<u>Recall</u>: Microlocal Morse theater:

$$\Lambda \longmapsto \mathcal{C}(\Lambda)$$

s.t. ① For $\Lambda = \mathcal{S}$ whitney Δ^n.

$\mathcal{C} \simeq \mathrm{Perf}$.

② Def^n: Morse characters at $p \in \Lambda$ sm pt:

$f: M \to \mathbb{R}$, $f(p)=0$ only Λ-crit value in $f^{-1}[-t,t]$

$\mathfrak{X}_{(p,f,\Lambda)}$ = image in $\mathcal{C}(\Lambda)$ of

$$\mathrm{Cone}(\mathbb{1}_{f^{-1}(-\infty,\epsilon)} \to \mathbb{1}_{f^{-1}(-\infty,\epsilon]}) \in \mathcal{C}(N^*\mathcal{S})$$

Then for $\Lambda \subset \Lambda'$,

$$\mathcal{C}(\Lambda')/\mathfrak{D} \xrightarrow{\sim} \mathcal{C}(\Lambda)$$

Last time: sheaves form morse theatre:

$\Lambda \longmapsto Sh_\Lambda(M)^c$ is a Morse theatre

<u>Today</u>: Fukaya categories:

Conventions: For L, K disjoint at ∞

$HF(L,K)$: gen by intersect^n points.

$HF(L,L) := HF(L^+, L)$, L^+: small pushoff of L

$\simeq H^*(L)$ has a unit ← continuat^n elt

img of unit in $HF(L,K)$

$$HW(L,K) = \varinjlim_{L \rightsquigarrow L^{++}} HF(L^{++},K) = \varinjlim_{\substack{L \rightsquigarrow L^{++} \\ K^{--} \rightsquigarrow K}} HF(L^{++}, K^{--})$$

$$= \varinjlim_{K^{--} \rightsquigarrow K} HF(L,K).$$

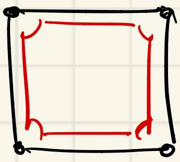
$\mathcal{W}(T^*M)$

L_t a cofinal seq, L_t disj from K for $t \gg 0$, then $HW(L,K) = HF(L_t, K)$, $t \gg 0$.

More generally only wrap in the complement of $\Lambda \subset \partial_\infty T^*M \rightsquigarrow \mathcal{W}(T^*M, \Lambda)$.

Let $\mathcal{S}$ be a Whitney stratification, U $\mathcal{S}$-constructible
$U^{-\epsilon}$ denote an "inward cornering" of U.

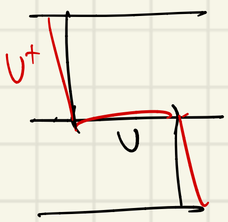
roughly delete union of tub nbd of strata.



Key property: $N^* U^{-\epsilon}$ ① flows into the stop $N^*\mathcal{S}$
② always disjoint from the stop.

§ Computing $\mathcal{W}(T^*M, N^*\mathcal{S})$, for $\mathcal{S}$ Whitney Δ^n:

Want $\mathbb{Z}[\mathcal{S}] \xrightarrow{\sim} \mathcal{W}(T^*M, N^*\mathcal{S})$



$U \rightsquigarrow L_U := N^*_+ U^{-\epsilon}$ + smoothened corners.
positive conormal

$U^+ \rightsquigarrow L_{U^+} = L_U^+$ $\quad$ $pt^+ \rightsquigarrow$ ball

$U^+ = \epsilon$-nbd of U

Defn: $U \underset{\text{open}}{\subset} M$ ball: $\bar{U}$ diffeo to cld ball

① U, V balls, $U \subset\subset V$, $HF(L_V, L_U) \simeq \mathbb{Z}$
canonically generated by continuation elt.

② U, V balls, W open
$U \subset\subset V \subset\subset W$,

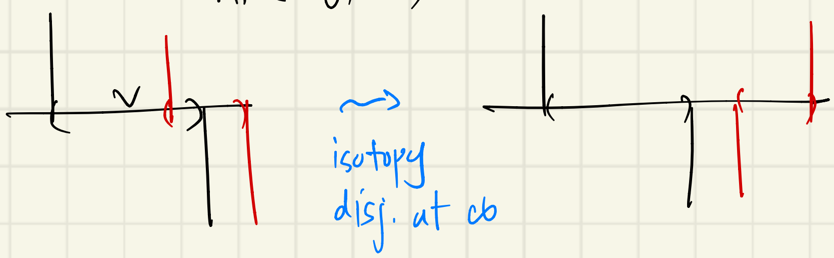
$HF(L_W, L_V) \xrightarrow{\sim} HF(L_W, L_U)$ multiplication by continuation elt
is an iso

③ $\exists$ canonical iso

$HF(L_w, L_U) \longrightarrow \mathbb{Z}$ respecting iso in ②

④ V open w/ sm ∂. U an ϵ-ball w/ center on ∂V

$HF(L_U, L_V) = 0$



② Simplifying assumption: $\mathcal{S}$ is smooth stratificat$^{\text{n}}$

all star(s), $s \in \mathcal{S}$ are balls

in general only know contractibility star(s)

"stable ball"

technical results: analogs of ①-④ for stable balls.

§ Floer cohomology for conormals of stars stopped at $N^*\mathcal{S}$

$\mathcal{S}$ Whitney Δ.

① L disj from $N^*\mathcal{S}$, U $\mathcal{S}$-constr.

$$CF(L_{U^{-\epsilon}}, L) \xrightarrow{\sim} CW(L_{U^{-\epsilon}}, L)_{N^*\mathcal{S}}$$

② Proposition Fully faithful.

$$HW(L_{star(s)}, L_{star(t)}) = \begin{cases} \mathbb{Z} & t \to s \\ 0 & o/w. \end{cases}$$

Pf:

$$HW(L_{star(s)}, L_{star(t)})_{N \ast S}$$

$$= HF(L^{-\delta}_{star(s)}, L^{-\epsilon}_{star(t)}),$$

$\underline{\epsilon} > 0$ fixed

$\underline{\delta} \to 0$

ⓐ $t \to s$, $L^{-\epsilon}_{st(t)} \subset L^{-\delta}_{st(s)} \Rightarrow \mathbb{Z}$

incl of balls

ⓑ $t \not\to s$, $st(t) \cap st(s) = \phi$: Done

ⓒ $t \not\to s$, $st(t) \cap st(s) \neq \phi$

$st(t) \cap st(s) = st(r)$, $r = span(s, t)$

$st^{-\epsilon}(t) \cap \partial st^{-\delta}(s)$ ball

③ L_s: conormal to small ball w/ center on strata $s \in \mathcal{S}$.

$$HW(L_U, L_s)_{N \ast \mathcal{S}} = \begin{cases} \mathbb{Z} & star(s) \subset U \\ 0 & o/w. \end{cases}$$

Pf: ⓐ $s \subset int(U)$.

L_s: pushoff of cotangent fiber

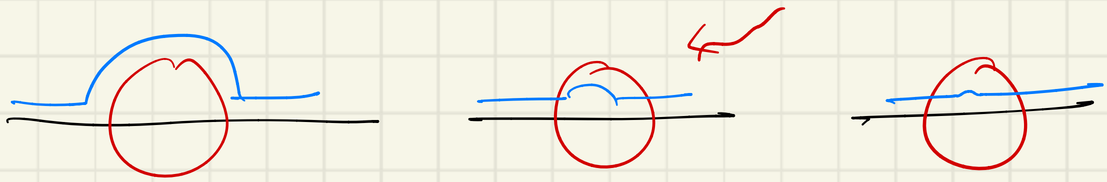
$HW(L_U, L_s)_{N \ast \mathcal{S}} = HF(L_U, L_s)$

can arrange single int. pt

ⓑ $s \not\subset \bar{U}$ $\rightsquigarrow 0$

ⓒ $s \subseteq \partial \bar{U}$: always disj. at ∞.

Cofinal wrapping



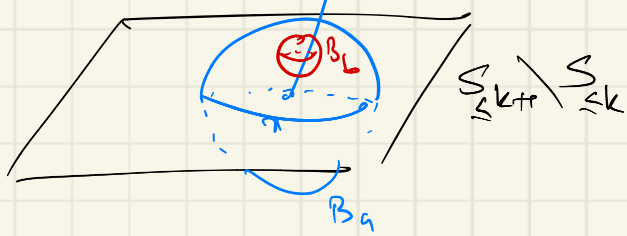
④ Prop^{n}: L_S : split generate $\mathcal{W}(T^*M, N^*S)$

$$\mathcal{W}(T^*M, N^*S) \to \cdots \mathcal{W}(T^*M, N^*S_{\leq k}) \to \mathcal{W}(T^*M, N^*S_{\leq k-1})$$

$$\cdots \to \mathcal{W}(T^*M)$$

localization at linking disks at $N S_{\leq k-1} \setminus N S_{\leq k}$.

(x, ξ) : pt in conormal

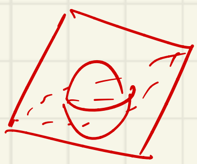


Wrapping exact triangle : $L_a \to L_B \to \begin{pmatrix} \text{linking} \\ \text{disk at} \end{pmatrix} \to 1$
(x, ξ)

$\Rightarrow$ L_a's split generate.

⑤ Prop^{n}: $L_{st(S)} \to L_S$ is an iso

Intuition :



ball inside star fills it

① codim 0 , Ball $\subset$ open set.

② codim > 0 , for codim t < codim s ,

$\mathrm{Hom}(L_{star(t)}, L_{star(s)}) = 0$

$\mathrm{Hom}(L_{star(t)}, L_s) = 0$

$\mathcal{W}(T^*M, N^*S) \to \mathcal{W}(T^*M, N^*S_{\leq \dim s})$

is quotient by L_t for codim t < codim s

Which vanish on hitting $L_{star(s)}$, L_s on right

Suffices to show $L_{star(s)} \to L_s$ iso in

$\mathcal{W}(T^*M, N^*S_{\leq \dim s})$

Follows by Mystery result 5-18 .

§ Thm: ① Get functor:

$$\textcircled{\star} \qquad H^* \mathbb{Z}[S] \longrightarrow H^* \mathcal{W}(T^*M, N^*S)^{op}.$$ Morita equiv.

fully faithful + essentially surj ← on Perf categories

Homological algebra (all morphism spaces live in deg 0)

② ∃ unique upto contractible choice lift

$$\mathbb{Z}[S] \longrightarrow \mathcal{W}(T^*M, N^*S).$$

§ Thm: $\Lambda \longmapsto \mathcal{W}(T^*M, N^*S)$ is a Morse theater, with Morse char = linking disc.

Pf: Previous thm: natural identificat

$$\mathrm{Perf}(\mathcal{A}) \longrightarrow \mathrm{Perf}\, \mathcal{W}(T^*M, N^*S)$$

$$\mathbb{1}_{f^{-1}(\dots)} \to L_{f^{-1}(\dots)} \longrightarrow \text{linking disk}$$

∴ suffices to show

① $\mathbb{1}_{f^{-1}(\dots)} \xrightarrow{\sim} L_{f^{-1}(\dots)} \in \mathrm{Perf}(T^*M, N^*S)$

② canonical map

$$\mathbb{1}_{f^{-1}(\dots)} \longrightarrow \mathbb{1}_{f^{-1}(\dots)}$$

sent by F_S to correct continuation elt.

$$L_{f^{-1}(\dots)} \longrightarrow L_{f^{-1}(\dots)}$$ ← involved in continuation exact △

① : Thm ① ✪ $\Rightarrow F_S$ Morita equiv. suffices to show

$$F_S^{*} L_w \simeq \mathbb{1}_w \underset{\text{Yoneda}}{\iff} \mathrm{Hom}(L_w, F_S(-)) \cong \mathbb{1}_w(-)$$

at level of obj : result above $\Rightarrow HW(L_w, L_{star(s)})_{N^*s} = \begin{cases} \mathbb{Z} & star(s) \subseteq W \\ 0 & o/w \end{cases}$

at level of morph: for $t \to s$

$$HW(L_w, L_{star(s)}) \longrightarrow HW(L_w, L_{star(t)})$$

given by continuation maps S_f and compatible with $\mathbb{Z} \longrightarrow \mathbb{Z}$

② $F_S : \mathbb{1}_{f^{-1}(\dots)} \longmapsto L_{f^{-1}(\dots)}$

To show $F_S : \left(\mathbb{1}_{f^{-1}(-\infty,-\epsilon)} \to \mathbb{1}_{f^{-1}(-\infty,\epsilon)}\right) \longmapsto \left(L_{f^{-1}(\dots)} \to L_{f^{-1}(\dots)}\right)$

in the exact seq.

ie. $\forall\, s \in S$, $\mathbb{1}_{f^{-1}(-\infty,-\epsilon)}(s) \to \mathbb{1}_{f^{-1}(-\infty,\epsilon)}(s)$

$$\begin{array}{c} ||| \\ HW(L_{f^{-1}(\dots)}, \underbrace{F_S(s)}_{L_{star(s)}})_{N^*s} \longrightarrow HW(L_{f^{-1}(\dots)}, \underbrace{F_S(s)}_{L_{star(s)}})_{N^*s} \end{array}$$

given by continuation $L_{f^{-1}(-\infty,-\epsilon)} \rightsquigarrow L_{f^{-1}(-\infty,\epsilon)}$

is comptible w/ identification

$$\begin{tikzcd} HW(L_{f^{-1}(\dots)}, F_S(s))_{N^*s} \arrow[r] \arrow[d, "S"] & HW(L_{f^{-1}(\dots)}, F_S(s))_{N^*s} \arrow[d, "S"] \\ \mathbb{Z} \arrow[r] & \mathbb{Z} \end{tikzcd}$$

Follows from $L_{star(s)} \xrightarrow{\sim} L_s$:

$$\begin{tikzcd} HW(\dots, F_S(s))_{N^*s} \arrow[r] \arrow[d, "s"] & HW(\dots, F_S(s))_{N^*s} \arrow[d, "s"] \\ HW(\dots, L_s)_{N^*s} \arrow[r] \arrow[d, "s"] & HW(\dots, L_s)_{N^*s} \arrow[d] \\ HF(\dots, L_s) \arrow[r] \arrow[d] & HF(\dots, L_s) \arrow[d] \\ \mathbb{Z} \arrow[r] & \mathbb{Z} \end{tikzcd}$$

identity since continuation happening away from L_s