Relative Ginzburg algebras and Chekanov–Eliashberg dg-algebras

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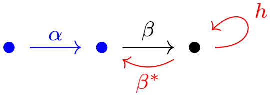
Main result

The relative Ginzburg algebra

Let $\boldsymbol{k}$ be a field, Q a quiver and $F \subset Q$ a subquiver.
Define $\overline{Q}_n(F)$ to be the graded quiver consisting of

- $g\colon v \to w$ in degree 0 for each $g\colon v \to w$ in Q_1.
- $g^*\colon w \to v$ in degree $2-n$ for each $g\colon v \to w$ in $Q_1 \setminus F_1$.
- $h_v\colon v \to v$ in degree $1-n$ for each $v \in Q_0 \setminus F_0$.

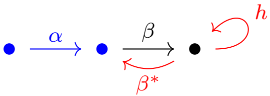


The relative Ginzburg algebra

The n-dimensional relative Ginzburg algebra $\mathcal{G}_n(Q,F)$ is the path algebra $\boldsymbol{k}\overline{Q}_n(F)$ equipped with the differential defined by

$$dg = dg^* = 0, \qquad dh_v := \sum_{g\colon v\to\bullet} gg^* - \sum_{g\colon \bullet\to v} g^*g$$

Example



$$d\alpha = d\beta = d\beta^* = 0, \qquad dh = \beta^*\beta.$$

□

There is in general a natural map $\mathcal{G}_{n-1}(F) \to \mathcal{G}_n(Q,F)$.

The Chekanov–Eliashberg algebra

The Chekanov–Eliashberg algebra is a dg-algebra defined for *Legendrian submanifolds* of *contact manifolds*.

- Quasi-isomorphism class: Invariant of the Legendrian isotopy class.
- Floer theory and Fukaya categories.
- “Computable”

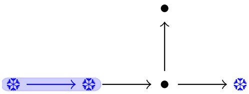
Main result

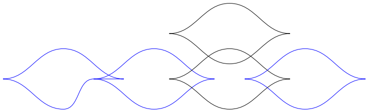
Theorem A (A.)

Let (Q, F) be a quiver pair and let $n \geq 4$. Then there exists a $(2n-1)$-dimensional contact manifold $Y(Q)$ and a singular $(n-1)$-dimensional Legendrian submanifold $\Lambda(Q, F) \subset Y(Q)$ such that:

1. *There is a quasi-isomorphism of dg-algebras $CE^*(\Lambda(Q, F); Y(Q)) \cong \mathcal{G}_n(Q, F)$.*
2. *There is a canonical dg-subalgebra $\mathcal{B} \subset CE^*(\Lambda(Q, F); Y(Q))$ and a quasi-isomorphism of dg-algebras $\mathcal{B} \cong \mathcal{G}_{n-1}(F)$ such that the following diagram commutes*

$$\begin{array}{ccc} \mathcal{B} & \hookrightarrow & CE^*(\Lambda(Q, F); Y(Q)) \\ \downarrow \cong & \circlearrowleft & \downarrow \cong \\ \mathcal{G}_{n-1}(F) & \longrightarrow & \mathcal{G}_n(Q, F) \end{array}$$





- The proof is constructive and every quasi-isomorphism in fact comes from an explicit chain homotopy equivalence.
- $\Lambda(Q, F)$ is singular if and only if $F_1 \neq \varnothing$.
- If $F_1 = \varnothing$, then there is a quasi-isomorphism $\mathcal{B} \cong C_{-*}(\Omega\Lambda(F))$ where $\Lambda(F) \subset \Lambda(Q, F)$ are the frozen components.

The natural map $\mathcal{G}_{n-1}(F) \to \mathcal{G}_n(Q, F)$ admits a *strong relative smooth n-Calabi–Yau structure*

Corollary

The canonical inclusion $\mathcal{B} \hookrightarrow CE^*(\Lambda(Q, F); Y(Q))$ admits a strong relative smooth n-Calabi–Yau structure.

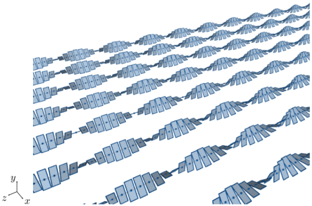
Known results

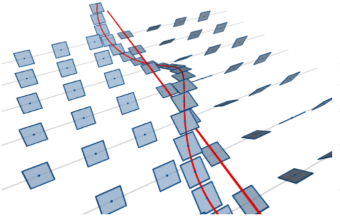
- Strong smooth n-Calabi–Yau structure on the wrapped Fukaya category (Ganatra 2012,2019, Shende–Takeda 2016 + Ganatra–Pardon–Shende 2024)
- Weak smooth (relative) n-Calabi–Yau structure on the Chekanov–Eliashberg algebra (Legout 2023, Dimitroglou Rizell–Legout in progress)
- Other related variants on augmentation categories (Chen in progress, Sabloff–Ma in progress)

Contact geometry and Legendrians

Contact geometry

- A contact manifold is a tuple (Y^{2n+1}, ξ) where ξ is a hyperplane field on Y, such that $\xi = \ker \alpha$ and $\alpha \wedge (d\alpha)^{\wedge n} \neq 0$.
- An n-dimensional submanifold $\Lambda \subset Y$ is *Legendrian* if and only if $T_x\Lambda \subset \xi_x$ for every x.





Figures from [Massot, Patrick. "Topological methods in 3-dimensional contact geometry." Contact and symplectic topology 26 (2014): 27–83.]

Contact geometry

Example

The standard contact structure on $\mathbb{R}^{2n+1} \ni (x_1, y_1, \ldots, x_n, y_n, z)$ is given by $\xi_{\text{std}} := \ker \alpha_{\text{std}}$ where

$$\alpha_{\text{std}} := dz - \sum_{i=1}^{n} y_i dx_i.$$

□

The *Reeb vector field* is the unique vector field R_α satisfying

1. $d\alpha(R_\alpha, -) = 0$
2. $\alpha(R_\alpha) = 1$

Example

The Reeb vector field associated to $(\mathbb{R}^{2n+1}, \alpha_{\text{std}})$ is given by $R_{\alpha_{\text{std}}} = \partial_z$.

Legendrian knots

Study Legendrian embeddings $S^1 \hookrightarrow \mathbb{R}^3$ via two projections

$$\pi_{\text{front}}(x, y, z) := (x, z), \qquad \pi_{\text{Lagrangian}}(x, y, z) := (x, y).$$

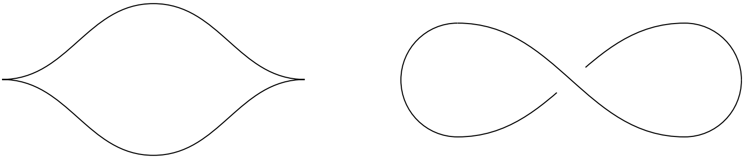


Figure: An unknot

- Front: No vertical tangencies! Legendrian if and only if $y(t) = \dot{z}(t)/\dot{x}(t)$.
- Legendrian condition ensures unique lifts (up to a shift in the Lagrangian case).

The unknot is not unique(!)

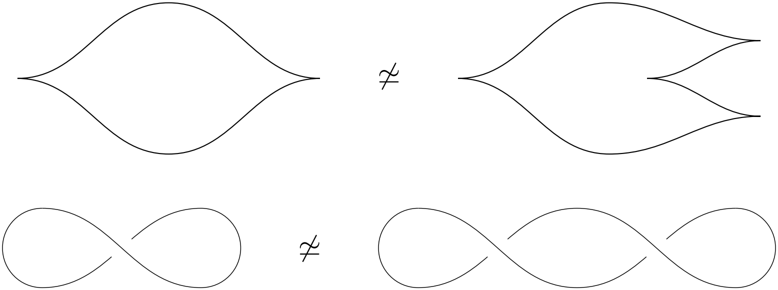
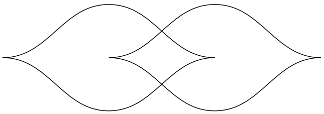
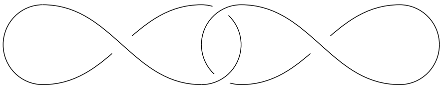


Figure: Two non-isotopic Legendrian representatives of the smooth unknot

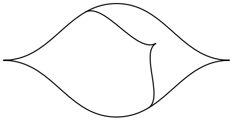
A Legendrian Hopf link

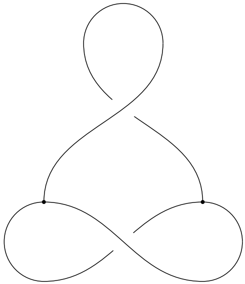




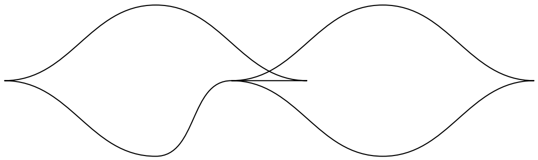
In the front $\times$ always means $\times$ due to the Legendrian condition $y = dz/dx$.

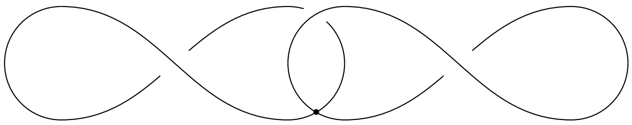
A Legendrian θ-graph





A singular Legendrian Hopf link





The Chekanov–Eliashberg algebra

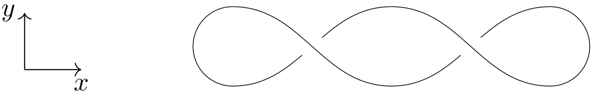
Setup

- (Y^{2n-1}, α) contact manifold that is the boundary of a *Weinstein domain* (a kind of symplectic manifold with boundary).
- $\Lambda \subset Y$ singular or smooth Legendrian.

A *Reeb chord* of Λ is a trajectory of the Reeb vector field that starts and ends at Λ.

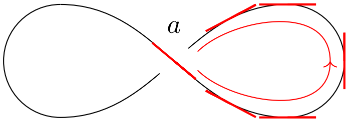
The algebra

$CE^*(\Lambda; Y)$ is the free associative non-commutative algebra generated by all Reeb chords of Λ.



Reeb chords in the Lagrangian projection correspond to double points!

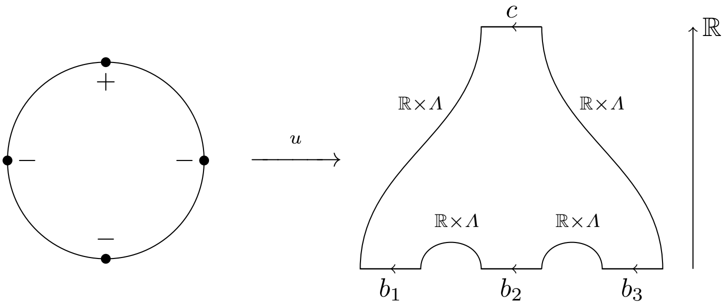
Grading



$$|a| = \frac{1}{2} - 2\mathsf{FracRot}(a) = \frac{1}{2} - 2 \cdot \frac{3}{4} = -1.$$

Differential

Defined by "counting" holomorphic maps from a boundary punctured disk to the almost complex manifold $\mathbb{R} \times Y$

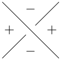


$\partial c = b_1 b_2 b_3 + \cdots$

$$\dim \overline{\mathcal{M}}(c; b_1 b_2 b_3) = -|c| + |b_1| + |b_2| + |b_3| - 1,$$

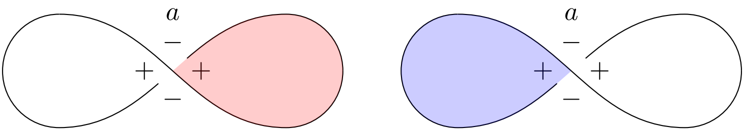
if dimension is $0 \rightsquigarrow$ count elements

Combinatorially

Decorate all crossings with signs 

and count "polygons" in the plane.

Example (Unknot)



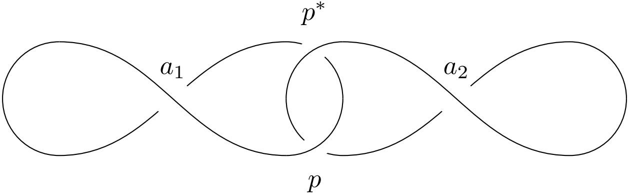
$$CE^*(\text{unknot}; \mathbb{R}^3) = \mathbb{Z}_2 \langle a \rangle , \quad |a| = -1, \quad da = \color{red}{1} + \color{blue}{1} = 0.$$

Note: This is equal to $\mathcal{G}_2(\bullet)$.

Hopf link

$$CE^*(\mathsf{Hopf};\mathbb{R}^3) = \mathbb{Z}_2\langle a_1, a_2, p, p^*\rangle$$

$$|a_1| = |a_2| = -1,\ |p| = |p^*| = 0$$



Hopf link

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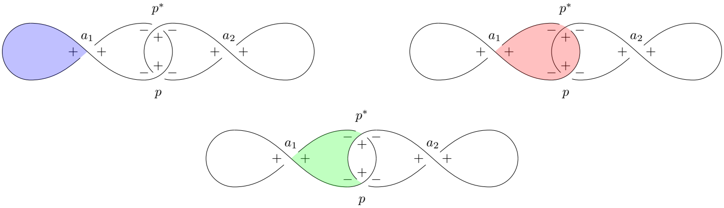


$$da_1 = 1 + 1 + pp^* = pp^*$$
$$da_2 = p^*p$$
$$dp = dp^* = 0.$$

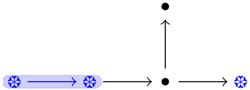
Note: This is equal to $\mathcal{G}_2(\overset{1}{\bullet} \xrightarrow{p} \overset{2}{\bullet})$.

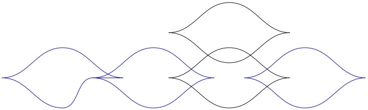
Construction and computations

Idea of construction (acyclic case)

Assuming Q is acyclic we take $Y(Q) = \mathbb{R}^{2n-1}$.
The Legendrian $\Lambda(Q, F)$ is as follows:

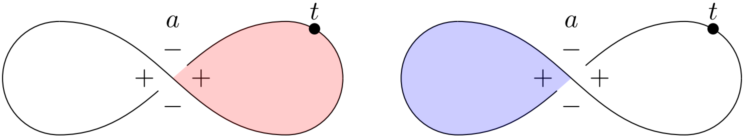
- Each non-frozen vertex $\leftrightarrow$ unknot
- Each frozen vertex $\leftrightarrow$ unknot "with basepoint"
- Each non-frozen arrow $\leftrightarrow$ two adjacent unknots link
- Each frozen arrow $\leftrightarrow$ two adjacent unknots intersect





Frozen vertex

$$CE^*(\text{based unknot}; \mathbb{R}^3) = \mathbb{Z}_2\langle a, t^{\pm 1}\rangle, \quad |a| = -1,\ |t^{\pm 1}| = 0$$



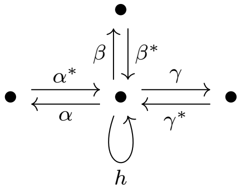
$$da = t + 1,\ dt^{\pm 1} = 0$$

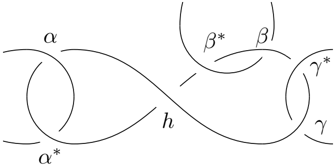
We then have a quasi-isomorphism

$$CE^*(\text{based unknot}; \mathbb{R}^3) \cong \mathbb{Z}_2 = \mathcal{G}_2(\bullet, \bullet).$$

Note: $\mathcal{B} = \mathbb{Z}_2\langle t^{\pm 1}\rangle \cong C_{-*}(\Omega S^1; \mathbb{Z}_2)$ and $\mathbb{Z}_2\langle t^{\pm 1}\rangle \subset CE^*(\text{based unknot}; \mathbb{R}^3)$ subalgebra is equal to $\mathcal{G}_1(\bullet)$.

Low dimension is tricky



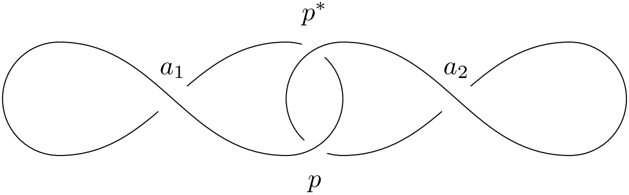


$$dh = \alpha\alpha^* + \beta\beta^* + \gamma\gamma^* \qquad dh = \alpha\alpha^* + \beta\beta^* + \gamma\gamma^* + \gamma\gamma^*\beta\beta^*$$

Good news

Extra term only exists when $n = 2$. This is the case of Legendrian knots in $\mathbb{R}^3$.

Hopf link again (in high dim)

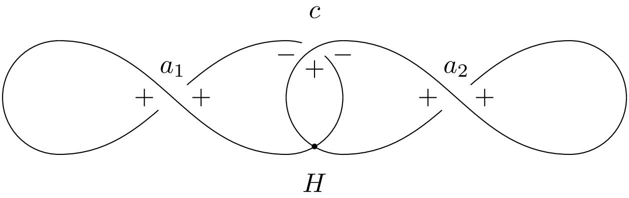


$$CE^*(\mathsf{Hopf}; \mathbb{R}^{2n-1}) = \mathbb{Z}_2 \langle a_1, a_2, p, p^* \rangle$$

$$|a_1| = |a_2| = 1 - n, \quad |p| = 0, \quad |p^*| = 2 - n$$

$$da_1 = pp^*, \quad da_2 = p^*p.$$

Singular Hopf link (in high dim)

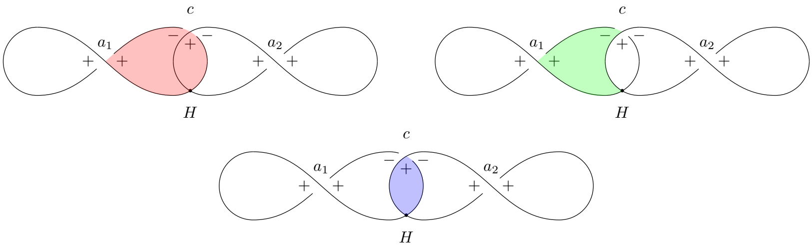


Hidden in the "point" H is $CE^*(\mathsf{Hopf}; \mathbb{R}^{2n-3})$. Denote those generators by b_1, b_2, q, q^*. Then

$$CE^*(\mathsf{SingHopf}; \mathbb{R}^{2n-1}) = \mathbb{Z}_2 \langle a_1, a_2, c, b_1, b_2, q, q^* \rangle$$

$$|a_1| = |a_2| = 1-n, \quad |b_1| = |b_2| = |c| = 2-n, \quad |q^*| = 3-n, \quad |q| = 0.$$

Singular Hopf link (in high dim)



$$da_1 = b_1 + qc, \quad db_1 = qq^*$$
$$da_2 = b_2 + cq, \quad db_2 = q^*q$$
$$dc = q^*, \quad dq = dq^* = 0$$

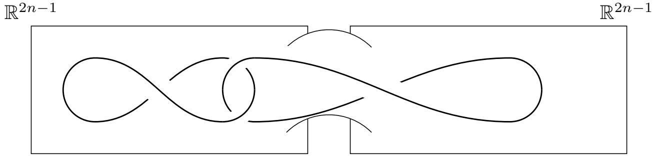
Quasi-isomorphic to $\mathcal{G}_n(\bullet \to \bullet, \bullet \to \bullet)$. Dg-subalgebra at H is quasi-isomorphic to $\mathcal{G}_{n-1}(\bullet \to \bullet)$.

Idea of construction and proof (general case)

1. Take one copy of $\mathbb{R}^{2n-1}$ for each vertex of Q, and for each arrow in Q, we take the "connected sum" between the various copies of $\mathbb{R}^{2n-1}$.
2. The Legendrian $\Lambda(Q, F)$ is constructed as described earlier, but stretches globally over the various copies of $\mathbb{R}^{2n-1}$.

Example

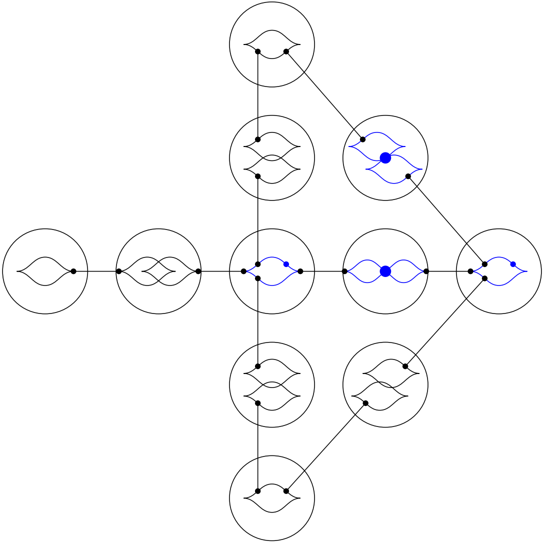
$$\bullet \longrightarrow \bullet$$



Idea of construction and proof (general case)

3. Main technical tool: Gluing formula for Chekanov–Eliashberg algebras to compute via local pieces that are similar to the examples we have seen.
4. Write down explicit chain homotopy equivalence

□



Thank you!