles and an open interval.

**Theorem 4.1.** There is an open neighborhood \( U_n \subset Q_n \) of the Lagrangian skeleton \( L_n \subset Q_n \) and an open symplectic embedding
\[ j : U_n \to T^*\mathbb{T}^n \] (52)
which restricts to an isomorphism
\[ j|_{L_n} : L_n \xrightarrow{\sim} \mathcal{L}_n. \] (53)

## 5 Contactification and symplectization

By Theorem 4.1, the symplectic geometry of a neighborhood \( U_n \subset Q_n \) of \( L_n \subset Q_n \) is equivalent to that of a neighborhood \( U_n \subset T^*\mathbb{T}^n \) of \( \mathcal{L}_n \subset T^*\mathbb{T}^n \).

We introduce the Liouville form \( \beta_n \) on the neighborhood \( U_n \subset T^*\mathbb{T}^n \) obtained by transporting the Liouville form \( \beta_{Q_n} \) restricted to the neighborhood \( U_n \subset Q_n \). Thus \( \beta_n \) provides a primitive to the restriction of the canonical symplectic form \( \omega_{T^*\mathbb{T}^n}|_{U_n} = d\beta_n \).

The Lagrangian subvariety \( \mathcal{L}_n \subset T^*\mathbb{T}^n \) is conic with respect to its associated Liouville vector field.

In general, let \( M \) be a Liouville manifold with Liouville form \( \alpha_M \). The **circular contactification** of \( M \) is the contact manifold \( N = M \times S^1 \), with contact form \( \lambda_N = dt + \alpha_M \), and contact structure \( \xi_N = \ker(\lambda_N) \). The **contactification** of \( M \) is the contact manifold \( \tilde{N} = M \times \mathbb{R} \), with contact form \( \lambda_{\tilde{N}} = dt + \alpha_M \), and contact structure \( \xi_{\tilde{N}} = \ker(\lambda_{\tilde{N}}) \). Note that there is a natural contact \( \mathbb{Z} \)-cover \( \tilde{N} \to N \).

**Definition 5.1.** A Lagrangian subvariety \( L \subset M \) is integral if there is a continuous function \( f : L \to S^1 \) such that the restriction of \( f \) to any submanifold of \( L \) is differentiable and a primitive for the restriction of \( \alpha_M \).

A Lagrangian subvariety \( L \subset M \) is exact if in addition there exists a lift of \( f : L \to S^1 \) to a continuous function \( \tilde{f} : L \to \mathbb{R} \).

**Remark 5.1.** A Lagrangian subvariety \( L \subset M \) is integral if and only if it admits a Legendrian lift \( \tilde{L} \subset \tilde{N} \). Similarly, \( L \subset M \) is exact if and only if it admits a Legendrian lift \( \tilde{L} \subset \tilde{N} \).
Return to the neighborhood $U_n \subset T^*\mathbb{T}^n$. It admits two Liouville forms: $\beta_n$ and the canonical Liouville form $\alpha_T \equiv \tau_n$. The Lagrangian subvariety $\mathcal{L}_n \subset U_n$ is conic with respect to the Liouville vector field associated to $\beta_n$, and thus exact with respect to $\beta_n$. On the other hand, if we construct $\mathcal{L}_n$ using $\chi > 0$ with $\hat{\chi} = \chi/(n+1)$ integral, then the function

$$\hat{f} : \mu_{\Delta}^{-1}(\chi) \cap \Lambda_{n+1} \to S^1, \hat{f} = \sum_{a=1}^{n+1} (\xi_a - \hat{\chi}) \theta_a$$

(54)

is invariant under $S^1_{\Delta}$-translations, hence descends to a function $f : \mathcal{L}_n \to S^1$. A straightforward computation shows that $f$ provides an integral structure of $\mathcal{L}_n$ for $\alpha_T \equiv \tau_n$.

Consider the circular contactification $(N_n, \lambda_n)$ of $U_n$. Denote by $\mathcal{L}_n$ the Legendrian lift of $\mathcal{L}_n$ to $(N_n, \lambda_n)$.

From now on we further specialize to $\chi = n + 1$ so that $\hat{\chi} = 1$. Introduce the conic open subspace and its spherical projectivization

$$\Omega_{n+1} - \mu_{\Delta}^{-1}(\mathbb{R}_{>0}) \subset T^*\mathbb{T}^{n+1}, \Omega_{n+1}^x - \Omega_{n+1}/\mathbb{R}_{>0} \subset S^x T^{n+1}.$$  

(55)

The natural projection gives an isomorphism of contact manifolds $\mu_{\Delta}^{-1}(\chi) \cong \Omega_{n+1}^x$.

**Lemma 5.1.** We have a finite contact cover

$$p_\chi : \Omega_{n+1}^x \cong \mu_{\Delta}^{-1}(\chi) \to T^*\mathbb{T}^n \times S^1$$

(56)

given by $p_\chi = p_\chi - \delta$, where $\delta : T^{n+1} \to S^1$ is the diagonal character.

The cover is trivializable over the neighborhood $N_n \subset T^*\mathbb{T}^n \times S^1$ of the Legendrian $\mathcal{L}_n \subset N_n$ with a canonical section $s : N_n \to \Omega_{n+1}^x$ such that $s(\mathcal{L}_n) = \Lambda_{n+1}^x$.

It follows that the contact geometry of the circular contactification $N_n \subset T^*\mathbb{T}^n \times S^1$ near the Legendrian lift $\mathcal{L}_n$ is equivalent to that of the open subspace $\Omega_{n+1}^x \subset S^x T^{n+1}$ near the Legendrian subvariety $\Lambda_{n+1}^x$.

Introduce the circular contactification $Q_n \times S^1$, and its symplectization $\tilde{Q}_n = Q_n \times S^1 \times \mathbb{R}$, with their natural projections

$$\tilde{Q}_n - Q_n \times S^1 \times \mathbb{R} \xrightarrow{c} Q_n \times S^1 \xrightarrow{\nabla} Q_n.$$  

(57)

The Lagrangian skeleton $L_n \subset Q_n$ lifts under $c$ to the Legendrian subvariety $L_n \times \{0\} \subset Q_n \times S^1$, and we can take its inverse image under $s$ to obtain a conic Lagrangian subvariety

$$\tilde{L}_n - s^{-1}(L_n \times \{0\}) \subset \tilde{Q}_n.$$  

(58)

The following is a consequence of Theorem 4.1.

**Theorem 5.1.** Fix $\chi = n + 1$. There is a conic open neighborhood $\tilde{U}_n \subset \tilde{Q}_n$ of the Lagrangian subvariety $\tilde{L}_n \subset \tilde{Q}_n$, a conic open neighborhood $\mathcal{T}_{n+1} \subset \Omega_{n+1}$ of the intersection $\Lambda_{n+1} \cap \Omega_{n+1}$, and an exact symplectomorphism

$$\tilde{j} : \tilde{U}_n \xrightarrow{\cong} \mathcal{T}_{n+1}$$

(59)

restricting to an isomorphism

$$\tilde{j}|_{L_n} : \tilde{L}_n \xrightarrow{\cong} \Lambda_{n+1} \cap \Omega_{n+1}.$$  

(60)

### 6 Mirror symmetry

We calculate (what is supposed to be) the dg category of wrapped microlocal sheaves on the pair-of-pants. Note that Theorem 5.1 allows us to define the dg category $\mu Sh_{L_n}(Q_n)$ of wrapped microlocal sheaves on $Q_n$ supported along $L_n$ to be the dg category of wrapped microlocal sheaves on $\Omega_{n+1}$ supported along $\Lambda_{n+1}$.
Lemma 6.1. There are mirror equivalences

\[ \text{Sh}_{\Lambda_{n+1}}(T^{n+1}) \cong \text{QCoh}(\mathbb{A}^{n+1}), \quad (61) \]
\[ \text{Sh}_{\Lambda_{n+1}}(T^{n+1}) \cong \text{Perf}_{\text{prop}}(\mathbb{A}^{n+1}), \quad (62) \]
\[ \text{Sh}_{\Lambda_{n+1}}(T^{n+1}) \cong \text{Coh}(\mathbb{A}^{n+1}). \quad (63) \]

Remark 6.1. Geometrically, \( \text{Sh}_{\Lambda_{1}}(S^1) \) and \( \text{Sh}_{\Lambda_{1}}^{w}(S^1) \) correspond respectively to the infinitesimal and partially wrapped Fukaya categories associated to the Landau-Ginzburg model \((\mathbb{C}^*, z)\), which is the mirror of \( \mathbb{A}^1 \).

Fix a subset \( I \subset \{1, \cdots, n+1\} \), with complement \( I^c = \{1, \cdots, n+1\} \setminus I \). Let \( T^I \subset T^{n+1} \) be the subtorus defined by \( \theta_a = 0 \) for \( a \in I^c \). Let \( \Lambda_I = (\Lambda_1)^I \subset T^*T^I \) be the product conic Lagrangian subvariety.

Consider the hyperbolic restriction

\[ \eta_I : \text{Sh}_{\Lambda_{n+1}}(T^n) \rightarrow \text{Sh}_{\Lambda_1}(T^I), \quad \eta_I(\mathcal{F}) := p_* q^! \mathcal{F} \quad (64) \]

built from the correspondence

\[ T^I \xrightarrow{p} T^I \times [0, 1/2]^{I^c} \xrightarrow{\theta} T^{n+1}, \quad (65) \]

where \( p \) is the projection and \( q \) is the inclusion.

Let \( f : \mathbb{A}^I \rightarrow \text{Spec} \mathbb{K}[t_a | a \in I] \rightarrow \mathbb{A}^n \rightarrow \text{Spec} \mathbb{K}[t_1, \cdots, t_n] \) be the affine subspace defined by \( t_a = 0 \) for \( a \in I^c \).

Lemma 6.2. The equivalences (61) and (62) fit into commutative diagrams

\[ \text{Sh}_{\Lambda_{n+1}}(T^{n+1}) \xrightarrow{\eta} \text{QCoh}(\mathbb{A}^{n+1}) \]
\[ \text{Sh}_{\Lambda_{n+1}}(T^I) \xrightarrow{\eta_I} \text{Perf}_{\text{prop}}(\mathbb{A}^{n+1}) \]

\[ \text{Sh}_{\Lambda_{n+1}}(T^{n+1}) \xrightarrow{\eta} \text{Perf}_{\text{prop}}(\mathbb{A}^I) \]

Theorem 6.1. There are mirror equivalences

\[ \mu \text{Sh}_{\Lambda_{n+1}}(\Omega_{n+1}) \cong \text{IndCoh}(X_n), \quad (68) \]
\[ \mu \text{Sh}_{\Lambda_{n+1}}(\Omega_{n+1}) \cong \text{Perf}_{\text{prop}}(X_n), \quad (69) \]
\[ \mu \text{Sh}_{\Lambda_{n+1}}^{w}(\Omega_{n+1}) \cong \text{Coh}(X_n). \quad (70) \]

Proof. Let \( \mathcal{J}_{n+1} \) be the category whose objects are subsets \( I \subset \{1, \cdots, n+1\} \), and morphisms \( I \rightarrow I' \) are inclusions \( I \subset I' \). Let \( \mathcal{J}_{n+1}^{0} \subset \mathcal{J}_{n+1} \) denote the full subcategory whose objects are proper subsets of \( \{1, \cdots, n+1\} \).

Define a functor \( A : \mathcal{J}_{n+1}^{0} \rightarrow \mathcal{X}_K \) as follows. For an object \( I \) of \( \mathcal{J}_{n+1}^{0} \), take the affine space \( A(I) = \mathbb{A}^I \), and for a morphism \( I \subset I' \), take the inclusion \( A(I, I') : \mathbb{A}^I \rightarrow \mathbb{A}^{I'} \), given by setting \( t_a = 0 \) for each \( a \in I^c \).

Recall the functor \( \text{IndCoh}^* : \mathcal{X}_K^{op} \rightarrow \text{dgSt}_K \) that assigns a scheme its ind-coherent sheaves and a proper morphism of schemes its \(*\)-pullback. Recall also the full subfunctor \( \text{Perf}_{\text{prop}}^{*} : \mathcal{X}_K^{op} \rightarrow \text{dgSt}_K \) of perfect complexes with proper support.
Consider the composite functor $\text{IndCoh}^* \circ A : (\mathcal{F}_{n+1})^{op} \to \text{dgSt}_{\mathcal{E}}$, and $\text{Perf}^*_{\text{prop}} \circ A(\mathcal{F}_{n+1})^{op} \to \text{dgSt}_{\mathcal{E}}$. By Proposition 1.2, the canonical maps are equivalences

$$\text{IndCoh}(\mathcal{X}) \xrightarrow{\cong} \lim_{(\mathcal{F}_{n+1})^{op}} \text{IndCoh}((\mathcal{A}^I)),$$

(71)

$$\text{Perf}^*_{\text{prop}}(\mathcal{X}) \xrightarrow{\cong} \lim_{(\mathcal{F}_{n+1})^{op}} \text{Perf}^*_{\text{prop}}((\mathcal{A}^I)),$$

(72)

$$\text{Coh}(\mathcal{X}) \xrightarrow{\cong} \text{colim}_{\mathcal{F}_{n+1}} \text{Coh}((\mathcal{A}^I)).$$

(73)

To prove the theorem, we will similarly identify $\mu \text{Sh}_{\Lambda_{n+1}}^\circ (\Omega_{n+1})$ as the limit of a functor

$$\mu \text{Sh}^\circ : (\mathcal{F}_{n+1})^{op} \to \text{dgSt}_{\mathcal{E}},$$

(74)

and then provide an equivalence of functors $\mu \text{Sh}^\circ \simeq \text{IndCoh}^* \circ A$. This will immediately prove the first and the third equivalences. For the second one, we observe that $\mu \text{Sh}_{\Lambda_{n+1}}^\circ (\Omega_{n+1})$ is the limit of a full subfunctor $\mu \text{Sh} \subset \mu \text{Sh}^\circ$, which is equivalent to the subfunctor $\text{Perf}^*_{\text{prop}} \circ A \subset \text{IndCoh}^* \circ A$.

For each $I \in \mathcal{F}_{n+1}$, introduce the conic open subspace $\Omega_I \subset \Omega_{n+1}$, cut out by the additional requirement $\zeta_a \neq 0$ for $a \notin I$. Thus for $I \subset I'$, we have the open inclusion $\Omega_I \subset \Omega_{I'}$, and for $I = \{1, \cdots, n+1\}$, we have $\Omega_I = \Omega_{n+1}$. Note that the collection $\{\Omega_I\}_{I \in \mathcal{F}_{n+1}}$ forms a conic open cover of $\Omega_{n+1}$ with the property $\Omega_{I \cap I'} = \Omega_I \cap \Omega_{I'}$. Define the functor $\mu \text{Sh}^\circ$ by

$$\mu \text{Sh}^\circ(I) = \mu \text{Sh}_{\Lambda_{n+1}}^\circ(\Omega_I),$$

(75)

with inclusions $I \subset I'$ taken to the restriction maps along the inclusions $\Omega_I \subset \Omega_{I'}$. Define the full subfunctor $\mu \text{Sh} \subset \mu \text{Sh}^\circ$ by $\mu \text{Sh}(I) = \mu \text{Sh}_{\Lambda_{n+1}}^\circ(\Omega_I)$.

Since $\mu \text{Sh}_{\Lambda_{n+1}}^\circ$ forms a sheaf, $\mu \text{Sh}_{\Lambda_{n+1}} \subset \mu \text{Sh}_{\Lambda_{n+1}}^\circ$ is a full subsheaf, and $\{\Omega_I\}_{I \in \mathcal{F}_{n+1}}$ is an open conic cover of $\Omega_{n+1}$, the canonical functors are equivalences

$$\mu \text{Sh}_{\Lambda_{n+1}}^\circ(\Omega_{n+1}) \xrightarrow{\cong} \lim_{(\mathcal{F}_{n+1})^{op}} \mu \text{Sh}_{\Lambda_{n+1}}^\circ(\Omega_I),$$

(76)

$$\mu \text{Sh}_{\Lambda_{n+1}}(\Omega_{n+1}) \xrightarrow{\cong} \lim_{(\mathcal{F}_{n+1})^{op}} \mu \text{Sh}_{\Lambda_{n+1}}(\Omega_I).$$

(77)

Next let us define an additional functor to interpolate between $\text{IndCoh}^* \circ A$ and $\mu \text{Sh}^\circ$.

For $I \in \mathcal{F}_{n+1}$, define the functor

$$\text{Sh}^\circ : (\mathcal{F}_{n+1})^{op} \to \text{dgSt}_{\mathcal{E}}, \quad \text{Sh}^\circ(I) = \text{Sh}_{\Lambda_I}^\circ(T^I)$$

(78)

with inclusions $I \subset I'$ taken to the hyperbolic restrictions

$$\eta_{I \subset I'} : \text{Sh}_{\Lambda_I}^\circ(T^{I'}) \to \text{Sh}_{\Lambda_I}^\circ(T^I), \quad \eta_{I \subset I'}(\mathcal{F}) = p_*q_! \mathcal{F}$$

(79)

built from the correspondence

$$T^I \xrightarrow{\cong} T^I \times [0,1/2]^{I \setminus I'} \xrightarrow{\cong} T^{I'},$$

(80)

where $p$ is the projection and $q$ is the inclusion. Define the full subfunctor $\text{Sh} \subset \text{Sh}^\circ$ by $\text{Sh}(I) = \text{Sh}_{\Lambda_I}(T^I)$.

Lemmas 6.1 and 6.2 imply that we have equivalences

$$\text{Sh}^\circ \simeq \text{IndCoh}^* \circ A, \quad \text{Sh} \simeq \text{Perf}^*_{\text{prop}} \circ A.$$ 

(81)

It remains to establish equivalences of functors

$$\text{Sh}^\circ \simeq \mu \text{Sh}^\circ, \quad \text{Sh} \simeq \mu \text{Sh}.$$ 

(82)

For any $I \in \mathcal{F}_{n+1}$, let us return to the hyperbolic restriction

$$\eta_I : \text{Sh}_{\Lambda_{n+1}}^\circ(T^{n+1}) \to \text{Sh}_{\Lambda_I}(T^I), \quad \eta_I(\mathcal{F}) = p_*q_! \mathcal{F}.$$ 

(83)
First, $\eta_I$ factors through the microlocalization

$$Sh_{\Lambda,n+1}^\circ(T^{n+1}) \to \mu Sh_{\Lambda_I}(\Omega_I) \xrightarrow{\tilde{\eta}_I} Sh_{\Lambda_I}(T^I)$$

(84)

since the hyperbolic restriction in the coordinate direction indexed by $a \in T^c$ vanishes on sheaves whose singular support does not intersect the locus $\{\xi_a > 0\} \subset T^*T^{n+1}$.

Next, for $I \subset I'$, the induced functors extend to natural commutative diagrams

$$\begin{array}{ccc}
\mu Sh_{\Lambda_I}(\Omega_I) & \xrightarrow{\tilde{\eta}_{I,I'}} & Sh_{\Lambda_{I'}}(T^{I'}) \\
\downarrow \rho_{I,I'} & & \downarrow \eta_{I,I'} \\
\mu Sh_{\Lambda_I}(\Omega_I) & \xrightarrow{\tilde{\eta}_I} & Sh_{\Lambda_I}(T^I)
\end{array}$$

(85)

Thus we have a map of functors $\tilde{\eta} : \mu Sh^\circ \to Sh^\circ$, restricting to a map of subfunctors $\mu Sh \to Sh$. It remains to show that $\tilde{\eta}$ is an equivalence. It suffices to show that

$$\tilde{\eta}_I : \mu Sh_{\Lambda_I}(\Omega_I) \to Sh_{\Lambda_I}(T^I)$$

(86)

is an equivalence for any $I \in I_{n+1}^c$. Note that it admits a inverse induced by the pushforward

$$j_{I*} : Sh_{\Lambda_I}(T^I) \to Sh_{\Lambda_{n+1}}(T^{n+1})$$

(87)

along the natural inclusion $j_I : T^I \to T^{n+1}$. To see this, note that $j_I$ is simply the product of inclusions in the coordinate directions indexed by $I^c$, and the identity in the coordinate directions indexed by $I$.

**Corollary 6.1.** There is a quasi-equivalence of differential $\mathbb{Z}/2$-graded categories

$$\mu Sh_{\Lambda_{n+1}}^\circ(\Omega_{n+1})_{\mathbb{Z}/2} \simeq MF(\mathbb{A}^{n+2}, W_{n+2}).$$

(88)

For a $\mathbb{Z}$-graded version of the above equivalence proved for the actual wrapped Fukaya category $W(P_n)$, see Lekili-Polischuk.