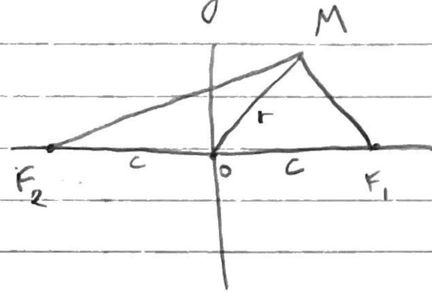
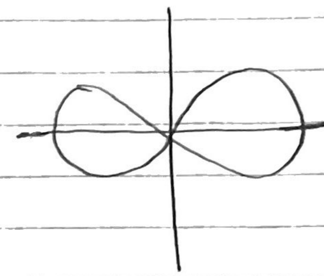
Background and History

Elliptic functions first show up in a work by Fagnano in work on the Bernoulli lemniscate.

Which is defined by



locus of the points s.t. $MF_1 \cdot MF_2 = c^2$

If we set $r_1 = MF_1$ $r_2 = MF_2$ $r = MO$,

$r_1^2 r_2^2 = c^4 \iff M$ lies on lemniscate

$$r_1^2 = r^2 - 2cx + c^2$$
$$r_2^2 = r^2 + 2cx + c^2$$

therefore

$$r^4 + 2c^2 r^2 - 4c^2 x^2 = 0$$

If we set $2c^2 = 1$, becomes:

$$r^4 + r^2 - 2x^2 = 0$$

In homogenous Cartesian coords: $(x^2+y^2)^2 + (y^2-x^2)z^2 = 0$

So, lemniscate is a quartic

we can see

Length of arc in first quadrant beginning at origin and terminating at point distance r from the origin is $s(r) = \int_0^r \frac{1}{\sqrt{1-x^4}}\,dx$

Notice a similar integral is $\arcsin r = \int_0^r \frac{1}{\sqrt{1-x^2}}\,dx$

we can rationalize the integrand by substituting $x = \frac{2w}{1+w^2} \Rightarrow = \int_0^1 \frac{2}{1+w^2}\,dw$

With inspiration from this substitution, we can try $x = \frac{w\sqrt{2}}{\sqrt{1+w^4}}$ and $w = \frac{y\sqrt{2}}{\sqrt{1-y^4}} \Rightarrow \frac{2y\sqrt{1-y^4}}{1+y^4} = x$

As a result, we obtain

$$\int_0^r \frac{1}{\sqrt{1-x^4}}\,dx = 2\int_0^v \frac{1}{\sqrt{1-y^4}}\,dy$$

$$\text{Where } r = \frac{2v\sqrt{1-v^4}}{1+v^4}$$

"Which shows the lemniscatic arc length for a point at distance r from origin is twice that for a point at distance v"
While Fagnano didn't solve the integral, he found this relation which doubles arclength.

Euler's contribution
How to add elliptic integrals, or choose r s.t.

$$\int_0^u \frac{du}{\sqrt{1-u^4}} + \int_0^v \frac{dv}{\sqrt{1-v^4}} = \int_0^r \frac{dr}{\sqrt{1-r^4}}$$

If we try $r = \dfrac{u\sqrt{1-v^4}+v\sqrt{1-u^4}}{1-u^2v^2}$, we find

in fact that we can show that

$$\frac{du}{\sqrt{1-u^4}} = -\frac{dv}{\sqrt{1-v^4}} \Rightarrow r \text{ constant}$$

So we have a general addition theorem for lemniscatic integrals.

Further relating to the arclength of a lemniscate are elliptic

# Elliptic Functions

## Liouville's Theorem

Let $f: \mathbb{C} \to \mathbb{C}$ holomorphic, $|f(z)| < K$ in $\mathbb{C}$

Then $f$ is a constant

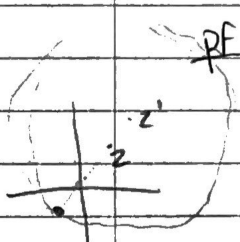
To prove this, we need

Cauchy's Integral Formula:

$$f(z) = \frac{1}{2\pi i} \int_C \frac{f(\zeta)}{\zeta - z} d\zeta$$

where $C$ is the boundary of some disk $D$, $z \in \text{int} D$

$\Rightarrow$ Holomorphic functions in $\mathbb{C}$ are determined by their behavior at this boundary




Pf of Liouville's Thm need to show $f(z) = f(z')$

Let $C$ be a circle of radius $r \geq 2|z - z'|$

$$f(z') - f(z) = \frac{1}{2\pi i}\left(\int_C \frac{f(\zeta)}{\zeta - z'} d\zeta - \int_C \frac{f(\zeta)}{\zeta - z} d\zeta\right)$$

$$= \frac{1}{2\pi i} \int_C \left(\frac{1}{\zeta - z'} - \frac{1}{\zeta - z}\right) f(\zeta) d\zeta$$

Setting $\zeta = z + re^{i\theta}$

$$|f(z') - f(z)| = \left|\frac{1}{2\pi} \int_0^{2\pi} \frac{z' - z}{\zeta - z'} f(\zeta) d\theta\right|$$

But $|\zeta - z'| \geq \big|\underbrace{|\zeta - z|}_{r} - \underbrace{|z - z'|}_{\leq r/2}\big| \geq r/2$

So $|f(z') - f(z)| \leq \frac{1}{2\pi} \int_0^{2\pi} \frac{|z' - z|}{r/2} K d\theta$

by chaining $\leq$

$$= \frac{2|z' - z|K}{r}$$

So $|f(z') - f(z)| = 0$

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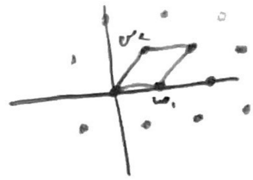

<u>Def</u>. Let $\omega_1, \omega_2 \in \mathbb{C}$ not collinear, let $\Lambda$ be the lattice $\Lambda = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$

$f : \mathbb{C} \to \mathbb{C} \cup \{\infty\}$ is <u>elliptic</u> if $f$ is meromorphic (holomorphic except for set of sparse points) and $\exists \Lambda$ s.t. $f(z+\omega) = f(z)$ $\forall z \in \mathbb{C}$ $\omega \in \Lambda$

"In fact suffices for $f(z+\omega_1) = f(z) = f(z+\omega_2)$"

$\Pi_{\omega_1,\omega_2} = (0, \omega_1, \omega_1+\omega_2, \omega_2) \setminus \omega_1, \omega_2$ and sides adjacent to $\omega_1, \omega_2$

(holomorphic and in $\mathbb{C}$)

<u>Thm</u> Every entire elliptic function is constant

Obvious because it must be bounded in $\Pi_{\omega_1,\omega_2}$

<u>Cor</u> If 2 elliptic functions of same lattice have the same poles in $\Pi$, their difference is a constant.

<u>Note</u> we can translate $\Pi$ s.t. $\alpha + \Pi$ has no poles on its boundary b/c $f$ is meromorphic.

<u>Thm</u>

Let $f$ elliptic, no poles on boundary of $\alpha + \Pi$, then sum of residues of $f$ in $\alpha + \Pi$ is zero

$\frac{1}{2\pi i}\int_{\alpha+\Pi} f(z)dz$

Pf is clear because if we integrate along sides of $\Pi + \alpha$, opposite sides cancel out, so integral is zero.

<u>Cor</u> $f$ non-constant and elliptic, $f$ either admits a multiple pole in $\alpha + \Pi$ or at least two simple poles of opposite residues.

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Def The number of poles (or zeros) of elliptic f. which lie in $\alpha + \Pi$ is called the order of f

if $f \neq c$, $\mathrm{ord}(f) \geq 2$

Thm

Let f elliptic on $\Lambda$, $\Pi + \alpha$ as above. Let $m_a$ denote orders of the zeros of f and $n_b$ denote orders of the poles of f in $\alpha + \Pi$.

Then $\sum m_a = \sum n_b$

pf $\sum m_a - \sum n_b = \frac{1}{2i\pi} \int_{\partial(\Pi + \alpha)} \frac{f'(z)}{f(z)} dz$ which is elliptic and therefore $0$

Thm $\overset{\text{zeroes}}{a_1 + \dots + a_n} \equiv (\overset{\text{poles}}{b_1 + \dots + b_n}) \bmod \Lambda$ including multiplicity

pf

We have

$$\sum_{v=1}^{n} (a_v - b_v) = \frac{1}{2\pi i} \int_{\partial(\Pi + \alpha)} z \frac{f'(z)}{f(z)} dz$$

But on opposite sides of $\Pi + \alpha$, values $z \frac{f(z)}{f'(z)}$ differ by $\pm \omega_j \frac{f'(z)}{f(z)}$ $j \in \{1, 2\}$

So above

$$= \frac{1}{2i\pi} \left( \int_0^1 -\omega_2 \frac{f'(\alpha + t\omega_1)}{f(\alpha + t\omega_1)} \omega_1 dt + \int_0^1 \omega_1 \frac{f'(\alpha + t\omega_2)}{f(\alpha + t\omega_2)} \omega_2 dt \right)$$

$$= \frac{1}{2i\pi} \left( -\omega_2 \left[ \log f(z) \right]_{\alpha}^{\alpha + \omega_1} + \omega_1 \left[ \log f(z) \right]_{\alpha}^{\alpha + \omega_2} \right)$$

Since $f(\alpha) = f(\alpha + \omega_j)$, the variation in log is an integer multiple of $2\pi i$, so

$$= n\omega_2 + m\omega_1 \quad n, m \in \mathbb{Z}$$

Cor

(i) $f(z) = c$ has $n$ solutions $z_1 \dots z_n$ in $\alpha + \Pi$ or a neighboring parallelogram

(ii) The sum $z_1 + \dots + z_n \bmod \Lambda$ does not depend on $\alpha$ or $c$

## Weierstrass Elliptic Functions

We want to construct a meromorphic function which is periodic on $\Lambda$

Consider $h(z) := \sum_{\omega \in \Lambda} \frac{1}{(z-\omega)^3}$ (lower than 3 series does not converge) and by prev. cor. 1 don't work

Pf This is uniformly convergent on every compact subset of $\mathbb{C}$

Lemma $\sum_{\omega \in \Lambda \setminus \{0\}} \frac{1}{|\omega|^s}$ is convergent $\forall s > 2$

If we consider $\mathbb{C}$ as $\mathbb{R}\omega_1 \oplus \mathbb{R}\omega_2$

$$\sum_{\omega \in \Lambda \setminus \{0\}} \frac{1}{|\omega|^s} \leq c \sum_{(m,n) \in \mathbb{Z}^2} \frac{1}{(\sup(m,n))^s}$$ c constant, similar to $\Delta$ inequality

Take $\|\cdot\|$ to denote $\|\omega\|_{\sup}$

we have $\sum_{\|\omega\| = N} \frac{1}{\|\omega\|^s} = \frac{8N}{N^s}$ so clearly converges

$$\Rightarrow \sum_{N=1}^{\infty} \left( \sum_{\|\omega\| = N} \frac{1}{\|\omega\|^s} \right)$$ converges $\forall s > 2$

Thm $h(z) = \sum_{\omega \in \Lambda} \frac{1}{(z-\omega)^3}$ is absolutely uniformly convergent on every compact subset of $\mathbb{C}$, and $h$ is an odd elliptic function, order 3

pf (i) By lemma, absolutely converges and is abs. uniformly convergent if $|z|$ is bounded above.

(ii) Since $\frac{1}{(z-\omega)^3}$ meromorphic $\forall \omega \in \Lambda$, $h$ is meromorphic $\forall$ compact subsets, therefore on all of $\mathbb{C}$

(iii) Since series is absolutely convergent, it is commutatively convergent and therefore periodic under lattice $\Lambda$

(iv) Only pole of $h$ in $\Pi$ is origin, which is a triple pole.

⑤

If we integrate $-2h(z) + \frac{2}{z^3}$, we get

$$H(z) = \sum_{\omega \in \Lambda \setminus \{0\}} \left( \frac{1}{(z-\omega)^2} - \frac{1}{\omega^2} \right)$$

By above, we know the series is uniformly convergent on every compact subset.

Define $P_\Lambda(z) = \frac{1}{z^2} + H(z)$ (extra $\frac{1}{z^2}$ is to get around $\omega = 0$ but mirror term in sum)

Then $P$ is meromorphic on $\mathbb{C}$ and $P'_\Lambda(z) = -2h(z)$
Call $P$ the Weierstrass function of $\Lambda$

<u>Thm</u>

(i) $P_\Lambda$ is an elliptic function of lattice $\Lambda$

(iii) $P_\Lambda$ has order 2

pf (ii) $P'(z) = -2h(z)$

$$\Rightarrow P'(z+\omega) = P'(z)$$

Take the generators of $\Lambda$ $(\omega_1, \omega_2)$ and integrate $P'(z+\omega) = P'(z)$
we get $P(z+\omega_i) = P(z) + C_i \quad i \in \{1, 2\}$

Since $H(z)$ is uniformly convergent on any compact subset, the only poles of $P$ are lattice points of $\Lambda$ so if we're in fundamental parallelogram $\Pi$, $0$ is the only pole of $P$. Since it is a double pole, it is of order 2 if $P$ elliptic (which we will show that it is).

If $z = \frac{-\omega_1}{2}$ or $\frac{-\omega_2}{2}$ the integration gives $P\left(\frac{\omega_i}{2}\right) = P\left(\frac{-\omega_i}{2}\right) + C_i$
Since $P$ is even, these are equal and $C_i = 0$.
$\Rightarrow P$ is periodic under lattice $\Lambda$ ∎

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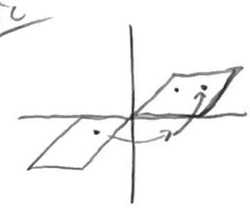
Pic



Now Prove Final Theorem

Let $\sigma$ be involution of $\mathbb{C} \setminus \Lambda$ which associates each $z$ to its opposite $-z$.

In $\Pi$, $\sigma$ can be represented by

$$\sigma(a) = -a \bmod \Lambda, \quad a \in \Pi$$

$\uparrow$ lies in $\Pi$

So, the group $\{id, \sigma\}$ acts on $\mathbb{C} \setminus \Lambda$ and if $a \notin \{0, \omega_1/2, \omega_2/2, (\omega_1+\omega_2)/2\}$ the orbit$(a)$ contains two elements $a, a'$

**Lemma** Let $f$ even elliptic function of lattice $\Lambda$ and let $a^* = \{a, a'\}$ be orbit under action of $\{id, \sigma\}$

Then (i) $f(a) = f(a')$ and $f$ has the same order at $a$ and $a'$

(ii) If $a = a'$, the order of $f$ at $a$ is even.

(i) Follows because $f$ is even and periodic under $\Lambda$

(ii) If $f(a+z) = a_m z^m + \dots$, then $f(-(a+z)) = a_m z^m + \dots$ since $f$ is even.

But $f(-a-z) = f(a'-z) = a'_{m'}(-z)^{m'} + \dots$

so $f$ has same order at $a$ and $a'$

If $a = a'$, then

$$a_m z^m + \dots = f(a+z) = f(a-z) = a_m(-z)^m + \dots$$

so $m$ is even.

⑦

**Cor** Every even elliptic function of $\Delta$ is a rational function of $P$.

pf Let $f$ be even elliptic and $a_\nu$ be zeros, $b_\nu$ be poles of $f$ in $\alpha + \Pi \setminus \{0\}$ mod $\{id, \sigma\}$, where we choose $|\alpha|$ very small but still s.t. $f$ has neither poles nor zeros on boundary of $\alpha + \Pi$.

Denote $r_\nu$ the order of multiplicity of $a_\nu$ and $s_\nu$ similar for $b_\nu$, divided by the order of its isotropy group (1 or 2) in $\{id, \sigma\}$

"So divisor will be 1 if $a_\nu$ or $b_\nu$ not in these special points $\{\omega_1/2, \omega_2/2, (\omega_1+\omega_2)/2\}$ otherwise its 2 by previous lemma"

Set

$$g(z) := \frac{\prod (P(z) - P(a_\nu))^{r_\nu}}{\prod (P(z) - P(b_\nu))^{s_\nu}}$$

"So we're constructing an elliptic function of order $n$ which imitates the behavior of $f$ at its zeros and poles"

Since $\operatorname{ord}(g) = n$, $f$ and $g$ have the same order at the origin.

$\Rightarrow f/g$ is an elliptic function under $\Delta$, holomorphic in $\Pi + \alpha$. Therefore it is constant.

Set $\frac{f}{g} = C$ and

$$\varphi(X) = C \frac{\prod (X - P(a_\nu))^{r_\nu}}{\prod (X - P(b_\nu))^{s_\nu}}$$

$$\Rightarrow f = \varphi(P)$$

"We have a rational function of $P$ to get any $f$ even and elliptic" ∎

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Last Thm

(i) Every elliptic $f$ can be written
$$f = \varphi(P) + P' \psi(P)$$
where $\varphi$ and $\psi$ are rational functions.

(ii) The field of elliptic functions of lattice $\Lambda$ is $\mathbb{C}(P, P')$

(i) Follows from previous corollary if $f$ is even

Set
$$g := \frac{f(z) + f(-z)}{2} \qquad h := \frac{f(z) - f(-z)}{2}$$

$\Rightarrow$ $g$ and $h$ are elliptic functions under $\Lambda$ and that either $g$ or $h$ is odd, other even

Thus WLOG $g = \varphi(P)$ and $\frac{h}{P'} = \psi(P)$ ■

"So we can construct any non-constant elliptic function from a Weierstrass function"