Mock Theta Functions and Ramanujan's Last Letter

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Abstract

In this seminar we have explored Fermat's famous claim that $x^n + y^n = z^n$ has no positive integer solutions for $n > 2$, a statement for which he provided no proof and which posthumously stimulated intense study by other mathematicians in attempts to determine its veracity. In fact, there is another strikingly similar story in the history of number theory, and one that concerns a topic we have already studied extensively: modular forms. In the months before his premature death at the age of 32, Indian mathematician Ramanujan penned a letter to his longtime mentor G.H. Hardy, in which he described what he called **mock theta functions**. He provided some examples, a few relations between them, and made some claims about their properties. Like Fermat, Ramanujan did not provide proof for his claims in the letter. In the decades after his death, mathematicians made various efforts to substantiate Ramanujan's statements, but the questions introduced in his letter were not wrapped up until the 21st century. In a 1935 address to the London Mathematical Society, British mathematician G.N. Watson called the letter Ramanujan's "Final Problem." What remarkable symmetry to Fermat! And how appropriate that symmetry would appear in the history of the study of modular forms.

This paper will provide a *very* brief overview of Ramanujan's initial claims about mock theta functions as well as subsequent work by Zwegers which constituted a breakthrough in their study and provided an alternative definition as holomorphic parts of harmonic Maass forms.

1 Ramanujan

> *"I am extremely sorry for not writing you...I discovered very interesting functions recently which I call "Mock" theta functions....they enter into mathematics as beautifully as the ordinary theta functions. I am sending you with this letter some examples."*

Ramanujan originally provided 17 examples of his mock theta functions in the form of q-hypergeometric series, or sums of the form $\sum_{n=0}^{\infty} A_n(q)$, where each $A_n(q) \in \mathbb{Q}$ and $(A_{n+1}(q)/A_n = R(q, q^n))\ \ \forall n \geq 1$ and for some fixed rational function $R(q, r) \in \mathbb{Q}(q, r)$. The Eisenstein series we have studied is an instance of such a sum, but there is no guarantee that a given q-hypergeometric series is modular or vice versa. We present three of Ramanujan's examples here.

$$f(q) = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(1+q)^2 \cdots (1+q^n)^2} \tag{1.1}$$

$$\phi(q) = \sum_{n=0}^{\infty} q^{n^2}(1+q)(1+q^3)\cdots(1+q^{2n-1}) \tag{1.2}$$

$$\psi(q) = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(1-q^{n+1})(1-q^{n+2})\cdots(1-q^{2n})} \tag{1.3}$$

He classified each function as order either 3, 5, or 7, and the above are functions of each order, respectively. He did not specify in his letter what the "order" of a mock theta function meant (in fact it is related to the level of the corresponding mock modular form), but did provide relations between the functions of each order, which were later proved by Watson and can be found in [10].

Ramanujan asserted vague principles about the behavior of these mock theta functions, refined in the literature into the following definition.

Definition 1.1 - Ramanujan's Mock Theta Functions

A mock theta function m is a function defined on $\mathbb{H}$ satisfying:

1. There are infinitely many roots of unity ζ for which $m(\tau)$ grows exponentially as $q = e^{2\pi i\tau}$ approaches ζ radially from inside the unit disk.
2. For every ζ, there exists a weakly holomorphic modular form B_ζ, and a rational α_ζ such that
$$m(\tau) - q^{a_\zeta} B_\zeta(\tau)$$
is bounded as $q \to \zeta$ from within the unit disk.
3. There does not exist a single weakly holomorphic modular form B which satisfies condition 2.

Griffin, Ono, and Rolen proved that Ramanujan's mock theta functions satisfied this definition only in 2013. [7]

2 Maass Forms and Mock Modular Forms

It is now necessary to define a type of modular form known as a **Maass form**, a real-analytic function that transforms like ordinary modular forms on a suitable subgroup of $\mathrm{SL}_2(\mathbb{Z})$.

Definition 2.1 - Maass forms

A complex-valued function $f(\tau)$ on the upper half-plane $\mathbb{H}$, with $\tau = x + iy$ is called a **Maass Form** if it satisfies:

1. For every element $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ in a subgroup $\Gamma \subset \mathrm{SL}_2(\mathbb{Z})$, f satisfies
$$f\left(\frac{a\tau + b}{c\tau + d}\right) = f(\tau)$$
2. f is an eigenfunction of the hyperbolic Laplacian Δ, where
$$\Delta = -y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)$$
3. There is some $N > 0$ such that $f(\tau) = O(y^n)$ as $y \to +\infty$

We generalize this notion of a Maass form by defining a **harmonic weak Maass form** of weight $k \in \frac{1}{2}\mathbb{Z}$. In reality a pure generalization would be to define simply weak Maass forms, which relax the growth conditions for Maass forms, but we will only work with *harmonic* weak Maass forms in this paper. To do this, we first define the hyperbolic Laplacian of weight k, or Δ_k by
$$\Delta_k = -y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) + iky\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)$$
and for odd integers d, define ϵ_d by
$$\epsilon_d = \begin{cases} 1 & \text{if } d \equiv 1 \pmod 4 \\ i & \text{if } d \equiv 3 \pmod 4 \end{cases}$$

We can now define harmonic weak Maass forms, also called simply harmonic Maass forms.

Definition 2.2 - Harmonic Maass forms

If $N \in \mathbb{N}$, a **weight k harmonic Maass form** on $\Gamma \in \{\Gamma_1(N), \Gamma_0(N)\}$ is a smooth function $M : \mathbb{H} \to \mathbb{C}$ satisfying:

1. For all $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$ and all $\tau \in \mathbb{H}$, we have

$$M\left(\frac{a\tau+b}{c\tau+d}\right) = \begin{cases} (c\tau+d)^k M(\tau) & \text{if } k \in \mathbb{Z} \\ \left(\frac{c}{d}\right)^{2k} \epsilon_d{-2k}(c\tau+d)^k M(\tau) & \text{if } k \in \frac{1}{2}\mathbb{Z} \setminus \mathbb{Z} \end{cases}$$

   where $\left(\frac{c}{d}\right)$ denotes the extended Legendre symbol.

2. $\Delta_k(M) = 0$

3. There exists a polynomial $P_M(\tau) \in \mathbb{C}[q^{-1}]$ such that

$$M(\tau) - P_M(\tau) = O(e^{-\epsilon v})$$

   as $v \to \infty$ and for some $\epsilon > 0$. We also require analogous conditions at the cusps.

If we investigate the Fourier expansions of these harmonic Maass forms, we find a very natural way decompose M into two parts.

Lemma 2.1

If M is a harmonic Maass form, with $k \in \frac{1}{2}\mathbb{Z}$, $k \neq 1$, and Γ as above, then M has Fourier expansion

$$M(\tau) = \sum_{n=r_M}^{\infty} c_M^+(n)q^n + \sum_{n=-\infty}^{-1} c_M^-(n)\Gamma(1-k, 4\pi|n|v)q^n$$

for some $r_M \in \mathbb{Z}$.

We now see that we can split a harmonic Maass form into two parts, M^+ equal to the left addend, which we call the **holomorphic part** of M, and M^- equal to the right, which we call the **non-holomorphic part**. M^+ is also called a **mock modular form**.

3 Zwegers

We return now to Ramanujan's mock theta functions and discover their connection to harmonic Maass forms through work by Zwegers in the early 2000s. Zwegers, like Watson and others before him, examined the relations between Ramanujan's theta functions of distinct order. For example, with f defined in Section 1, Zwegers proved that

$$F(\tau) = q^{-\frac{1}{24}} f(q) + g(\tau) \tag{3.1}$$

is a harmonic Maass form of weight 1/2 for some g which satisfies

$$\xi_{\frac{1}{2}} g(\tau) = \frac{\sqrt{6}}{3} \sum_{n \in 1+6\mathbb{Z}} n q^{\frac{n^2}{24}} \tag{3.2}$$

Where $\xi_k g(\tau) = 2iy^k \overline{\partial_{\bar{\tau}} g(\tau)}$. Zwegers's full proof for each mock theta function is outside the scope of this paper, but more generally, Zwegers's work implies the following pivotal theorem.

Theorem 3.1. *If m is a Ramanujan mock theta function, then*

$$m(\tau) = q^{\alpha} M^+(\tau)$$

for some $\alpha \in \mathbb{Q}$, where M^+ is the holomorphic part of a harmonic Maass form of weight 1/2.

Thus Ramanujan's ancient mock theta functions are shown to be mock modular forms! As stated previously, it was not yet proven that Ramanujan's functions fit the definition we gave in Section 1, which was only shown in 2013. [7]

So, almost a century after the fact, Ramanujan's vague claims about his mock theta functions were proven true, and indeed those 17 functions were a subset of the broader category of mock modular forms.

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