Siegel Modular Forms

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1 Introduction

In this seminar, we have investigated the notion of a modular form. These are certain holomorphic functions on the upper half-space $\mathbb{H}$ that transform in a nice way with respect to the group $SL_2(\mathbb{Z})$ acting on $\mathbb{H}$. Modular forms appear in many fields across number theory, the Langlands program, combinatorics, algebraic geometry, and even physics. They admit various generalizations: they can be seen as sections of line bundles on modular curves, or automorphic forms and functions on adelic quotients.

In this article, we will introduce one generalization of modular forms, called *Siegel modular forms*. These generalize classical modular forms to higher-dimensional analogues of the half-plane $\mathbb{H}$ and group $SL_2(\mathbb{Z})$. It turns out that the proper generalization is to consider functions on a space of matrices, called the *Siegel upper half-plane*, equipped with an action of the *symplectic group* $Sp(2g, \mathbb{Z})$. We develop these notions and define a Seigel modular form in Section 2.

The Fourier theory of modular forms is also important, including their q-expansions and Fourier coefficients. In Section 3, we will show how this theory generalizes to higher dimensions and give some applications. These include the *Koecher principle*, which shows how higher rank Siegel modular forms need no additional growth condition at the "cusp" at infinity.

This paper makes the major sin of containing no examples. As with modular forms, two main constructions of Siegel modular forms are Eisenstein series and theta functions. While their theory is interesting and deep, the details are unfortunately technical and complicated. We will content ourselves with showing how some aspects of the general theory generalizes to the higher-dimensional setting, with the assurance that this theory may be applied to many interesting examples. It is my hope that the theory for higher ranks elucidates some aspects of the classical $g = 1$ case. As one example application, we mention that Siegel modular forms may be seen as functions on the moduli space of principally-polarized complex abelian varieties, with the usual interesting difficulties of dealing with orbifold points and compactifications of the moduli space.

Our presentation closely follows [vdG08], which is our main reference. For applications of Siegel modular forms and much more of the theory, we refer the reader to [vdG08].

2 Siegel Modular Forms

2.1 Symplectic Group and Siegel Upper Half Space

The starting point for generalizing modular forms to higher dimensions will be the following observation. The group $SL_2(\mathbb{Z})$ is the automorphism group of the lattice $\mathbb{Z}^2$ with alternating form $\langle -,-\rangle$ defined by

$$\langle (a,b),(c,d)\rangle = ad - bc.$$

In higher dimensions, we generalize the group $SL_2(\mathbb{Z})$ with the *symplectic group* $Sp(2g,\mathbb{Z})$ for $g \geq 1$. This is by definition the automorphism group of the lattice $\mathbb{Z}^{2g}$ equipped with symplectic form $\langle -,-\rangle$ as follows: letting $e_1,\ldots,e_g,f_1,\ldots,f_g$ be a basis for $\mathbb{Z}^{2g}$, we define

$$\langle e_i,e_j\rangle = 0, \quad \langle f_i,f_j\rangle = 0, \quad \langle e_i.f_j\rangle = \delta_{ij}. \tag{1}$$

In other words, the matrix for the alternating form $\langle -,-\rangle$ in this basis is, in block form,

$$\Omega = \begin{pmatrix} 0 & I_g \\ -I_g & 0 \end{pmatrix} \tag{2}$$

where I_g is the $g \times g$ identity matrix. Thus, we may write the symplectic group as

$$Sp(2g,\mathbb{Z}) = \left\{ M \in M_{2g\times 2g}(\mathbb{Z}) \mid M^T \Omega M = \Omega \right\}. \tag{3}$$

In the sequel, we will often write $g \times g$ matrices in the block form

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \tag{4}$$

for $g \times g$ integer matrices A,B,C,D. If M is represented in such a block form, we will even write $M = (A,B;C,D)$. If $M = (A,B;C,D)$, it is easy to see that $M \in Sp(2g,\mathbb{Z})$ if and only if the three conditions

$$AB^T = BA^T, \quad CD^T = DC^T, \quad AD^T - BC^T = I_g \tag{5}$$

are satisfied. By taking a transpose and multiplying by -1, one can see that these conditions are equivalent to

$$C^T A - A^T C = 0, \quad D^T B - B^T D = 0, \quad D^T A - B^T C = I_g. \tag{6}$$

In particular, when $g = 1$ we recover the group $SL_2(\mathbb{Z})$.

We may similarly generalize the upper half-plane.

Definition 2.1. The *Siegel upper half-plane* $\mathbb{H}_g$ is defined as

$$\mathbb{H}_g = \left\{\tau \in M_{g\times g}(\mathbb{C}) \mid \tau^T = \tau, \, \mathrm{Im}(\tau) > 0\right\}, \tag{7}$$

i.e. the complex symmetric matrices with positive-definite imaginary part.

Clearly $\mathbb{H}_1 = \mathbb{H}$. We can define an action of $Sp(2g, \mathbb{Z})$ on $\mathbb{H}_g$ by setting, for $M = (A, B; C, D) \in Sp(2g, \mathbb{Z})$ and $\tau \in \mathbb{H}_g$,

$$M \cdot \tau := (A\tau + B)(C\tau + D)^{-1}. \tag{8}$$

While this definition is clear when $g = 1$, it is not clear that it is well-defined for higher g; for example, $C\tau + D$ may not be invertible.

Lemma 2.2. *If $M = (A, B; C, D) \in Sp(2g, \mathbb{Z})$, then $C\tau + D$ is invertible and*

$$M \cdot \tau = (A\tau + B)(C\tau + D)^{-1} \in \mathbb{H}_g.$$

Proof. Write $\tau = x + iy$ with x and y symmetric real $g \times g$ matrices. In particular, y is positive-definite. We will show that $\det(C\tau + D) \neq 0$. Using the fact that $M \in Sp(2g, \mathbb{Z})$, a computation shows that

$$(C\overline{\tau} + D)^T(A\tau + B) - (A\overline{\tau} + B)^T(C\tau + D) = \tau - \overline{\tau} = 2iy.$$

If the equation $(C\tau + D)\xi = 0$ had a nonzero solution $\xi \in \mathbb{C}^g$, then plugging ξ into the above identity implies $\overline{\xi}^T y\xi = 0$, contradicting the assumption that y is positive-definite.

Thus, the expression $M \cdot \tau$ makes sense. We will show it is symmetric and has positive-definite imaginary part. For this we compute an additional identity

$$\begin{aligned}(C\tau + D)^T(M \cdot \tau - (M \cdot \tau)^T) &= (C\tau + D)^T(A\tau + B) - (A\tau + B)^T(C\tau + D)\\ &= \tau - \tau^T = 0.\end{aligned}$$

Canceling $(C\tau + D)^T$ on the left hand side, we see that $M \cdot \tau$ is symmetric. Finally, let $y' = \mathrm{Im}(M \cdot \tau)$. By combining the two identities, we see that

$$(C\overline{\tau} + D)^T y'(C\tau + D) = \frac{1}{2i}(C\overline{\tau} + D)^T(M \cdot T - (\overline{M \cdot \tau})^T)(C\tau + D) = y$$

and since y is positive-definite, so is y'. □

Thus $M \cdot \tau$ defines an action $\mathbb{H}_g$. As with the $g = 1$ case, the matrix $-I_{2g}$ acts trivially on $\mathbb{H}_g$, but the quotient $Sp(2g, \mathbb{Z})/\{\pm I_{2g}\}$ acts effectively on $\mathbb{H}_g$.

Remark 2.3. As with the half-plane $\mathbb{H}$, the Siegel half-spaces $\mathbb{H}_g$ have an analogous "Poincare disk model" and the structure of a homogeneous symmetric space. Moreover, the symplectic group $Sp(2g, \mathbb{Z})$ admits a level set structure, analogous to the congruence subgroups in $SL_2(\mathbb{Z})$. For more about these ideas, see [vdG08].

2.2 Siegel Modular Forms

We now come to modular forms. In the original $g = 1$ case, the "automorphy factor" $(cz + d)^k$ was relevant. The key insight is that the number k is really the weight of a representation of $GL_1(\mathbb{C})$. Higher dimensional Siegel modular forms are defined with respect to representations, but for higher ranks.

Definition 2.4. Let $\rho : GL_g(\mathbb{C}) \to GL(V)$ be a representation, where V is a finite-dimensional complex vector space. A holomorphic map $f : \mathbb{H}_g \to V$ is a *Siegel modular form of weight* ρ if

$$f(M \cdot \tau) = \rho(C\tau + D)f(\tau) \tag{9}$$

for all $M = (A, B; C, D) \in Sp(2g, \mathbb{Z})$ and all $\tau \in \mathbb{H}_g$. If $g = 1$, we further require that f is holomorphic at the cusps.

Naturally, one might wonder why there is no requirement to be "holomorphic at the cusps" when $g \geq 2$. The reason for this is the *Koecher principle*, proved in the following section. This says that, with no additional assumptions, Siegel modular forms are bounded on subsets of the form $\{\tau \in \mathbb{H}_g \mid \mathrm{Im}(\tau) > cId_g\}$, i.e. as they approach the "cusp" at infinity. Thus in higher ranks the holomorphicity at the cusps is automatic.

One might notice that general Siegel modular forms are functions into a vector space. If we consider the determinant representation $\det : GL_g(\mathbb{C}) \to GL_1(\mathbb{C})$ and its tensor products (powers), then we obtain complex-valued modular forms.

Definition 2.5. A *classical Siegel modular form of weight* k is a holomorphic function $\mathbb{H}_g \to \mathbb{C}$ such that

$$f(M \cdot \tau) = \det(c\tau + d)^k f(\tau) \tag{10}$$

for all $M = (a, b; c, d) \in Sp(2g, \mathbb{Z})$ and all $\tau \in \mathbb{H}_g$. If $g = 1$, we also require that f is holomorphic at the cusps.

3 Fourier Expansion of Siegel Modular Forms

We now begin generalizing the Fourier theory to Siegel modular forms. We will give the very beginnings of the theory here. One primary difference in the $g \geq 2$ case is the *Koecher principle*, which we prove here. The proof makes clear why a similar principle does not hold in the $g = 1$ case.

The "exponents" of our Fourier expansions will be half-integral matrices, which we define now.

Definition 3.1. A symmetric $g \times g$ matrix $N \in GL_g(\mathbb{Q})$ is *half-integral* if $2N$ is a matrix with integer entries, whose diagonal entries are even (i.e. the diagonal entries of N are integers).

Let τ_{ij}, $1 \leq i, j \leq g$ be the coordinates of $\mathbb{H}_g$. Any half-integral matrix N defines a linear form with integral coefficients in the coordinates τ_{ij}, defined by

$$\mathrm{Tr}(N\tau) = \sum_{i=1}^{g} n_{ii}\tau_{ii} + 2 \sum_{1\le i<j\le g} n_{ij}\tau_{ij}. \tag{11}$$

Since τ is symmetric, every linear integral combination of the coordinates τ_{ij} comes from such a half-integral matrix N.

Now, write $\tau = x + iy$ with x, y real symmetric $g \times g$ matrices. By the usual Fourier theory, a function $f : \mathbb{H}_g \to \mathbb{C}$ with the periodicity property $f(\tau + s) = f(\tau)$ for all integral symmetric $g \times g$ matrices s has a Fourier expansion

$$f(\tau) = \sum_{N \text{ half-integral}} a(N) e^{2\pi i \,\mathrm{Tr}(N\tau)} \tag{12}$$

where $a(N) \in \mathbb{C}$ is defined by

$$a(N) = \int_{x \mod I_n} f(\tau) e^{-2\pi i \,\mathrm{Tr}(N\tau)} dx, \tag{13}$$

where dx is the Euclidean volume element with respect to x on the space of real symmetric $g \times g$ matrices and the integral is over the box $-1/2 \le x_{ij} \le 1/2$. The Fourier expansion converges uniformly on compact subsets of $\mathbb{H}_g$.

Similarly, if f is a vector-valued Siegel modular form of weight ρ, then we can write

$$f(\tau) = \sum_{N \text{ half-integral}} a(N) e^{2\pi i \,\mathrm{Tr}(N\tau)} \tag{14}$$

where $a(N) \in V$ is defined in a similar matter. If we use the notation $q^N = e^{2\pi i \,\mathrm{Tr}(N\tau)}$, we get

$$f(\tau) = \sum_{N \text{ half-integral}} a(N) q^N, \tag{15}$$

generalizing the familiar q-expansions for modular forms.

Lemma 3.2. *For all $U \in GL_g(\mathbb{Z})$,*

$$a(U^T N U) = \rho(U^T) a(N). \tag{16}$$

Proof. Set $M = \begin{pmatrix} U & 0 \\ 0 & (U^{-1})^T \end{pmatrix} \in Sp(2g, \mathbb{Z})$. Then $M \cdot \tau = U\tau U^{-1}$ and

$$\begin{aligned} a(U^T N U) &= \int_{x \mod 1} f(\tau) e^{-2\pi i \,\mathrm{Tr}\left(U^T N U \tau\right)} dx \\ &= \rho(U^T) \int_{x \mod 1} f(\tau) e^{-2\pi i \,\mathrm{Tr}\left(N U \tau U^T\right)} dx \\ &= \rho(U^T) a(N). \end{aligned}$$

□

This property of the Fourier coefficients leads immediately, by parity conditions, to a restriction on the weights of forms.

Corollary 3.3. *A classical Siegel modular form of weight k with kg odd, vanishes.*

Proof. Letting $U = -I_g$, we get

$$a(N) = (-1)^{kg} a(N).$$

Since kg is odd, we see that $a(N) = 0$ for all N. □

We finally come to the *Koecher principle.*

Theorem 3.4 (Koecher principle)**.** *Let f be a Siegel modular form of weight ρ with q-expansion $f = \sum_N a(N)q^N$. Then if the half-integral matrix N is not positive semi-definite, we have $a(N) = 0$.*

Proof. For $g = 1$, this is the assumption that the Fourier expansion of f has no negative terms. So we suppose $g \geq 2$. Setting $\tau = iI_g$ in the expansion $f = \sum_N a(N)e^{2\pi i \operatorname{Tr}(N\tau)}$, the fact that f converges absolutely in $\mathbb{H}_g$ shows that there is a constant $C > 0$ such that for all half-integral matrices N, $|a(N)| \leq Ce^{2\pi i \operatorname{Tr}(N)}$.

Suppose N is not positive semi-definite. General results about half-integral matrices yield a primitive (the entries are coprime) column vector ξ such that $\xi^T N\xi < 0$. By the theory of unimodular matrices (those with determinant ± 1), we may complete ξ to a unimodular matrix U such that ξ is the first column of U. Using $a(U^T NU) = \rho(U^T)a(N)$ and replacing N by $U^T NU$, we may assume that the entry N_{11} of N is negative.

Now for $m \in \mathbb{Z}$, let V be the matrix

$$V = \begin{pmatrix} 1 & m & \\ 0 & 1 & \\ & & I_{g-2} \end{pmatrix} \in GL_g(\mathbb{Z}).$$

Then,

$$|a(N)| = |\rho(V^T)^{-1}||a(V^T NV)| \leq Cme^{2\pi \operatorname{Tr}\left(V^T NV\right)}.$$

But now $\operatorname{Tr}\left(V^T NV\right) = \operatorname{Tr}(V) + n_{11}m^2 + 2n_{12}m$. Letting $m \to \infty$, we see that $\operatorname{Tr}\left(V^T NV\right) \to -\infty$ and so $|a(n)| = 0$. □

Corollary 3.5. *Let f be a Siegel modular form of weight ρ. Then f is bounded on any subset of the form $\mathbb{H}_g$ of the form $\{\tau \in \mathbb{H}_g \mid \operatorname{Im}(\tau) > cId_g\}$ with $c > 0$.*

Proof. As in the proof of the Koecher principle, the $g = 1$ case follows from the assumption of holomorphicity at the cusps. So we assume $g \geq 2$ and write $f(\tau) = \sum_{N \geq 0} a(N)e^{2\pi i \operatorname{Tr}(N\tau)}$, where we can assume $N \geq 0$ by the Koecher principle. We can now estimate, for $\mathrm{Im}(\tau) > cI_g$,

$$|f(\tau)| \leq \sum_{N \geq 0} |a(N)| e^{-2\pi \operatorname{Tr}(N \operatorname{Im}(\tau))} \leq \sum_{N \geq 0} |a(N)| e^{-2\pi \operatorname{Tr}(Nc)}.$$

This latter sum is finite since it is the absolutely-convergent version of the series for $f(cI_g)$, so we are done. □

References

[vdG08] Gerard van der Geer. *Siegel Modular Forms and Their Applications*, pages 181–245. Springer Berlin Heidelberg, Berlin, Heidelberg, 2008.