# A Short Introduction to the Langlands Correspondence

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Much of this is taken from Edward Frenkel's "Lecture on the Langlands Program and Conformal Field Theory." The goal of this exposition will be to understand the motivation behind the Langlands program and how the Geometric Langlands Conjecture emerges from the corresponding problem in number theory. For the sake of intuition and (relative) brevity, I focus on $\mathbb{Q}$ as our number field of concern. Especially to understand the geometric part, it is very helpful to have some knowledge of topology and manifolds–especially Riemann surfaces–as well as basic Riemannian geometry (up to connections). I've tried to include the details of as much of the relevant algebraic geometry as I can in a succinct way.

## 1 The Meaning of Langlands Correspondence and its Motivations

The Langlands Correspondence is a statement that declares a relationship between $n$-dimensional representations of $\mathrm{Gal}(\bar{F}/F)$ and representations of $GL(n, \mathbb{A}_F)$ in functions of $GL(n, \mathbb{A}_F)/GL(n, F)$. This first section will explain in detail the case $F = \mathbb{Q}$ and how we arrive at this statement.

### 1.1 Galois Field Extensions

We recall from abstract algebra the first definitions from field extension theory:

We consider *algebraic* extensions over a field $F$, which are fields $E \supset F$ such that for all $x \in E$, there exists a polynomial $f$ with coefficients in $F$ such that $x$ is a zero of $f$. Such an $E$ is also an $F$-vector space; we define the *degree* of the extension, denoted $[E : F]$, to be the dimension of this vector space; if this number is finite, then we say the extension is *finite*. We say that a field extension is *Galois* if it is:

- *Normal*: If $E$ is a splitting field for $f \in F[x]$ over $F$, meaning that $f$ splits completely into linear factors in $E[x]$, and all roots of $f$ generate the extension $E$ as an $F$-vector space.
- *Separable*: If for every extension $K/F$, there exists an extension $L/K$ so that there are exactly $[E : F]$ homomorphisms $E \to L$ with $\varphi(x) = x$ for all $x \in F$.

The Galois group $\mathrm{Gal}(E/F)$ is the set of field automorphisms $\sigma : E \to E$ such that $\sigma(x) = x$ for all $x \in F$. It turns out that we may equivalently say that $E$ is Galois over $F$ if $\#\mathrm{Gal}(E, F) = [E : F]$. The group operation here is composition; the identity element is the identity map $\mathrm{Id} : E \to E$, and all automorphisms have unique inverses (and these will preserve the property of fixing elements of $F$).

An important example of Galois field extension is the cyclotomic extension of degree $n$ over the rational numbers $\mathbb{Q}(\zeta_n)$, formed by adjoining $\zeta_n = e^{2\pi/n}$. $\zeta_n$ is a zero of the polynomial $x^n - 1$; all of the zeros are $\zeta_n^k$ for $k \leq n-1 :$ $\gcd(k, n) = 1$, which generate $\mathbb{Q}(\zeta_n)$ as a $\mathbb{Q}$-vector space. It can that $\mathbb{Q}(\zeta_n)$ is a

$$\Phi_n(x) = \prod_{k \leq n-1,\ \gcd(k, n-1)=1} (x - \zeta_n^k)$$

is a polynomial with integer coefficeints, which means that $\mathbb{Q}(\zeta_n)$ is a splitting field for $\Phi_n$ over $\mathbb{Q}$. Furthermore, $\mathbb{Q}(\zeta_n)$ is an algebraic extension over $\mathbb{Q}$, a field of characteristic 0, $\mathbb{Q}(\zeta_n)$ is automatically separable.

One of the major properties of Galois extensions is the correspondence between subgroups of the Galois group and intermediate field extensions–and additionally, normal subgroups of the Galois group correspond to normal intermediate extensions.

Galois theory has provided many valuable insights. It provides a method of proving the fundamental theorem of algebra, it tells us about when polynomial equations are solvable by radicals, and there are valuable analogies between Galois extensions in field theory and covering spaces and deck transformations on Riemann surfaces (referred to as *Galois coverings*) which will become relevant later when we discuss geometric Langlands.

## 1.2 Abelian Class Field Theory

One major concern in number theory is, given a field $F$, is to understand its *algebraic closure* $\bar{F}$, an extension over $F$ formed by adjoining all roots of every polynomial with coefficients in $F$. To find detailed information about Galois groups of algebraic closures is in general very difficult. For example, think of the fundamental theorem of algebra, which states that $\bar{\mathbb{Q}} \subset \mathbb{C}$. This is statement is quite nontrivial. $\mathrm{Gal}(\mathbb{C}/\mathbb{Q})$ is a massive group, and no known proof of the fundamental theorem of algebra involves purely algebraic methods.

But we can say something about the abelianization[1] of $\mathrm{Gal}(\bar{F}/F)$, which is identifiable with the maximal field $F^{\mathrm{ab}} \subset \bar{F}$ where $\mathrm{Gal}(F^{\mathrm{ab}}/F)$ is abelian–this is called the maximal abelian extension. For example, recall the cyclotomic extension $\mathbb{Q}(\zeta_n)/\mathbb{Q}$ from before. One can check that $\mathrm{Gal}(\mathbb{Q}(\zeta_n)/\mathbb{Q}) \cong (\mathbb{Z}/n\mathbb{Z})^*$, which means that $\mathrm{Gal}(\mathbb{Q}(\zeta_n)/\mathbb{Q})$. If we adjoin multiple roots of unity[2] to $\mathbb{Q}$, the Galois group remains abelian; in fact, if we adjoin all roots of unity to $\mathbb{Q}$, we get the maximal abelian extension $\mathbb{Q}^{\mathrm{ab}}$. This is the result of the Kronecker-Weber Theorem.

The abelian class field theory describes the Galois group $\mathrm{Gal}(F^{\mathrm{ab}}/F)$, or equivalently the ablelianization of $\mathrm{Gal}(\bar{F}/F)$. Returning to the example of $\mathbb{Q}^{\mathrm{ab}}/\mathbb{Q}$, we see that

$$\mathrm{Gal}(\mathbb{Q}^{\mathrm{ab}}/\mathbb{Q}) \cong \varprojlim (\mathbb{Z}/n\mathbb{Z})^* = \prod_{n\in\mathbb{N}} (\mathbb{Z}/n\mathbb{Z})^* \Big/ \{x \sim y \iff \exists p_{mn} : p_{mn}(x) = y\}$$

where $p_{nm} : (\mathbb{Z}/n\mathbb{Z})^* \to (\mathbb{Z}/m\mathbb{Z})^*$ with $m|n$ is defined by taking choosing a representative $x$ of $[x]_n$ and then finding $[x]_m$.[3]. This group can be nicely descibed in terms of the $p$-adic numbers as well. Recall that a $p$-adic number for $p$ prime is an infinite series $\sum_{n=k}^{\infty} a_n p^n$ where $a_k \in [0, p-1] \cap \mathbb{Z}$ and $k \in \mathbb{Z}$ is chosen so that $a_k \neq 0$; the set $\mathbb{Q}_p$ of $p$-adic numbers turns out to be an analytic completion of $\mathbb{Q}$. It turns out that

$$\mathrm{Gal}(\mathbb{Q}^{\mathrm{ab}}/\mathbb{Q}) \cong \prod_p \mathbb{Z}_p^*$$

where $\mathbb{Z}_p$, the collection of $p$-adic integers, is the set of elements of $\mathbb{Q}_p$ with $k \in \mathbb{Z}_{\geq 0}$; one finds this by considering prime factorizations.

Note that all analytic completions of $\mathbb{Q}$ are either $\mathbb{R}$ or $\mathbb{Q}^p$. If $P$ is the set of primes, the *adèle* of $\mathbb{Q}$, denoted $\mathbb{A}_{\mathbb{Q}}$, consists of tuples $((x_p)_{p\in P}, x)$ with $x_p \in \mathbb{Q}_p$, $x \in \mathbb{R}$, and $x_p \in \mathbb{Z}_p$ for all but finitely many $p$; this gives us

$$\mathbb{A}_Q \cong (\hat{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{Q}) \times \mathbb{R}$$

where $\hat{\mathbb{Z}} = \prod_p \mathbb{Z}_p$. The $p$-adic norm on $\mathbb{Q}$ is defined by $\left|\frac{p^k a}{b}\right|_p = p^{-k}$, where $a, b \in \mathbb{Z}$ are coprime to $p$[4]; observe that

[1] In general, while some aspects of the structure of the original group may be lost through abelianization, the result still can possess very useful properties, as demonstrated in other areas of mathematics:

- In algebraic topology, the first homology class happens to be the abelianization of the fundamental group and is still a homotopy-invariant object.
- The classification of compact Lie groups involves understanding how the Lie algebra is decomposed into root spaces and the Lie algebra of a maximal torus–that is, a maximal abelian sub Lie group.
- Faltings's theorem is a major result in algebraic geometry draws a connection between the finiteness of rational points in an abelian variety and the absence of subabelian varieties; Olivier Debarre and Matthew Klassen used this in the 1990s to prove powerful results about smooth plane curves.

We will see shortly that maximal abelian extensions can be used to describe representations of $\mathrm{Gal}(\bar{F}/F)$. (A representation is a homomorphism $\rho : \mathrm{Gal}(\bar{F}/F) \to GL(V)$, or equivalently a linear action $\mathrm{Gal}(\bar{F}/F) \times V \to V$, on some vector space $V$.).

[2] A $n$th root of 1 for any integer $n$ is referred to as a *root of unity*.

[3] In such a system, we have $p_{nn} = \mathrm{Id} : (\mathbb{Z}/n\mathbb{Z})^* \to (\mathbb{Z}/n\mathbb{Z})^*$ and $p_{nm} \circ p_{mr} = p_{nr}$.

[4] One can express any rational number as $\frac{p^k a}{b}$ for some $a, b \in \mathbb{Z}$ relatively prime to $p$.

$\mathbb{Q}_p$ is the analytic completion of $\mathbb{Q}$ under this norm. With this norm, we can define a metric space topology on $\mathbb{Q}_p$; we can then say $\mathbb{Z}_p = \{x \in \mathbb{Q}_p : ||x||_p \leq 1\}$. It can be shown that as a metric space, $\mathbb{Q}_p$ has the Heine-Borel property and that $\mathbb{Z}_p$ is a compact set.

Then, we assign to $\hat{\mathbb{Z}}$ the product topology, $\mathbb{Q}$ the discrete topology, and $\mathbb{R}$ the standard Euclidean topology. We identity $\mathbb{Q}$ as a subset of $\mathbb{A}_{\mathbb{Q}}$ through a diagonal embedding and find that

$$\mathbb{A}_{\mathbb{Q}}/\mathbb{Q} \cong \hat{\mathbb{Z}} \times (\mathbb{R}/\mathbb{Z})$$

$\mathbb{R}/\mathbb{Z}$ is homeomorphic to a circle, and by Tychonoff's theorem, $\hat{\mathbb{Z}}$ is compact, so the quotient above is compact. Now if we take the multiplicative groups, we get that similarly

$$\mathbb{A}_{\mathbb{Q}}^*/\mathbb{Q}^* \cong \prod_p \mathbb{Z}_p^* \times \mathbb{R}^+$$

$\prod_p \mathbb{Z}_p^*$ is totally disconnected and $\mathbb{R}$ is connected, so the group of connected components is isomorphic to $\prod_p \mathbb{Z}_p^*$. We saw previously that this is isomorphic to $\mathrm{Gal}(\mathbb{Q}^{\mathrm{ab}}/\mathbb{Q})$.

### 1.3 Restrictions on this Isomorphism from the Abelian Class Field Theory

For an arbitrary number field $F$, the set of its adèles $\mathbb{A}_F$ is defined quite analogously; the norms are defined through prime ideals of the ring of integers of $F$–roots of monic polynomials with coefficients in $F$–and the completions are all isomorphic to a finite extension of some $\mathbb{Q}_p$. In general the abelian class field theory says that for an arbitrary field $F$, $\mathrm{Gal}(F^{\mathrm{ab}}/F)$ is isomorphic to the group of connected components of $\mathbb{A}_F^*/F^*$. This is really useful, especially since we do not have an analogue of Kronecker-Weber for arbitrary number fields.

Furthermore, this isomorphism satisfies some restrictions. Recall from abstract algebra that we can classify all finite extensions of finite fields $\mathbb{F}_p$ up to isomorphism: An extension of degree $m$ must be isomorphic to $\mathbb{F}_q$ with $q = p^m$, and $\mathrm{Gal}(\mathbb{F}_q/\mathbb{F}_p) \cong \mathbb{Z}/m\mathbb{Z}$ is generated by the Frobenius automorphism $\mathbb{F}_q \to \mathbb{F}_q$, defined by $x \mapsto x^p$, which gets identified with $[1]_m$ in $\mathbb{Z}/m\mathbb{Z}$. If we take the union of all such $\mathbb{F}_q = \mathbb{F}_{p^m}$, we get the algebraic closure $\bar{\mathbb{F}}_p$; Notice that $\mathrm{Gal}(\bar{\mathbb{F}}_p/\mathbb{F}_p) \cong \lim_{\leftarrow} \mathbb{Z}/m\mathbb{Z} \cong \hat{\mathbb{Z}}$. This holds true for any $p$ that is a power of a prime. Now, recalling the definition of the inverse limit from before, there is an element of $\hat{\mathbb{Z}}$ that identifies with $[1]_m$ under the projection $\hat{\mathbb{Z}} \to \mathbb{Z}/m\mathbb{Z}$, which then gets identified with the Frobenius automorphism of $\mathrm{Gal}(\mathbb{F}_{p^m}/\mathbb{F})$. We say that this element of $\hat{\mathbb{Z}}$ corresponds to the Frobenius automorphism of $\bar{\mathbb{F}}_p$.

There is a relation between $\mathrm{Gal}(\bar{F}/F)$ and $\mathrm{Gal}(\bar{\mathbb{F}}_p/\mathbb{F}_p)$. We again look at the case $F = \mathbb{Q}$. We have seen that $\mathbb{Q}(\zeta_n)$ is an abelian extension of degree $\varphi(n)$, where $\varphi$ is the Euler-$\varphi$ function. Let $p \in \mathbb{Z}$ be a prime number. We consider the ideal $(p)$ in $\mathbb{Z}[\zeta_n]$. By Dedekind's theorem, $(p)$ is a product of prime ideals of $\mathbb{Z}[\zeta_n]$. Let $\mathfrak{q}$ be one of these ideals.

We call $\mathbb{Z}/(p) \cong \mathbb{F}_p$ the *residue field*. One can also verify that $\mathbb{Z}[\zeta_n]/\mathfrak{q} \cong \mathbb{F}_{p^m}$ for some $m$. We have from before that $\mathrm{Gal}(\mathbb{F}_{p^m}/\mathbb{F}_p) = \mathbb{Z}/m\mathbb{Z}$. Now, we call $D_\mathfrak{q}$, the subgroup of $\sigma \in \mathrm{Gal}(\mathbb{Q}(\zeta_n)/\mathbb{Q})$ such that $\sigma(x) \in \mathfrak{q}$ for all $x \in \mathfrak{q}$, the *decomposition group of* $\mathfrak{q}$. Any element of $D_\mathfrak{q}$ will preserve $(p)$, as an ideal of $\mathbb{Z} \subset \mathbb{Q}$, because it already preserves $\mathbb{Q}$; this observation leads to a natural surjective homomorphism $D_\mathfrak{q} \to \mathrm{Gal}(\mathbb{F}_{p^m}/\mathbb{F}_p)$.

Note that if $p$ does not divide $n$, the kernel of the above homomorphism, denoted $I_\mathfrak{q}$, otherwise known as the *inertia group of* $\mathfrak{q}$, must be trivial, since $\mathrm{Gal}(\mathbb{F}_{p^m}/\mathbb{F}_p)$ is generated by the Frobenius automorphism $x \mapsto x^p$. We say in this case that $\mathbb{Q}(\zeta_n)/\mathbb{Q}$ is *unramified at p*. So then we have isomorphisms

$$D_\mathfrak{q} \cong \mathrm{Gal}(\mathbb{F}_{p^m}/\mathbb{F}_p) \cong \mathbb{Z}/m\mathbb{Z}$$

and we can identify the Frobenius automorphism of $\mathrm{Gal}(\mathbb{F}_{p^m}/\mathbb{F}_p)$ uniquely with an element $\mathrm{Fr}[\mathfrak{q}]$ of $D_\mathfrak{q}$. For all choices of prime ideals $\mathfrak{q}$, the corresponding subgroups $D_\mathfrak{q}$ and elements $\mathrm{Fr}[\mathfrak{q}]$ are conjugate to each other, so in our case, the conjugacy class of $\mathrm{Fr}[\mathfrak{q}]$, denoted $\mathrm{Fr}(p)$, is well-defined in $\mathrm{Gal}(\mathbb{Q}(\zeta_n)/\mathbb{Q})$ and depends only on $p$. The action of $\mathrm{Fr}(p)$ on $\zeta_n$ sends $\zeta_n$ to $\zeta_n^p$. Recall that $\mathrm{Gal}(\mathbb{Q}(\zeta_n)/\mathbb{Q}) \cong (\mathbb{Z}/n\mathbb{Z})^*$; $\mathrm{Fr}(p)$ is identified with $[p]_n \in (\mathbb{Z}/n\mathbb{Z})^*$.

Now, what if we try to define the conjugacy class $\mathrm{Fr}(p)$ in $\mathrm{Gal}(\mathbb{Q}^{\mathrm{ab}}/\mathbb{Q})$? To avoid troubles, we reframe this question by instead considering $\mathrm{Gal}(\mathbb{Q}^{ab,p}/\mathbb{Q})$ where $\mathbb{Q}^{ab,p}$ is the maximal abelian extension that is unramified at $p$, obtained by adjoining all roots of unity such that $p$ does not divide the degree of the root. Since we are in effect "avoiding $p$," it turns out that

$$\mathrm{Gal}(\mathbb{Q}^{\mathrm{ab},p}/\mathbb{Q}) \cong \prod_{q \neq p} \mathbb{Z}_q^*$$

which is isomorphic to the group of connected components of $(\mathbb{A}_{\mathbb{Q}}^*/\mathbb{Q}^*)/\mathbb{Z}_p^*$ (we treat this as the set of double cosets, where the quotient by $\mathbb{Q}^*$ is a left quotient and the quotient by $\mathbb{Z}_p^*$ is a right quotient). Since $\mathrm{Gal}(\mathbb{Q}^{\mathrm{ab},p}/\mathbb{Q}) \cong \mathrm{Gal}(\mathbb{Q}^{\mathrm{ab}}, \mathbb{Q})/I_p$, we see that $I_p \cong \mathbb{Z}_p^*$. So then we can identify under this isomorphism the inverse of $\mathrm{Fr}(p)$, denoted $\mathrm{Fr}_p$, with the double coset of $(1, \ldots, 1, p, 1, \ldots) \in \mathbb{A}_{\mathbb{Q}}^*$ where $p$ occurs in the entry corresponding to $\mathbb{Q}_p^*$ in the group of conencted components of $(\mathbb{A}_{\mathbb{Q}}^*/\mathbb{Q}^*)/\mathbb{Z}_p^*$.

Observing the nature of the Frobenius conjugacy class as elements of these Galois groups can lead to interesting consequences–in particular, *reciprocity* laws, which are particular arithmetic properties about $p$ in relation to a given finite field extension. For example, in the case $n = 4$, one can use this to derive the theorem by Fermat which states that an odd prime $p$ as an element of the Gaussian integers $\mathbb{Z}[i]$ is representable as the sum of two integer squares if and only if $p \equiv 1 \mod 4$. We may go through a similar process for an arbitrary number field $F$ and obtain a similar result, which results in analogous reciprocity laws.

### 1.4 De-Abelianization and the Langlands Correspondence

Unlike what may be extrapolated from the title above, there is no real way to "de-abelianize" here, but the point is that we would like to be able to say something about $\mathrm{Gal}(\bar{F}/F)$. Fortunately, we can make the following observations: The 1-dimensional representations of $\mathrm{Gal}(\bar{F}/F)$ are the same as the 1-dimensional representations of $\mathrm{Gal}(F^{\mathrm{ab}}/F)$; the 1-dimensional representations is enough to completely retrieve $\mathrm{Gal}(F^{\mathrm{ab}}/F)$ itself. So we can rewrite the abelian class field theory as saying that there is a bijective correspondence between one-dimensional representations of $\mathrm{Gal}(\bar{F}/F)$ and one-dimensional representations of the group of connected components of $\mathbb{A}_F^*/F^*$. The group of connected components of $\mathbb{A}_F^*/F^*$ is closely related to $\mathbb{A}_F^*/F^*$ itself, and we can view one-dimensional representations of these as representations of $GL(1, \mathbb{A}_F)$ in functions on $GL(1, \mathbb{A}_F)/GL(1, F)$. So the abelian class field theory, which we have spent the last few sections on, can be reformulated as describing a relationship between one-dimensional representations of $\mathrm{Gal}(\bar{F}/F)$ and representations of $GL(1, \mathbb{A}_F)$ in functions on $GL(1, \mathbb{A}_F)/GL(1, F)$. Robert Langlands conjectured the Langlands correspondence, which declares a relationship between $n$-dimensional representations of $\mathrm{Gal}(\bar{F}/F)$ and irreducible representations of $GL(n, \mathbb{A}_F)$ in functions on $GL(n, \mathbb{A}_F)/GL(n, F)$– called *automorphic* representations.

### 1.5 Hecke Eigenvalues

Before, we saw that the isomorphism from the abelian class field theory had an extra condition involving the Frobenius conjucacy classes. Indeed, the Langlands correspondence demands a correspondence between Frobenius conjucacy classes (in the context of Galois groups–found by going through the same process as before but this time applied to the infinite extension $\bar{F}$) and *Hecke eigenvalues* (in the context of automorphic representations). In this section, we will define what these are in the case of 2-dimensional representations of $GL(2, \mathbb{A}_{\mathbb{Q}})$; indeed, we return to the case $F = \mathbb{Q}$.

Let $K = \prod_p GL(2, \mathbb{Z}_p) \times O(2)$, a subgroup of $GL(2, \mathbb{A}_{\mathbb{Q}})$. The center $\mathfrak{z}$ of the universal enveloping algebra of the complexified Lie algebra $\mathfrak{gl}_2$ is generated as a polynomial algebra by the identity matrix $I$ and

$$C = \frac{1}{4}X_0^2 + \frac{1}{2}\left(X_+X_- + X_-X_+\right)$$

$$X_0 = \begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix} \quad X_\pm = \frac{1}{2}\begin{bmatrix} 1 & \mp i \\ \mp i & -1 \end{bmatrix}$$

We define a function on $GL(2, \mathbb{A}_{\mathbb{Q}})/GL(2, \mathbb{Q})$ to be *smooth* if it is locally constant as a function on $GL(2, \mathbb{A}')$, where $\mathbb{A}' = \prod_p' \mathbb{Q}_p$[5] and smooth as a function on $GL(2, \mathbb{R})$. We can define a group action of $GL(2, \mathbb{A}_{\mathbb{Q}})$ on the space

[5]The notation denotes a restricted product, where all but finitely many entries in the tuples are $p$-adic integers.

of smooth functions $f : GL(2,\mathbb{A}_{\mathbb{Q}})/GL(2,\mathbb{Q}) \to \mathbb{C}$ by

$$(g \cdot f)(h) = f(hg) \quad g \in GL(2,\mathbb{A}_{\mathbb{Q}})$$

One representation of the group $GL(2,\mathbb{A}') = \prod'_{p \text{ prime}} GL(2,\mathbb{Q}_p)$ and $\mathfrak{gl}_2$ (which corresponds to $GL(2,\mathbb{R})$) is given by the space $\mathfrak{C}_{\chi,\rho}(GL(2,\mathbb{A}_{\mathbb{Q}})/GL(2,\mathbb{Q}))$ of smooth functions satisfying

- The aforementioned left action by elements of $K$ span a finite-dimensional vector space.[6]
- For a character $\chi : Z(\mathbb{A}_{\mathbb{Q}}) \to \mathbb{C}^*$ and $\rho \in \mathbb{C}$, $f(gz) = \chi(z)f(g)$ for all $g \in GL(2,\mathbb{A}_{\mathbb{Q}})$, $z \in Z(\mathbb{A}_{\mathbb{Q}})$, and $C \cdot f = \rho f$.[7]
- $f$ is bounded on $GL(n,\mathbb{A}_{\mathbb{Q}})$.
- $\int_{\mathbb{A}_{\mathbb{Q}}/\mathbb{Q}} f\left(\begin{bmatrix}1 & u\\ 0 & 1\end{bmatrix} g\right) du = 0$

$\mathfrak{C}_{\chi,\rho}(GL(2,\mathbb{A}_{\mathbb{Q}})/GL(2,\mathbb{Q}))$ is completely reducible in terms of $GL(2,\mathbb{A}') \times \mathfrak{gl}_2$, with each irreducible representation occurring once in the direct sum; we may write these so-called *cuspidal automorphic representations of* $GL(2,\mathbb{A}_{\mathbb{Q}})$ as

$$\pi = \bigotimes_{p \text{ prime}}' \pi_p \otimes \pi_\infty$$

where $\pi_p$ is an irreducible representation of $GL(2,\mathbb{Q}_p)$, $\pi_\infty$ is one for $\mathfrak{gl}_2$, and for all but finitely many $p$, $\pi_p$ fixes a unique line under the action of $GL(2,\mathbb{Z}_p)$, in which case $\pi_p$ is called *unramified.*

Suppose $\pi_p$ is ramified at $p$, so that all lines change under action by $GL(2,\mathbb{Z}_p)$, and $\chi \equiv \mathrm{Id}$. Then there is a line whose image under $\pi_p$ is invariant under action of

$$K_p' = \{A \in GL(2,\mathbb{Z}_p) : A_{21} \equiv 0 \mod p^{n_p}\mathbb{Z}_p\}$$

for some $n_p \in \mathbb{Z}^+$. For all primes $p$ where $\pi$ is ramified, we choose this vector $\mathbf{v}_p$ and set $n_p = 0$ where $\pi_p$ is unramified, to obtain that the vector space whose image through $\pi$ is preserved under action of $K'$ is

$$\tilde{\pi}_\infty = \bigotimes_p \mathbf{v}_p \otimes \pi_\infty$$

Up to action by $GL(\mathbb{A}_{\mathbb{Q}})$, this subspace contains the information of the entire space.

If $n = \prod_p p^{n_p}$ and

$$\Gamma_0(N) = \{A \in SL(2,\mathbb{Z}) : A_{21} = 0 \mod n\mathbb{Z}\}$$

Then, by the strong approximation theorem,

$$(GL(2,\mathbb{A}_{\mathbb{Q}})/GL(2,\mathbb{Q}))/K' \cong GL^+(2,\mathbb{R})/\Gamma_0(n)$$

Note that $\Gamma_0(N)$ is one of the types of Hecke subgroups from earlier in the course.

If we consider the action of $GL(2,\mathbb{R})$ on $\mathfrak{C}(GL(2,\mathbb{A}_{\mathbb{Q}})/GL(2,\mathbb{Q}))$, we see that differentiating at the identity matrix gives us an action of $\mathfrak{gl}_2$ and that the subgroup $O(2)$ also acts. These actions are therefore compatible, and we therefore get a $(\mathfrak{gl}_2, O(2))$-module. This action carries over to the subspace of elements $\tilde{\pi}_\infty$. A result from representation theory is that irreducible representations of $(\mathfrak{gl}(2,\mathbb{C}), O(2))$ are all principle series, discrete series, limits of discrete series, or finite-dimensional representatoins.

If $\pi_\infty$ is a representation of the discrete series of $(\mathfrak{gl}_2(\mathbb{C}), O(2))$, we get that $\rho = k(k-2)/4$ for some $k > 1$, and it follows from the classification of representations of $\mathfrak{sl}_2$ that since $\mathfrak{sl}_2$ acts on $\mathfrak{C}(GL(2,\mathbb{A}_{\mathbb{Q}})/GL(2,\mathbb{Q}))$ through the representation $\pi_\infty$, $\pi_\infty$ can be expressed as a direct sum of (1) the irreducible module generated by the highest weight vector $\mathbf{v}_\infty$ such that $X_0 \cdot \mathbf{v}_\infty = -k\mathbf{v}_\infty$, $X_+ \cdot \mathbf{v}_\infty = 0$, and (2) the irreducible module generated by the lowest weight vector $\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix} \cdot \mathbf{v}_\infty$. These are otherwise known as the irreducible Verma module with highest weight $-k$

[6]Also note that $K$ is a compact subgroup.
[7]The center $Z(\mathbb{A}_{\mathbb{Q}})$ consists of diagonal matrices and is isomorphic to $\mathbb{A}_{\mathbb{Q}}^*$.

and the irreducible Verma module with lowest weight $k$, respectively.

The entire $\mathfrak{gl}_2(\mathbb{R})$-module $\pi_\infty$ is generated by $\mathbf{v}_\infty$; let $\varphi_\pi$ be the corresponding function on $SL_2(\mathbb{R})$ (modulo $\Gamma_0(n)$) corresponding to $\mathbf{v}_\infty$. We have

$$\varphi_\pi(\gamma g) = \varphi_\pi(g), \quad \gamma \in \Gamma_0(n)$$

$$\varphi_\pi\left(g\begin{bmatrix}\cos\theta & \sin\theta \\ -\sin\theta & \cos\theta\end{bmatrix}\right) = e^{ik\theta}\varphi_\pi(g) \quad 0 \le \theta \le 2\pi$$

Then, we assign $\varphi_\pi$ to $f_\pi : \mathbb{H} \to \mathbb{H}$ using the correspondence $\mathbb{H} \cong SL(2,\mathbb{R})/SO(2)$ given by

$$\begin{bmatrix}a & b \\ c & d\end{bmatrix} \longmapsto \frac{ai+b}{ci+d}$$

Define $f_\pi$ by

$$g \longmapsto \varphi_\pi(g)(ci+d)^k$$

This satisfies

$$f_\pi\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^k f_\pi(\tau) \quad \begin{bmatrix}a & b \\ c & d\end{bmatrix} \in \Gamma_0(n)$$

$X_+ \cdot \mathbf{v}_\infty = 0$ is equivalent to holomorphicity of $f_\pi$ (one can check that the condition $\frac{\partial f_\pi}{\partial \bar{\tau}} = 0$ holds). These conditions are exactly what it means for $f_\pi$ to be a modular form of weight $k$ and level $n$. One can also check that $f_\pi$ is also a cusp form.

The *spherical Hecke algebra* $\mathcal{H}_p$ is the algebra of compactly supported $GL(2,\mathbb{Z}_p)$ bi-invariant functions on $GL(2,\mathbb{Q}_p)$ with respect to the convolution product, given by

$$(\varphi * \psi)(g) = \int_{GL(2,\mathbb{Q}_p)} \varphi(gx^{-1})\psi(x)dx$$

Let $\rho_p : GL(2,\mathbb{Z}_p) \to \mathrm{End}\pi_p$ be the representation homomorphism,

$$M_2^1(\mathbb{Z}_p) = GL(2,\mathbb{Z}_p)\begin{bmatrix}p & 0 \\ 0 & 1\end{bmatrix}GL(2,\mathbb{Z}_p) \quad M_2^1(\mathbb{Z}_p) = GL(2,\mathbb{Z}_p)\begin{bmatrix}p & 0 \\ 0 & p\end{bmatrix}GL(2,\mathbb{Z}_p)$$

Consider the operators $H_{1,p}$ and $H_{2,p}$ whose action on $\mathbf{v}_p$ is given by

$$H_{1,p} \cdot \mathbf{v}_p = \int_{M_2^1(\mathbb{Z}_p)} \rho_p(g) \cdot \mathbf{v}_p dg$$

$$H_{2,p} \cdot \mathbf{v}_p = \int_{M_2^2(\mathbb{Z}_p)} \rho_p(g) \cdot \mathbf{v}_p dg$$

$H_{1,p} \cdot \mathbf{v}_p$ and $H_{2,p} \cdot \mathbf{v}_p$ are both scalar multiples of $\mathbf{v}_p$ because $\mathbf{v}_p$ is an eigenvector of the spherical Hecke algebra; those integrals are taken over $GL_2(\mathbb{Z}_p)$ cosets. Under the assumption that $\chi \equiv \mathrm{Id}$, , $H_{2,p} \cdot \mathbf{v}_p = \mathbf{v}_p$, but the eigenvalue $h_{1,p}$ of $H_{1,p}$ is in general nontrivial; such eigenvalues are defined to be the Hecke eigenvalues.

Also note that the cosets $M_2^i(\mathbb{Z}_p)$ generalize our double cosets of $(\mathbb{A}_\mathbb{Q}^*/\mathbb{Q}^*)/\mathbb{Z}_p^*$ from earlier, illustrating the correspondence between Hecke eigenvalues and Frobenius conjugacy classes.

We have seen how modular forms arise from the automorphic representation side of the correspondence. It turns out that elliptic curves emerge from the Galois representation; this is how Weil's proof of Fermat's theorem, which involves relations between modular forms and elliptic curves, has become considered one of the first major manifestations of the Langlands program.

# 2 The Geometric Langlands Correspondence

While we have made some headway in understanding what the Langlands correspondence with regard to number fields could be, this is has still a very difficult question to tackle.

In this section, we shall turn to the *geometric* version of the Langlands correspondence, which is a valuable analogue to the number-theoretic correspondence. In this section, we explain how the each side of the Langlands correspondence becomes geometrized. Our number fields will become function fields over a Riemann surface, and the correspondence will now involve holomorphic vector bundles and Hecke eigensheaves.

## 2.1 Galois Groups and Galois Coverings

One of the key insights that leads to the geometrization of the Galois side of the Langlands correspondence is the relationship between Galois extensions of number fields and Galois topological coverings.

First, we review some basic material from the theory of covering spaces in topology. Given a topological space $X$, a *covering* consists of a topological *covering space* $Y$ together with a *covering map* $p : Y \to X$ satisfying the condition that every $x \in X$ is contained in an open neighborhood $U$ that is *evenly covered* by $p$–that is, $p^{-1}(U) = \bigsqcup_{i \in I} V_i$ for a collection of disjoint sets $V_i$ each homeomorphic to $U$ through $p$. The group of *deck transformations*, denoted $\mathrm{Aut}(p)$, is the collection of homeomorphisms $f : Y \to Y$ such that $p \circ f = p$. These can often be thought of as ways to permutate the $V_i$'s. The group operation is composition. The covering is *finite* if the indexing set $I$ is finite.

It is well-known that if $Y$ is simply connected[8] and $X$ is path connected, then this group of deck transformations is isomorphic to the *fundamental group of* $X$, the set of loops[9] in $X$ modulo homotopy (or "wiggling").

Now suppose $Y$ and $X$ are connected manifolds where $(Y, p)$ is a finite covering of $X$. Recall from Galois theory of field extensions that a finite extension is Galois if the order of its Galois group is as large as it can be–namely, the degree of the extension. Analogously, a $(Y, p)$ is a *Galois covering* of $X$ if the order of $\mathrm{Aut}(p)$ is as large as it can be. One can define a group action of $\mathrm{Aut}(p)$ on $Y$ by stipulating that on each fiber $p^{-1}(x)$,

$$(f, y) \mapsto f(y) \qquad f \in \mathrm{Aut}(p) \quad y \in p^{-1}(x)$$

The maximum order condition for the deck transformations is achieved only if $p : Y \to X$ descends to a homeomorphism $\bar{p} : Y/\mathrm{Aut}(p) \to X$; equivalently, $\mathrm{Aut}(p)$ must act transitively on all fibers of $p$. These definitions make sense also if we remove the finiteness condition.

In the case that $(Y, p)$ is a Galois cover, we denote $\mathrm{Aut}(p)$ instead by $\mathrm{Gal}(Y/X)$. Recall also from the Galois theory of field extensions that there are correspondences between (1) subgroups of the Galois group and intermediate field extensions and (2) normal subgroups of the Galois group and normal extensions. Similarly, in the Galois theory of covering spaces, there are correspondences between (1) subgroups of $\mathrm{Gal}(X/Y)$ and intermediate coverings up to equivalence[10] and (2) normal subgroups of $\mathrm{Gal}(Y/X)$ and intermediate Galois coverings.

But it turns out that the connection between the Galois theory of field extensions and the Galois theory of covering maps is even more intimate than this. If $X$ is now a projective algebraic curve defined over the complex numbers (or, equivalently, a Riemann surface), and $(Y, p)$ is a Galois covering, the function field $\mathbb{C}(Y)$ is a Galois extension of $\mathbb{C}(X)$ and $\mathrm{Gal}(Y/X) \cong \mathrm{Gal}(\mathbb{C}(Y)/\mathbb{C}(X))$.

## 2.2 Galois Representations and Holomorphic Vector Bundles

Suppose that $p : Y \to X$ is a Galois covering of connected Riemann surfaces. Recall that covering spaces are in general also a type of fiber bundle: Letting $x \in X$ and $U \ni x$ be the neighborhood evenly covered by $p$, we see that

[8] *Simply connected* means that any loop can be continuously "wiggled" (i.e. defomed via homotopy) down to a single point. Examples are Euclidean space and any sphere of dimension $\geq 2$. The case where $Y$ is simply connected is unique up to homeomorphism; in this case, $Y$ is referred to as the *universal cover* of $X$.

[9] The group operation here is concatenation.

[10] Two intermediate covers $W, Z$ are equivalent if they cover each other in such a way that fits in with the other covering maps in the compositions $Y \to W \to Z \to X$ and $Y \to Z \to W \to X$

we have the local trivialization property described by the commutative diagram below:

$$\begin{array}{ccc} p^{-1}(U) & \xrightarrow{\cong} & U \times p^{-1}(x) \\ {\scriptstyle p}\big\downarrow & \swarrow_{\mathrm{pr}_1} & \\ U & & \end{array}$$

Here, the fiber is the discrete preimage of $x$. What makes it a fiber bundle is the *local trivialization*, which is the homeomorphism $p^{-1}(U) \to U \times p^{-1}(x)$

But now, we consider representations of the Galois group $\mathrm{Gal}(Y/X)$; here, an $n$-dimensional representation is a homomorphism $\rho : \mathrm{Gal}(Y/X) \to GL(n, \mathbb{C})$. Representations will transform permutations of the fiber elements $p^{-1}(x)$ into invertible linear transformations on $\mathbb{C}^n$. So, one may guess that what we will end up with as our analogue to $n$-dimensional representations of Galois groups is a type of fiber bundle whose fibers are vector spaces–in particular, copies of $\mathbb{C}^n$–instead of the discrete fiber $p^{-1}(x)$. These are *holomorphic vector bundle of rank* $n$, a pair $(E, \pi)$ where $E$ is the *total space*, $\pi : E \to X$ is the *projection*, and for an open cover $\{U_\alpha\}_{\alpha \in A}$ of $X$, the following commutative diagram is satisfied:

$$\begin{array}{ccc} \pi^{-1}(U_\alpha) & \xrightarrow{h_\alpha} & U \times \mathbb{C}^n \\ {\scriptstyle \pi}\big\downarrow & \swarrow_{\mathrm{pr}_1} & \\ U_\alpha & & \end{array}$$

where the *local trivialization* $h_\alpha : \pi^{-1}(U) \to U \times \mathbb{C}^n$ is biholomorphic, and additionally, on nonempty overlaps $U_\alpha \cap U_\beta$, we have that the transition maps $h_{\alpha\beta} : h_\beta \circ h_\alpha^{-1} : U_\alpha \cap U_\beta \to U_\alpha \cap U_\beta$ are biholomorphisms of the form

$$(x, \mathbf{v}) \longmapsto (x, g_{\alpha\beta}(x)\mathbf{v})$$

Where $g_{\alpha\beta} : U_\alpha \cap U_\beta \to GL(n, \mathbb{C})$ is holomorphic. Lastly, a *holomorphic section* $s$ is a family of locally defined holomorphic maps $s_\alpha : U_\alpha \to E$ such that $\pi \circ s_\alpha = \mathrm{Id}_{U_\alpha}$. These sections form an $n$-dimensional $\mathcal{O}_X$-module, where $\mathcal{O}_X$ denotes the set of holomorphic functions $X \to \mathbb{C}$.

Now that we have a guess for the spaces which are analogues to Galois representations, let's understand how we reach this analogy. We first develop the language of sheaves.

A *sheaf* is a functor $\mathcal{F}$ that takes the data of our manifold $X$ and turns it into something special, such as a group or a ring. In other words, for our purposes, for every open set $U$, $\mathcal{F}(U)$ is a $n$-dimensional $\mathbb{C}$-vector space, whose elements are *sections of* $\mathcal{F}$ *over* $U$. These sections will form our aforementioned "special" set–such as a group or ring. $\mathcal{F}(X)$ consists of the *global sections*. If $U \subseteq V$ are two open sets, we have a corresponding restriction map $\mathrm{res}_{V,U} : \mathcal{F}(V) \to \mathcal{F}(U)$ which is the image of the inclusion map $\iota : U \to V$ under the functor $\mathcal{F}$. Functors between categories preserve morphisms: This means that, for example, in the case of a sheaf of groups, $\mathrm{res}_{V,U}$ will be a group homomorphism, or in the case of a sheaf of rings, $\mathrm{res}_{V,U}$ will be a ring homomorphism. There are two required conditions: (1) $\mathrm{res}_{U,U}$ is the identity on $\mathcal{F}(U)$, and (2) if $U \subseteq V \subseteq W$, then we have the following commutative diagram:

$$\begin{array}{ccc} \mathcal{F}(W) & \xrightarrow{\mathrm{res}_{W,V}} & \mathcal{F}(V) \\ & {}_{\mathrm{res}_{W,U}}\searrow & \big\downarrow{\scriptstyle \mathrm{res}_{V,U}} \\ & & \mathcal{F}(U) \end{array}$$

All of this makes $\mathcal{F}$ a *presheaf*: this becomes a *sheaf* with two more ingredients: If $\{O_\alpha\}_{\alpha \in A}$ is an open cover of $U$, then

- $s_1, s_2 \in \mathcal{F}(U)$, and $\mathrm{res}_{U,O_\alpha}(s_1) = \mathrm{res}_{U,O_\alpha}(s_2)$ for all $\alpha \in A$, then $s_1 = s_2$.
- If we have a collection $\{s_\alpha \in \mathcal{F}(O_\alpha)\}_{\alpha \in A}$ such that $\mathrm{res}_{U,O_\alpha \cap O_\beta}(s_\alpha) = \mathrm{res}_{U,O_\alpha \cap O_\beta}(s_\beta)$ for all $\alpha, \beta \in A$, then there exists a unique $s \in \mathcal{F}(U)$ such that $\mathrm{res}_{U,O_\alpha}(s_\alpha) = s$.

We can see immediately, for example, that $\mathcal{O}_X$ can be thought of as a sheaf whose sections are holomorphic functions.

If we have another sheaf $\mathcal{G}$, then a *morphism of sheaves* $\varphi : \mathcal{F} \to \mathcal{G}$ is such that the induced morphism of objects[11] $\varphi(U) : \mathcal{F}(U) \to \mathcal{G}(U)$ obeys the following commutative diagram, if $U \subseteq V$ are open sets:

$$\begin{array}{ccc} \mathcal{F}(V) & \xrightarrow{\varphi(V)} & \mathcal{G}(V) \\ \big\downarrow \text{res}_{V,U} & & \big\downarrow \text{res}_{V,U} \\ \mathcal{F}(U) & \xrightarrow{\varphi(U)} & \mathcal{G}(U) \end{array}$$

This morphism is an isomorphism if on all open sets $U \subset X$ $\varphi(U) : \mathcal{F}(U) \to \mathcal{G}(U)$ are isomorphisms which obey the above property. Then, thinking about the fact that as we move a point around a manifold, a section of an $n$-dimensional holomorphic vector bundle will manifest as a $n$-tuple of holomorphic functions, we can intuit that a holomorphic vector bundle can be equivalently expressed as a sheaf of *modules*–in particular, a *locally free sheaf of rank n*–meaning that $\mathcal{F} \cong \mathcal{O}_X^{\oplus n}$.

Now that we have developed this language, we can move back to discussing how to get from Galois representations to holomorphic vector bundles–or, as we have now established, locally free sheaves.

For this, we consider a such a sheaf $\mathcal{F}$ with an extra condition that makes it a *locally constant sheaf*. This means that for we have a covering $\{U_\alpha\}_{\alpha \in A}$ of $X$ such that $\mathcal{F}|_{U_\alpha}$ is isomorphic to the *constant sheaf* $\mathcal{K}$, where $\mathcal{K}(U) = \mathbb{C}^n$ constantly for all open sets $U$ and all restriction maps are isomorphisms.[12] This means in particular that for nonempty $U_\alpha \cap U_\beta$, $\mathcal{F}|_{U_\alpha}$ and $\mathcal{F}|_{U_\beta}$ are naturally identified via a constant linear transformation $g_{\alpha\beta} \in GL_n(\mathbb{C})$; this demands, in the language of vector bundles, that our transition maps $h_{\alpha\beta}$ be of the form $(x, \mathbf{v}) \mapsto (x, g_{\alpha\beta}\mathbf{v})$ for a *constant* general linear matrix $g_{\alpha\beta}$. It turns out that this condition is equivalent to, in the language of vector bundles, the existence of a *flat* connection–one that satisfies $[\nabla_X, \nabla_Y] = \nabla_{[X,Y]}$ for any vector fields $X, Y$–in other words, a connection that yields zero Riemannian curvature.

Returning to the topic of Galois covers, suppose $\tilde{X}$ is the universal cover of $X$ (meaning that $\tilde{X}$ is simply connected). Then $\tilde{X}$ is automatically a Galois cover. Furthermore, $\tilde{X}$ covers every other covering space of $X$; as a result, for every intermediate Galois cover $Y$, $\text{Gal}(Y/X)$ is a normal subgroup of $\text{Gal}(\tilde{X}/X)$. In general, every intermediate cover has an associated subgroup of $\text{Gal}(\tilde{X}/X)$. Also, $\text{Gal}(\tilde{X}/X) \cong \pi_1(X)$, where $\pi_1(X)$ denotes the fundamental group of $X$. So our Galois representations are now homomorphisms $\rho : \pi_1(X) \to GL(n, \mathbb{C})$.

Here, it is useful to define the *stalk of $\mathcal{F}$ at $x \in X$*:

$$\mathcal{F}_x = \{(s, U) : U \text{ open}, x \ni U, s \in \mathcal{F}(U)\} \Big/ \{(s, U) \sim (t, V) \iff \exists W \subset U, V : \text{res}_{U,W}(s) = \text{res}_{V,W}(t)\}$$

In the language of vector bundles, this is the set of *germs*, where we identify under an equivalence relation sections which agree on all neighborhoods containing $p$.

By definition of our locally constant sheaf, if $x_0$ and $x_1$ are contained in the same neighborhood isomorphic $U$ where $\mathcal{F}(U) = \mathbb{C}^n$, then we can draw a path from $x_0$ and $x_1$ and transport a vector via a section by moving along the path between $x_0$ and $x_1$ and looking at the values of the section on $x_0$ and $x_1$. The result is an invertible linear map $\mathcal{F}_{x_0} \to \mathcal{F}_{x_1}$, or equivalently an element of $GL(n, \mathbb{C})$ that depends only on the homotopy class of the path. Now, setting $x_0 = x_1$, we can identify each loop modulo homotopy with an element of $GL(n, \mathbb{C})$; equivalently, we get a representation $\rho : \pi_1(X) \to GL(n, \mathbb{C})$. To make this more concrete with an example, think of the tangent bundle with a flat affine connection; the representation $\rho$ can be expressed exactly in terms of *holonomy*, where the resulting linear transformation is the parallel transport of a tangent vector along a loop.

So any locally constant sheaf of rank $n$ gives us an $n$-dimensional Galois representation. If we instead start with a Galois representation $\rho : \text{Gal}(\tilde{X}/X) \to GL(n, \mathbb{C})$. Recalling the group action of $\text{Gal}(\tilde{X}, X)$ on $\tilde{X}$, we see that

$$\tilde{X} \times \mathbb{C}^n \Big/ \{(\tilde{x}, \mathbf{v}) \sim (f(\tilde{x}), \rho(f)\mathbf{v}) \; \forall f \in \text{Gal}(\tilde{X}/X)\}$$

This demonstrates a one-to-one correspondence between Galois representations of dimension $n$ and locally constant sheaves of rank $n$–or equivalently, holomorphic vector bundles with a flat connection.

[11] "Objects" is in reference to the type of sheaf. So if $\mathcal{F}, \mathcal{G}$ are sheaves of groups, for example, the induced morphism of objects $\varphi(U)$ on each open set is a group homomorphism. This is a category-theoretic term.

[12] $\mathcal{F}|_U$ denotes the restriction of the $\mathcal{F}$ to $U$, where we consider a single fixed open set $U$ as the global topological space.

## 2.3 Hecke Eigensheaves and the Geometric Langlands Correspondence

The last two sections were dedicated to understanding how we get from Galois representations to holomorphic vector bundles. Now for the other component of the Langlands correspondence, we will briefly summarize how we get from automorphic representations to Hecke Eigensheaves, which will come from moduli spaces of rank $n$ vector bundles. Recall from the first part that we are looking specifically at automorphic representations of $GL(\mathbb{A}_K)$ for some field $K$. As we pointed out in Section 2.1, for Geometric Langlands, instead of a number field $K$, we now have a function field $K$. There are analogous definitions for ramified and unramified representations; we will assume that we are dealing with irreducible unramified automorphic representations.

If we have a Riemann surface $X$, we have the associated function field $\mathbb{C}(X) = F$, and we define the set of adèles $\mathbb{A}_F$ in exactly the same way as we did before. Recall from Section 1.5 that out of an automorphic representation, we get a modular form $f_\pi$ that is an eigenfunction of the spherical Hecke algebra. Analogously, If we fix a point $x \in X$, if we are given an automorphic representation on $GL(n, \mathbb{A}_K)$, we can similarly derive from this a function $f_\pi$ on $(GL(n, \mathbb{A}_F)/GL_n(\mathcal{O}))/GL(n, F)$, where $\mathcal{O} = \prod_{x \in X} \mathcal{O}_x$ is the ring of integers of the completion $F_x$ of $F$ at $x$; this function $f_\pi$ is an eigenfunction of every spherical Hecke algebras $\mathcal{H}_x$, where $x$ ranges over $X$.

From here, we invoke a crucial observation in the form of a theorem from Weil: There is a one-to-one correspondence between $(GL(n, \mathbb{A}_F)/GL_n(\mathcal{O}))/GL(n, F)$ and rank $n$ holomorphic vector bundles on $X$ up to isomorphism.

The first observation instructs us to view the set $(GL(n, \mathbb{A}_F)/GL_n(\mathcal{O}))/GL(n, F)$ as a moduli space of rank $n$ vector bundles on $X$. Heck eigensheaves turn out to be the analogue to our $f_\pi$; these are defined as perverse sheaves defined on the moduli stack on the space of isomorphism classes of vector bundles, denoted $\text{Bun}_n$, with associated "eigenvalue" $E$–a holomorphic vector bundle.

So the geometric Langlands correspondence declares a relation between holomorphic rank $n$ bundles together with a flat connection and Hecke eigensheaves on the moduli stack $\text{Bun}_n$. As of quite recently, this correspondence has been proven by the mathematicians D. Arinkin, D. Beraldo, J. Campbell, L. Chen, J. Faergeman, D. Gaitsgory, K. Lin, S. Raskin and N. Rozenblyum.

# References


[1] Keith Conrad. "Factoring after Dedekind". In: *URL https://kconrad. math. uconn. edu/blurbs/gradnumthy/dedekindf. pdf* ().

[2] Edward Frenkel. "Lectures on the Langlands program and conformal field theory". In: *Frontiers in Number Theory, Physics, and Geometry II: On Conformal Field Theories, Discrete Groups and Renormalization.* Springer, 2007, pp. 387–533.

[3] Yves Hellegouarch. *Invitation to the mathematics of Fermat-Wiles.* Elsevier, 2001.

[4] Elias M Stein and Rami Shakarchi. *Complex analysis.* Vol. 2. Princeton University Press, 2010.

[5] Terrence Tao. *Trying to understand the Galois correspondence.* `https://terrytao.wordpress.com/2018/08/28/trying-to-understand-the-galois-correspondence/`. [Accessed 12-12-2024].

[6] Ravi Vakil. "The rising sea: Foundations of algebraic geometry". In: *preprint* (2017).