Fiber Floer cohomology and conormal stops

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Introduction

Let S be a closed, orientable and spin n-manifold. Consider the Liouville manifold (T^*S, λ), where $\lambda := pdq$ and $\omega := d\lambda$ in coordinates $(q, p) \in T^*S$. The Liouville vector field is defined as

$$\omega(X, -) = \lambda\,.$$

In this case $X = p\partial_p$.

- ω is symplectic (ω closed and non-degenerate)
- T^*S can be exhausted by domains with convex boundary (with respect to X)

Introduction

The $(2n-1)$-dimensional boundary ST^*S is a contact manifold with contact form $\alpha := \lambda|_{ST^*S}$.

$$\alpha \wedge d\alpha^{\wedge(n-1)} \neq 0\,.$$

The Reeb vector field R is defined by

- $d\alpha(R, -) = 0$
- $\alpha(R) = 1$

In this case $R = p\partial_q$.

Introduction

Let $F = T^*_\xi S$ be a cotangent fiber. The wrapped Floer cochain complex $CW^*(F,F)$ is generated by

- Reeb chords of $\partial F \subset ST^*S$
- One generator corresponding to a minimum of Morse function on F

The differential $\mu^1\colon CW^*(F,F) \longrightarrow CW^*(F,F)$ is defined by counting holomorphic disks with 2 boundary punctures.

Maps $\mu^k\colon CW^*(F,F)^{\otimes k} \longrightarrow CW^*(F,F)$ counting holomorphic disks with $k+1$ boundary punctures turn $CW^*(F,F)$ into an A_∞-algebra.

Introduction

Abbondandolo–Schwarz (2008) and Abouzaid (2012)

There is an isomorphism $HW^*(F,F) \cong H_{-*}(\Omega S)$ which intertwines the triangle product μ^2 on $HW^*(F,F)$ with the Pontryagin product on $H_{-*}(\Omega S)$.

Introduction

Let $K \subset S$ be a submanifold and consider its unit conormal bundle

$$\Lambda_K := \{(x,p) \mid x \in K, |p| = 1, \langle p, T_x K\rangle = 0\} \subset ST^*S\,.$$

The smooth topology of K can be studied via Λ_K:

- *Legendrian contact homology* (LCH) of Λ_K is related to the Alexander polynomial of $K \subset \mathbb{R}^3$ and furthermore detects the unknot and torus knots (Ng 2008, Ekholm–Ng–Shende 2016, Cieliebak–Ekholm–Latschev–Ng 2016)
- Refined version of LCH of Λ_K is a complete knot invariant of $K \subset \mathbb{R}^3$ (ENS 2016)
- Legendrian isotopy class of Λ_K is a complete knot invariant of $K \subset \mathbb{R}^3$ (Shende 2016, ENS 2016)

Main results

Questions

1. How is $HW^*(F,F)$ affected by adding a stop at Λ_K?
2. Does the Legendrian isotopy class of Λ_K know about the underlying smooth codimension 2 knot $K \subset S^n$ for $n \geq 4$?

The based loop space ΩS is homotopy equivalent to the space BS of smooth piecewise geodesic loops.

Theorem A

Let $M_K := S \setminus K$. There is an isomorphism of A_∞-algebras $\Psi\colon CW^*_{\Lambda_K}(F,F) \longrightarrow C^{\mathrm{cell}}_{-*}(BM_K)$.

In particular we have $HW^*_{\Lambda_K}(F,F) \cong H_{-*}(\Omega M_K)$.

Main results

Theorem B

Ψ induces an isomorphism of $\mathbb{Z}[\pi_1(M_K)]$-modules $HW^*_{\Lambda_K}(F,F) \longrightarrow H_{-*}(\Omega_\xi M_K)$.

Application

For certain codimension 2 knots $K \subset S^n$, $HW^*_{\Lambda_K}(F,F)$ is related to the Alexander invariant of K.

Theorem C

Let $n = 5$ or $n \geq 7$. Then there exists a codimension 2 knot $K \subset S^n$ with $\pi_1(M_K) \cong \mathbb{Z}$ such that $\Lambda_K \cup \Lambda_x \not\simeq \Lambda_{\mathsf{unknot}} \cup \Lambda_x$ where $x \in M_K$.

Wrapped Floer cohomology without Hamiltonian

Let $CW^*(F, F)$ denote the wrapped Floer cochain complex of the fiber $F \subset T^*S$.

Generators

- Reeb chords of $\partial F \subset ST^*S$
- One generator corresponding to a minimum of Morse function on F

Grading

(Negative of the) Conley–Zehnder index

Wrapped Floer cohomology without Hamiltonian

Parallel copies of F

Choose family of positive Morse functions

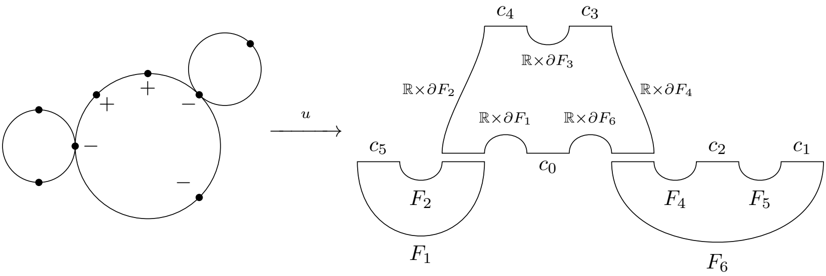
$$G_k \colon F \longrightarrow \mathbb{R}$$
$$g_k \colon \partial F \longrightarrow \mathbb{R}$$

- G_k has one minimum on F and no other critical points
- F_k = flow of F by the Hamiltonian vector field associated to G_k
- ∂F_k is a small pushoff of ∂F in the positive Reeb direction

Wrapped Floer cohomology without Hamiltonian

A_∞-structure

$\mu^k\colon CW^*(F,F)^{\otimes k} \longrightarrow CW^*(F,F)$ counts solutions to the equation $\overline{\partial}_J u = \frac{1}{2}\left(\frac{\partial u}{\partial s} + J(u)\frac{\partial u}{\partial t}\right) = 0$ with k inputs, 1 output and switching boundary conditions

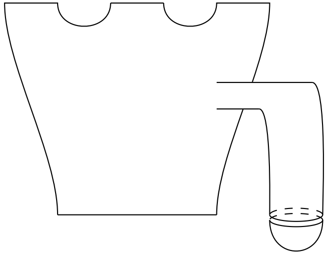


No cuvers can be multiply covered! Boundary bubbling is precluded for topological reasons.

Wrapped Floer cohomology without Hamiltonian

Anchoring

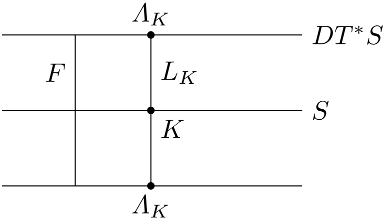
Disks are allowed to have internal punctures asymptotic to Reeb orbits. Anchoring means that we cap off the Reeb orbits by holomorphic planes.

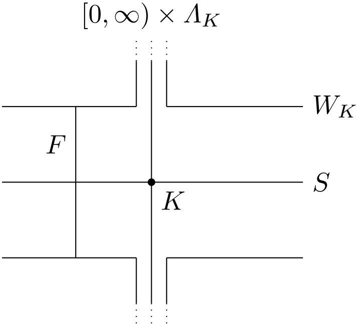


Transversality of holomorphic planes requires abstract perturbations.

Partially wrapped Floer cohomology via surgery

Let $K \subset S$ be any submanifold and consider its unit conormal bundle $\Lambda_K \subset ST^*S$.

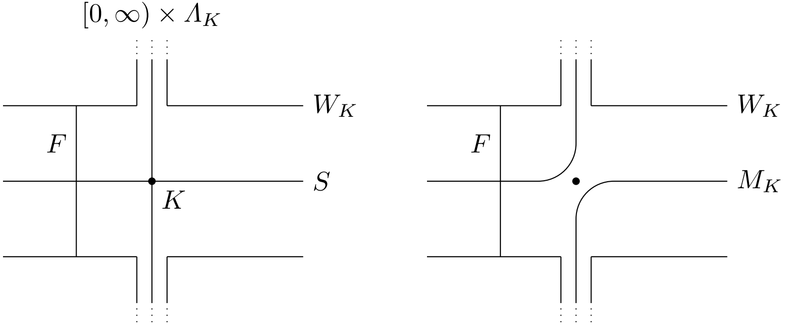




Attach a handle along Λ_K modeled on $D_\varepsilon T^*([0,\infty) \times \Lambda_K)$.
$CW^*_{\Lambda_K}(F,F)$ is defined as $CW^*(F,F)$ computed in W_K

The complement Lagrangian

$\mathrm{Skel}(W_K) = S \cup L_K$, and $S \cap L_K = K$ is a clean intersection.
Lagrange surgery along K gives an exact Lagrangian $M_K \cong S \setminus K$

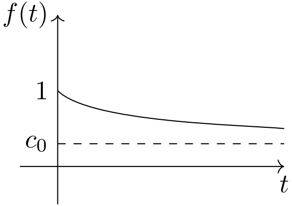


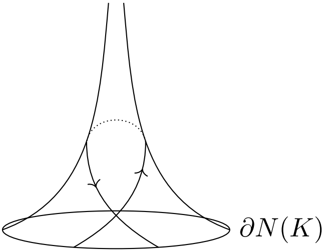
The complement Lagrangian

Metric on M_K

Pick a generic metric g in S away from $N(K)$. Then define

$$h = \begin{cases} dt^2 + f(t)\, g|_{\partial N(K)} & \text{in } [0,\infty) \times \partial N(K) \\ g & \text{in } S \setminus N(K) \end{cases}$$



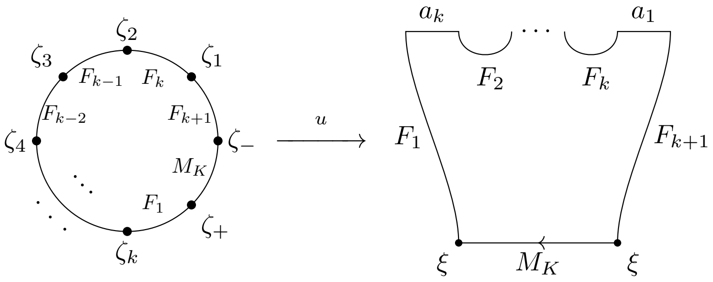


Moduli space of half strips

Let

- $k \geq 1$
- $\boldsymbol{a} = a_1, \ldots, a_k$ generators of $CW^*_{\Lambda_K}(F, F)$

Consider $\mathcal{M}(\boldsymbol{a})$ moduli space of holomorphic disks



Moduli space of half strips

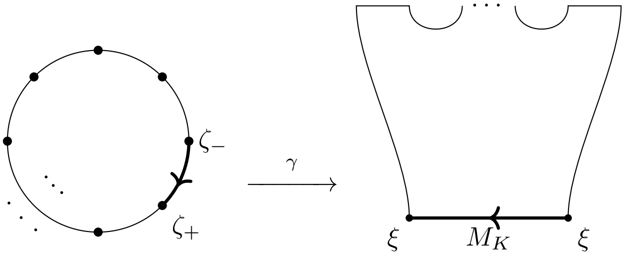
The moduli space $\mathcal{M}(\boldsymbol{a})$ has the following properties:

- Transversely cut out.
- Compact after adding broken disks. Denote the compactification by $\overline{\mathcal{M}}(\boldsymbol{a})$.
- $\dim \overline{\mathcal{M}}(\boldsymbol{a}) = -1 + k - \sum_{i=1}^{k} |a_i|$
- There exists a family of fundamental chains $[\overline{\mathcal{M}}(\boldsymbol{a})]$, which is compatible with orientations and the boundary stratification.

The evaluation map

$$\begin{aligned} \mathrm{ev}\colon \overline{\mathcal{M}}(\boldsymbol{a}) &\longrightarrow \Omega_\xi M_K \\ u &\longmapsto \gamma\,. \end{aligned}$$

Restriction of u to the boundary arc between the punctures $\zeta_\pm$



The A_∞-homomorphism

We define

$$\begin{aligned} \Psi_k \colon CW^*_{\Lambda_K}(F,F)^{\otimes k} &\longrightarrow C^{\mathsf{cell}}_{-*}(\Omega_\xi M_K) \\ a_k \otimes \cdots \otimes a_1 &\longmapsto \mathrm{ev}_*([\overline{\mathcal{M}}(\boldsymbol{a})]) \end{aligned}$$

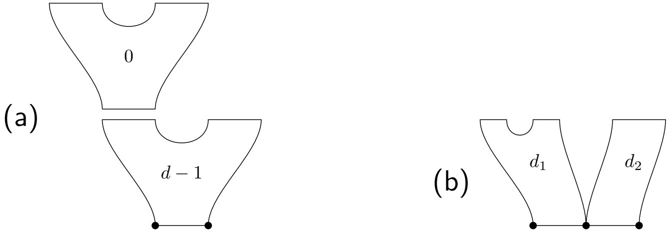
Proposition

$\Psi := \{\Psi_k\}_{k=1}^{\infty}$ is an A_∞-homomorphism.

The A_∞-homomorphism

Proof.

We look at the boundary of $\overline{\mathcal{M}}(\boldsymbol{a})$ of dimension d. The codimension 1 boundary is covered by strata of the form:



Strata of the form (a) contributes to $\partial\Psi_k$ with terms of the form

$$\sum_{r+s+t=k} \Psi_{r+1+t}(\mathrm{id}^{\otimes r} \otimes \mu^s \otimes \mathrm{id}^{\otimes t}) \,.$$

The A_∞-homomorphism

Proof.

Strata of the form (b) contributes to $\partial \Psi_k$ with terms of the form

$$\sum_{k_1+k_1=k} P(\Psi_{k_2} \otimes \Psi_{k_1}) .$$

All the terms together make up the codimension 1 boundary of a compact oriented manifold, so

$$\partial \Psi_k = \sum_{r+s+t=k} \Psi_{r+1+t}(\mathrm{id}^{\otimes r} \otimes \mu^s \otimes \mathrm{id}^{\otimes t}) + \sum_{k_1+k_1=k} P(\Psi_{k_2} \otimes \Psi_{k_1})$$

□

Correspondence between generators

Each geodesic in M_K of index λ corresponds to a generator of $C^{\text{cell}}_{-*}(BM_K)$ in degree λ.

Length filtrations

- For $c \in CW^*_{\Lambda_K}(F,F)$ a Reeb chord, define $\mathfrak{a}(c) := \int_c \lambda$
- For $\sigma \in C^{\text{cell}}_{-*}(BM_K)$, define $\mathfrak{a}(\sigma) := \max_{x \in [0,1]^*} L(\sigma(x))$, where

$$L(\gamma) = \int_0^1 \left|\gamma(t)\right| dt \,.$$

Lemma

There is a one-to-one correspondence

$$\left\{ \begin{array}{c} \textit{Reeb chords of } \partial F \\ \textit{of index } -\lambda \textit{ and action } A \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \textit{Geodesic loops in } M_K \\ \textit{of index } \lambda \textit{ and length } A \end{array} \right\}$$

Correspondence between generators

Proof.

We consider the "trivial holomorphic strip" over a Reeb chord a:

$T_{(q,p)} \operatorname{im} u_0 = \operatorname{span}\{\text{Reeb}, \text{Liouville}\}$

Consider the 2-form $d\beta$ where

$$\beta := \text{cutoff}(|p|) \cdot \frac{pdq}{|p|}.$$

Integrating $d\beta$ over any holomorphic half strip u asymptotic to a yields $\mathfrak{a}(a) - L(\gamma) \geq 0$. In the case of u_0 we have equality. □

Towards the proof of the main theorem

Lemma

u_0 is transversely cut out

Corollary

For any $a \in CW^*_{\Lambda_K}(F, F)$ we have $\mathfrak{a}(a) = \mathfrak{a}(\Psi_1(a))$.

Proof of lemma.

Let $v \in \ker D_{u_0}$ and define for $\varepsilon > 0$

$$\begin{cases} u_\varepsilon := \exp_{u_0}(\varepsilon v) \\ \gamma_\varepsilon := \mathrm{ev}(u_\varepsilon) \end{cases} .$$

Towards the proof of the main theorem

Proof of lemma.

Then

1. From u_0: $\mathfrak{a}(a) = L(\gamma_0)$
2. Can show: $\mathfrak{a}(a) - L(\gamma_\varepsilon) > 0$

This implies

$$L(\gamma_\varepsilon) - L(\gamma_0) < 0 \implies E_{**}(\pi_* v, \pi_* v) < 0\,.$$

By unique continuation, the restriction

$$\pi_* \colon \ker D_{u_0} \longrightarrow \left\{ w \in T_\gamma BM_K \mid E_{**}(w,w) < 0 \right\}\,,$$

is injective, and finally $\dim \ker D_{u_0} \leq \operatorname{ind} \gamma_0 = \operatorname{ind} D_{u_0}$. □

Outline of proof of the main theorem

Lemma

The evaluation

$$\mathrm{ev}\colon \overline{\mathcal{M}}(a) \longrightarrow BM_K\,,$$

is a submersion at $u_0 \in \overline{\mathcal{M}}(a)$.

Theorem

For any $A > 0$, there is an isomorphism of A_∞-algebras $\mathcal{F}_A CW^*_{\Lambda_K}(F, F) \cong \mathcal{F}_A C^{\mathit{cell}}_{-*}(BM_K)$.

Finally take colimits as $A \to \infty$, and we obtain the main theorem.

$\mathbb{Z}[\pi_1(M_K)]$-module structure in cohomology

In cohomology we can equip $HW^*_{\Lambda_K}(F,F)$ and $H_*(\Omega M_K)$ with $\mathbb{Z}[\pi_1(M_K)]$-module structures.

$$\begin{aligned} \pi_1(M_K) \times H_*(\Omega M_K) &\longrightarrow H_*(\Omega M_K) \\ ([\gamma], \sigma) &\longmapsto P(\sigma \otimes \sigma_\gamma) \end{aligned}$$

$$\begin{aligned} \pi_1(M_K) \times HW^*_{\Lambda_K}(F,F) &\longrightarrow HW^*_{\Lambda_K}(F,F) \\ ([\gamma], a) &\longmapsto \mu^2(a \otimes a_\gamma) \end{aligned}$$

$\mathbb{Z}[\pi_1(M_K)]$-module structure in cohomology

Theorem B

Ψ induces an isomorphism of $\mathbb{Z}[\pi_1(M_K)]$-modules $HW^*_{\Lambda_K}(F,F) \longrightarrow H_{-*}(\Omega M_K)$.

The Alexander invariant

We now let $n = 5$ or $n \geq 7$ and $K \subset S^n$ a codimension 2 knot.

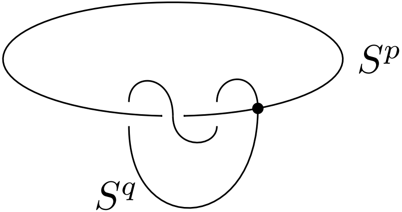
- Alexander invariant of K is $H_*(\widetilde{M}_K)$ viewed as a $\mathbb{Z}[t^{\pm 1}]$-module
- It is used to show non-triviality of certain knots K with $\pi_1(M_K) \cong \mathbb{Z}$.

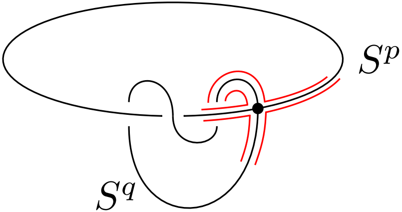
Plumbing of spheres

Let $n = p + q + 1$ where $p \geq 2$, $q \geq 4$ or $p = q = 2$. Consider $S^p, S^q \subset S^n$.

The Alexander invariant

Plumbing of spheres





Define K to be the boundary of the plumbing above.

The Alexander invariant and partially wrapped Floer cohomology

Theorem C

Let $n = p + q + 1$ with p and q as above. Then there exists a codimension 2 knot $K \subset S^n$ with $\pi_1(M_K) \cong \mathbb{Z}$ such that $\Lambda_K \cup \Lambda_x \not\simeq \Lambda_{\text{unknot}} \cup \Lambda_x$ where $x \in M_K$.

Sketch of proof.

By using the Leray–Serre spectral sequence, we can compute that in either case

$$H_p(\widetilde{M}_K) \cong \mathbb{Z}[t^{\pm 1}]/(t-1) \otimes_{\mathbb{Z}[t^{\pm 1}]} HW^{1-p}_{\Lambda_K}(F,F)\,.$$

Implies $HW^{1-p}_{\Lambda_K}(F,F) \ncong HW^{1-p}_{\Lambda_{\text{unknot}}}(F,F)$ and hence $\Lambda_K \cup \Lambda_x \not\simeq \Lambda_{\text{unknot}} \cup \Lambda_x$ where $x \in M_K$. □

Thank you!