Elliptic Functions II

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These notes are an expansion on Sections 2.6-2.12 of [1]; most of the Sections have been named accordingly.

1 Review of Elliptic Functions I

First, we review some previous material.

1.1 Fundamentals of Complex Analysis

Definition 1.1. We define $f : \mathbb{C} \to \mathbb{C}$ to be *holomorphic* if the following limit exists (where the limit as $h \to 0$ is taken over the complex numbers):

$$\lim_{h \to 0} \frac{f(z+h) - f(z)}{h}$$

Equivalently, we can use the following:

Theorem 1.2. *$f : \mathbb{C} \to \mathbb{C}$ can be written as $f = u + iv$. f is holomorphic if an only if the following Cauchy-Riemann equations hold:*

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

The following propositions may be useful from time to time:

Proposition 1.3. *(Liouville / Cauchy) A bounded holomorphic function (i.e. $|f(z)| \leq b$ for some $b \in \mathbb{R}_{\geq 0}$) must be constant.*

Proposition 1.4. *The limit of a uniformly convergent sequence of holomorphic functions is holomorphic (given an open set), and the differential and limit operators are interchangeable in this case.*

Definition 1.5. A *meromorphic function* $f : \mathbb{C} \to \mathbb{C} \cup \infty$ is the quotient of two holomorphic functions, that is $f = \frac{g}{h}$ for g, h holomorphic. Consider a point p such that $f(z) = (z-p)^n h(z)$ where h is a holomorphic function. p is a zero of order or multiplicity n if $n > 0$, and p is a pole of order or multiplicity $-n$ if $n < 0$.

As we define differentiation of complex functions, we can also define integration of complex functions. Given a curve γ in $\mathbb{C}$ where $z : [a, b] \to \mathbb{C}$ is the parameter, and f is a continuous complex function, the integral of f along γ is

$$\int_\gamma f(z)dz = \int_a^b f(z(t))z'(t)$$

This looks quite similar to path integrals in real domains. To define integrals over areas rather than paths, we can write continuous $f : \mathbb{C} \to \mathbb{C}$ as $f = u + iv$ for two real continuous functions u, v, and define $\int f = \int u + i \int v$.

The *Laurent series* for a complex function are the complex analogue of Taylor series, but this time we include negative powers:

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z-c)^n$$

If c is a pole of f with order n, it is known from complex analysis that we can write $f(z)$ like this (where G is holomorphic):

$$f(z) = \frac{a_{-n}}{(z-c)^n} + \frac{a_{-n+1}}{(z-c)^{n+1}} + \ldots + \frac{a_{-1}}{z-c} + G(z)$$

$a_{-1} = \text{res}_c f$ is called the *residue* of f at c.

Proposition 1.6. *One can compute the residue of f at a singularity c by evaluating the following integral, where C is a circle such that c is the only pole of f contained in C:*

$$\frac{1}{2\pi i}\int_C P(z)dz = a_{-1}$$

This formula comes from Cauchy's integral.

Another result from complex analysis is the residue formula:

Theorem 1.7. *Given an open set U which contains a circle C, if f contains a finite number of poles $z_1, \ldots, z_N$ inside C and is holomorphic everywhere else in U, we have*

$$\int_U f(z)dz = 2\pi i \sum_{k=1}^{n} \text{res}_{z_k} f$$

This formula is an application of Stokes theorem, which relates the integral over some region to the integral over the boundary of said region. We can replace the circle with anything homeomorphic to it, or more specifically, a "toy countour," which is useful for integration techniques in complex analysis.

1.2 Defining Elliptic Functions

Definition 1.8. A *lattice* Λ is the $\mathbb{Z}$-span of two $\mathbb{R}$-linearly independent $\omega_1, \omega_2 \in \mathbb{C}$, i.e.

$$\Lambda = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$$

If $\Lambda = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$, the *fundamental parallelogram* Π_{ω_1,ω_2} is the filled-in parallelogram formed by the points 0, ω_1, $\omega_1 + \omega_2$, and ω_2 minus the boundary. We can translate a parallelogram by adding a fixed α to every element of Π; we denote this by $\alpha + \Pi$.

Definition 1.9. *Elliptic functions* are meromorphic $f : \mathbb{C} \to \mathbb{C} \cup \{\infty\}$ where there exists a lattice Λ so that $f(z + \omega) = f(z)$ for all $\omega \in \Lambda$.

Below is another important result from complex analysis:

Proposition 1.10. *If f is a meromorphic function on $\mathbb{C} \cup \infty$, that the number of zeroes of f equals the number of poles f counted with multiplicity.*

This follows from evaluating the integral $\int \frac{f'(z)}{f(z)}dz$ over $\mathbb{C} \cup \infty$ using Stokes Theorem and the residue formula.

One can translate this to elliptic functions:

Proposition 1.11. *If f is an elliptic function of lattice Λ which has no poles on the boundary of $\alpha + \Pi$, then the number of zeroes and poles contained in $\alpha + \Pi$ are the same when counted with multiplicity. Furthermore, we have that if $a_1, \ldots, a_n$ are the zeroes and $b_1, \ldots, b_n$ are the poles in $\alpha + \Pi$, then*

$$\sum_{j=1}^{n} a_j \equiv \sum_{j=1}^{n} b_j \mod \Lambda$$

Definition 1.12. The number of poles (or zeroes) of an elliptic function f of a lattice Λ which lie in a parallelogram of $\alpha + \Pi$ is *the order of f*.

1.3 Weierstrass Theory

The theory of Weierstrass functions, which we is a way to find meromorphic functions at specific poles and zeroes.

Definition 1.13. Given a lattice Λ, we define the *Weierstrass function of the lattice* Λ to be

$$\varrho_\Lambda(z) = \frac{1}{z^2} + H(z) \quad H(z) = \sum_{\omega \in \Lambda \setminus \{0\}} \left[\frac{1}{(z-\omega)^2} + \frac{1}{\omega^2} \right]$$

Proposition 1.14. *This meromorphic function has the following properties:*

(i) $H(z)$ *is holomorphic.*

(ii) $\varrho'_\Lambda(z) = -2h(z)$ *where*

$$h(z) := \sum_{\omega \in \Omega} \frac{1}{(z-\omega)^3}$$

h is an odd elliptic function of lattice Λ *with order* 3.

(iii) $\varrho_\Lambda(z)$ *is an elliptic function of lattice* Λ.

(iv) If $\lambda \in \mathbb{C}^*$, *then we have*

$$\varrho_\Lambda(z) = \lambda^2 \varrho_{\lambda\Lambda}(\lambda z)$$

(v) $\varrho_\Lambda(z)$ *is an even function of order 2.*

Then comes a very important result:

Theorem 1.15. *Every elliptic function* f *of lattice* Λ *can be written in terms of the Weirstrass function (where* φ, ψ *are rational functions):*

$$f = \varphi(\varrho_\Lambda) + \varrho'_\Lambda \psi(\varrho_\lambda)$$

Furthermore, the field of elliptic functions of lattice Λ *is* $\mathbb{C}(\varrho_\Lambda, \varrho'_\Lambda)$.

2 Eisenstein Series

As with any meromorphic function, we would like to know the Laurent expansion of ϱ_Λ is, given a lattice Λ.

For $m \geq 3$, consider

$$G_m(\Lambda) := \sum_{\omega \in \Lambda \setminus \{0\}} \frac{1}{\omega^m}$$

If m is odd, then noting that if $\omega \in \Lambda$, $-\omega \in \Lambda$, we observe that $G_m(\Lambda) = 0$. So G_m is interesting only for even $m = 2k$.

Definition 2.1. G_{2k} is the *Eisenstein series of index* $2k$ *of the lattice* Λ.

Recall from the previous section that

$$\varrho_\Lambda(z) - \frac{1}{z^2} = \sum_{\omega \in \Lambda \setminus \{0\}} \left[\frac{1}{(z-\omega)^2} + \frac{1}{\omega^2} \right]$$

If we take $z \neq \omega$, we see that $\frac{1}{(z-\omega)^2}$ is holomorphic, so only the principal part of the Laurent expansion remains.

$$\frac{1}{(z-\omega)^2} = \sum_{j=2}^{\infty} \frac{(j-1)z^{j-2}}{\omega^j}$$

We can then write this in terms of the Eisenstein series. Given our observation about Eisenstein series of odd indices, the Weierstrass equation becomes

$$\varrho_\Lambda(z) = \frac{1}{z^2} + \sum_{j=1}^{\infty} (2k+1) G_{2k+2} z^{2k} = \frac{1}{z^2} + 3G_4 z^2 + 5G_6 z^4 + 7G_8 z^6 + \dots$$

Using the Laurent series for ϱ, we can derive the Laurent series for ϱ' by differentiating with respect to z:

$$\varrho'(z) = -\frac{2}{z^3} + 6G_4 z + 20G_6 z^3 + 42G_8 z^5 + \dots$$

Theorem 2.2. *Let $G_4 = G_4(\Lambda)$, $G_6 = G_6(\Lambda)$ for some lattice Λ and ϱ be the Weierstrass function of Λ. Then ϱ is the solution to the differential equation*

$$Y'^2 = 4Y^3 - 60G_4Y - 140G_6$$

Proof. Replace Λ with $\alpha\Lambda$ and z with αz for some $\alpha \in \mathbb{C}^*$. It follows from Proposition 1.14 that ϱ becomes $\alpha^{-2}\varrho$. Furthermore, G_4 becomes $\alpha^{-4}G_4$ and G_6 becomes $\alpha^{-6}G_6$; this can be seen just by looking at each Eisenstein series. In other words, ϱ is of weight 2, G_4 is of weight 4, and G_6 is of weight 6.

From example 2.5.1 of [1], we have the following relation between ϱ' and ϱ:

$$\varrho'^2 = 4\varrho^3 + a\varrho^2 + b\varrho + c$$

Note from before that the Laurent series for ϱ' has only terms of odd power (this can also be seen from the fact that $\varrho' = -2h(z)$ and $h(z)$ is an odd function, which means that so is ϱ'). So ϱ'^2 has terms of power $2 \mod 4$. The first term of ϱ^2 is $\frac{1}{z^4}$, so we can conclude $a = 0$.

Out of convenience, we experiment with the case that the terms $4\varrho^3 + a\varrho^2 + b\varrho + c$ are all of the same weight (i.e. homogeneous). For this to happen, b must be of weight 4 and c must be of weight 6, since ϱ^3 is of weight 6. Using our observations about the weights of G_4 and G_6 and the Laurent series expansions of ϱ and ϱ', we can see that it is most likely that

$$b = \lambda G_4 \quad c = \mu G_6$$

We compute λ and μ using the Laurent series expansion for ϱ and ϱ'.

We compare terms our two different versions of the Laurent expansion of $\varrho'(z)$. In particular, to compute λ, we pay attention to the terms involving G_4; the expression on the left corresponds with the Laurent expansion of $\varrho'(z)$ obtained by taking the derivative of $\varrho(z)$, and the expression on the right corresponds to the Laurent expansion for $4\varrho^3 + \lambda G_4\varrho + \mu G_6$.

$$\left(-\frac{2}{z^3} + 6G_4z\right)^2 + \dots \quad 4\left(\frac{1}{z^2} + 3G_4z^2\right)^3 + \lambda\frac{G_4}{z^2} + \dots$$

Examining the terms with $\frac{1}{z^2}$, we see that the expression on the left has the term $\frac{-24G_4}{z^2}$ and the expression on the right has the term $\frac{(36+\lambda)G_4}{z^2}$. Hence, we see that $\lambda = -60$.

To compute the terms involving μ, we pay attention to the terms involving G_6.

$$\left(-\frac{2}{z^3} + 20G_6z^3\right)^2 + \dots \quad 4\left(\frac{1}{z^2} + 5G_6z^4\right)^3 + \mu G_6 + \dots$$

Examining the constant terms, we see that the expression on the left has the term $-80G_6$ and the expression on the right has the term $60G_6 + \mu G_6$. Hence we get that $\mu = -140$. □

3 The Weierstrass Cubic

Before we proceed further, it may be in our best interest to introduce some aspects of geometry regarding projective curves.

3.1 Complex Projective Space

I introduce the geometry of projective space quite loosely here. I have purposefully left out a handful of terminology.

Definition 3.1. We define *complex projective n-space* $\mathbb{P}^n_{\mathbb{C}}$ to be $\mathbb{C}^{n+1}/\{\mathbf{x} \sim \lambda\mathbf{x}\ \forall\lambda \in \mathbb{C}^*\}$.

Its elements are written as $[z_1 : z_2 : \ldots : z_n : z_{n+1}]$, recognizing that $[z_1 : z_2 : \ldots : z_n : z_{n+1}] \sim [\lambda z_1 : \lambda z_2 : \ldots : \lambda z_n : \lambda z_{n+1}]$. Note that z_i cannot not all be 0.

For our purposes, it is ideal to understand what $\mathbb{P}^1_{\mathbb{C}}$ and $\mathbb{P}^2_{\mathbb{C}}$ look like.

The elements of $\mathbb{P}^1_{\mathbb{C}}$ look like $[z_1 : z_2]$. Let's consider the set of points in $\mathbb{P}^1_{\mathbb{C}}$ where $z_2 \neq 0$. Scaling our coordinates by z_2 gives us $[z_1 : z_2] = [z : 1]$, where $z = \frac{z_1}{z_2}$. Note that every point in $\{z_2 \neq 0\}$ can be expressed $[z : 1]$ for some $z \in \mathbb{C}$. Hence, we have a bijective correspondence $\{z_2 \neq 0\} \to \mathbb{C}$ given by $[z : 1] \mapsto z$. If $z_2 = 0$, the only such point in $\mathbb{P}^1_{\mathbb{C}}$ is $[1 : 0]$. So, we see that

$$\mathbb{P}^1_{\mathbb{C}} = \{z_2 \neq 0\} \cup \{[1 : 0\} \cong \mathbb{C} \cup \infty$$

In other words, $[1 : 0]$ serves as our point at infinity, and $\mathbb{P}^1_{\mathbb{C}}$ is the[1] one-point compactification of the complex plane–in other words, the unit sphere[2] $\mathbb{S}^2$.

The elements of $\mathbb{P}^2_{\mathbb{C}}$ look like $[z_1 : z_2 : z_3]$. Similarly, we can consider the set $\{z_3 \neq 0\}$; when we scale by z_3 respectively, we find a correspondence $\{z_3 \neq 0\} \to \mathbb{C}^2$. But this time, we need more than one "point at infinity," namely points of the form $[a : b : 0]$ to complete $\mathbb{P}^2_{\mathbb{C}}$. But note that this set is identifiable with $\mathbb{P}^1_{\mathbb{C}}$. So we have $\mathbb{P}^2_{\mathbb{C}} \cong \mathbb{C}^2 \sqcup \mathbb{P}^1_{\mathbb{C}}$

Observe that we have seen that locally, $\mathbb{P}^1_{\mathbb{C}}$ mostly looks like $\mathbb{C}^1$ and $\mathbb{P}^2_{\mathbb{C}}$ mostly looks like $\mathbb{C}^2$, except for the fact that these projective spaces are compact. In general, $\mathbb{P}^n_{\mathbb{C}}$ will mostly look like $\mathbb{C}^n$–this is why we call it projective n-space, even though we define the equivalence relation on $\mathbb{C}^{n+1}$.

3.2 Elliptic Curves

Definition 3.2. Let $60G_4 = g_2$ and $140G_6 = g_3$. We define

$$Y^2 = 4X^3 - g_2X - g_3$$

to be the *Weierstrass cubic associated to the lattice* Λ.

This is the first example we have seen of an elliptic curve!

Definition 3.3. *Elliptic curves* are of the form $E = \{y^2 = f(x)\}$ where the polynomial $f(x)$ is a cubic with no repeated roots.

This is the expression for the curve in terms of *affine coordinates*; i.e. we consider this algebraic set as living in $\mathbb{C}^2$. But, it often behooves us to homogenize this equation. In other words, if we replace x with X/Z, y with Y/Z and clear denominators, we get, supposing that $f(x) = ax^3 + bx^2 + cx + d$, that

$$Y^2Z = aX^3 + bX^2Z + cXZ^2 + dZ^3$$

We now view this as a subset of $\mathbb{P}^2_{\mathbb{C}}$, i.e. we consider $[X : Y : Z]$ which satisfy the above relation of the form $F(X, Y, Z) = 0$. Indeed, if we scaled X, Y, Z by some constant λ, the solutions to the above equation would remain untouched. In other words, we are looking at this set now as a *projective curve*. In particular, we are looking at the *projective completion*, denoted $\bar{E}$, of our curve E.

We can also look at this curve locally by de-homogenizing the curve to other subsets of $\mathbb{P}^2_{\mathbb{C}}$. For example, if we consider the subset $\{X \neq 0\}$ and let $\frac{Y}{X} = y$ and $\frac{Z}{X} = z$, after scaling our coordinates by X, our equation becomes

$$y^2z = a + bz + bz^2 + bz^3$$

Or we can also consider the subset $\{Y \neq 0\}$ and, after scaling by Y and suitably replacing coordinates, our equation becomes

$$z = ax^3 + bx^2z + bxz^2 + cz^3$$

There are three non-homogenous equations associated with our curve, of the form $f_1(x, y) = 0$, $f_2(y, z) = 0$, and $f(x, z) = 0$; they are said to represent the three different *affine charts* of our curve, in the same way that $\{X \neq 0\}$,

[1] One-point compactifications are unique up to homeomorphisms; this is a result from point-set topology.

[2] Imagine folding the distant edges of the complex plane into one point. Another way to see this is through stereographic projection: In which we view the unit sphere as $\{x^2 + y^2 + z^2 = 1\} \subset \mathbb{R}^3$ and define a map $f : \mathbb{S}^2 \setminus \{(0, 0, 1)\} \to \mathbb{R}^2$ which sends a point $p \in \mathbb{S}^2 \setminus \{(0, 0, 1)\}$ to the point where the xy-plane intersects the line through $(0, 0, 1)$ and p. This shows that $\mathbb{S}^2 \setminus \{(0, 0, 1)\} \cong \mathbb{R}^2 \cong \mathbb{C}$–the sphere minus a point may be identified with the complex plane. Adding $(0, 0, 1)$ turns the complex plane into a compact set.

$\{Y \neq 0\}$, and $\{Z \neq 0\}$ are three affine charts of $\mathbb{P}^2_{\mathbb{C}}$, each identifiable with $\mathbb{C}^2$ (which is what we call "affine space"), which together cover all of $\mathbb{P}^2_{\mathbb{C}}$.

Definition 3.4. A projective curve is *smooth* if at each affine chart, not all partial derivatives of the defining polynomial (which in our case would be f_i) are simultaneously zero at any point.

Proposition 3.5. *An elliptic curve $y^2 = f(x)$ forms a smooth projective curve.*

I will leave this to you as an exercise. You can either check every affine chart, or you can verify that for our homogeneous equation $F(X, Y, Z) = 0$ that $\frac{\partial F}{\partial x} = \frac{\partial F}{\partial y} = \frac{\partial F}{\partial z} = 0$ implies that $X = Y = Z = 0$, which is impossible for projective space.

In particular, this smooth projective curve, when defined over the complex numbers, forms a torus. This algebraic set also turns out to possess a group law which is compatible with the complex geometric structure, which makes it a *Lie group.*

3.3 Back to Weierstrass

We return to the Weierstrass cubic associated to the lattice Λ, defined

$$Y^2 = 4X^3 - g_2X - g_3 \qquad 60G_4 = g_2 \qquad 140G_6 = g_3$$

We call this curve E and its projective competion $\bar{E}$. Now, we define a map $f : \mathbb{C}/\Lambda \to \mathbb{P}^2_{\mathbb{C}}$ by

$$f(z) = \begin{cases} [\varrho(z) : \varrho'(z) : 1] & z \notin \Lambda \\ [0 : 1 : 0] & z \in \Lambda \end{cases}$$

Proposition 3.6. *The map $f : \mathbb{C}/\Lambda \to \mathbb{P}^2_{\mathbb{C}}$ is a bijective holomorphism onto $\bar{E}$.*

Remark. Notice that the lattice points are sent to points at infinity. In broad strokes, this map depicts the formation of a torus in $\mathbb{P}^2_{\mathbb{C}}$ by gluing together the lattice points together at one of the "points at infinity" in $\mathbb{P}^2_{\mathbb{C}}$. $\mathbb{C}/\Lambda$ itself forms a torus; opposite edges of the fundamental parallelogram get identified to glue the torus together. The geometry of an elliptic curve is determined by its lattice Λ. The construction of this lattice follows from the Riemann bilinear relations. All of this material may be covered in a course about Riemann surfaces.

Proof. Recall that we use Π to denote the fundamental parallelogram. Since we know that lattice points get mapped to a point at infinity, we just need to show that $\Pi \setminus \{0\}$ gets mapped bijectively onto E.

To show injectivity, suppose $f(z_1) = f(z_2)$.

If $\varrho'(z_1) \neq 0$, we have by definition of f and scaling that $\varrho(z_1) = \varrho(z_2)$, which implies $z_2 \in \{z_1, \sigma(z_1)\}$ where $\sigma(z) = -z \mod \Lambda$ is the representative that lies inside Π. $\varrho'(z_1) = \varrho'(z_2)$ then gives us $z_1 = z_2$ because ϱ' is an odd function.

If $\varrho'(z_1) = 0$, then from section 2.5 of the [1] (i.e. the previous lecture), z_1 can be one of three possible values: $\frac{\omega_1}{2}, \frac{\omega_2}{2}, \frac{\omega_1+\omega_2}{2}$, but these are all distinct mod Λ, so we will also conclude $z_1 = z_2$.

Again, because we know that the preimage of $[0 : 1 : 0]$ is $0 \mod \Lambda$, it suffices to show surjectivity of f restricted to $\Pi \setminus \{0\}$ onto E.

If we consider $(a, b) \in E$, which satisfy $b^2 = 4a^3 - g_2a - g_3$, we notice that since ϱ has order 2, we can find z_0 so that $\varrho(z_0) = a$ and $\varrho(\sigma(z_0)) = a$. Since we are considering points in E, we observe that $\varrho'(z_0)^2 = \varrho'(\sigma(z_0))^2 = b^2$. Since ϱ' is odd, $\varrho'(\sigma(z_0)) = -\varrho'(z_0)$, so we see that we have either $f(z_1) = [a : b : 1]$ or $f(\sigma(z_1)) = [a : b : 1]$ (and by injectivity, only one of these will be the solution).

$z \mapsto f(z)$ is holomorphic on $\mathbb{C} \setminus \Lambda$, so we need to prove that it is holomorphic on our lattice points. In order to do this, we need to shift our affine chart so that the "points at infinity" are somewhere else. In other words, our projective equation for $\bar{E}$ is given by

$$Y^2Z = 4X^3 - g_2XZ^2 - g_3Z^3$$

We need to consider $\{Y \neq 0\}$, which is where $[0:1:0]$ is located, so we dehomogenize to the affine chart given by

$$Z = 4X^3 - g_2 X Z^2 - g_3 Z^3$$

In dehomogenizing, we scaled by the Y-coordinate, which means that our function can be re-written as

$$f(z) = \left[\frac{\wp(z)}{\wp'(z)} : 1 : \frac{1}{\wp' z}\right]$$

In the language of affine space, we write

$$f(z) = \left(\frac{\wp(z)}{\wp'(z)}, \frac{1}{\wp' z}\right)$$

0 is not a zero of $\wp'$ in Π, so this function is holomorphic at lattice points as well. □

$\mathbb{C}/\Lambda$ has a group; correspondingly, $\bar{E}$ has a group structure, and its identity is given by $[0:1:0]$, which is predictable given our bijection f. We can define a group law $+$ on $\bar{E}$: let P and Q be two points in $\bar{E}$ and u, v be such that $f(u) = P$ and $f(v) = Q$; we define $P + Q := f(u + v)$.

A consequence of the relation between $\mathbb{C}/\Lambda$ and $\bar{E}$ is the following theorem by Abel:

Theorem 3.7. *Three points P, Q, R of $\bar{E}$ are collinear if and only if $P + Q + R = O$.*

Remark. Three points are collinear if they lie on the same line $aX + bY + cZ$. The proof of this can be found in the [1] on page 90.

4 Loxodromic Functions

There are multiple constructions of elliptic functions. We learned about Weierstrass-$\wp$ functions. Other approaches include the theory of θ functions (developed by Jacobi) and loxodromic functions. Here, we will discuss loxodromic functions.

Definition 4.1. A meromorphic map $f : \mathbb{C}^* \to \mathbb{C} \cup \{\infty\}$ is a *loxodromic function of multiplicator* $q \in \mathbb{C}^*$, where $|q| < 1$ if f satisfies $f(q\zeta) = f(\zeta)$ for all $\zeta \in \mathbb{C}^*$.

The set of loxodromic functions of multiplicator p is a field, denoted $\mathcal{L}_q$.

This is analagous to our previous discussion in the following way: let $g(z) = f\left(e^{2i\pi z/\omega_1}\right)$. This is periodic along $\mathbb{Z}\omega_1$. If we add the condition that $f(q\zeta) = f(\zeta)$ for every $\zeta \in \mathbb{C}^*$ for some $q = e^{2i\pi\omega_2/\omega_1}$ so that ω_2 and ω_2 are linearly independent, we see that g is doubly periodic with respect to the lattice $\mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$.

Instead of considering fundamental parallelograms Π, here we will consider annuli $\overline{C_q(\varepsilon)} = \{z \in \mathbb{C} : |q|\varepsilon \leq |z| \leq \varepsilon\}$. Loxodromic f of multiplicator q is determined by its behavior on $\overline{C_q(\varepsilon)}$. We have the following:

Proposition 4.2. *Let $f \in \mathcal{L}_q$.*

(i) If f is entire, then f is constant.

(ii) If f has neither zeroes nor poles on the boundary of $C_q(\varepsilon) = \{z \in \mathbb{C} : |q|\varepsilon \leq |z| < \varepsilon$, then the sum of the residues of the poles of $f(z)/z$ lying in $C_q(\varepsilon)$ is zero.

(iii) The number of poles and the number of zeroes of f, when counted with multiplicity, are the same.

Like the versions of these propositions for elliptic functions of a lattice, these theorems follow from results in complex analysis discussed in Section 1. We also have the following corollary:

Corollary 4.2.1. *Every non-constant $f \in \mathcal{L}_q$ has at least two poles (and two zeroes) in every annulus $C_q(\varepsilon)$.*

Now, we construct an example of a loxodromic function.

Define $S : \mathbb{C}^* \to \mathbb{C}$ to be

$$S(z) = \prod_{n=0}^{\infty}(1 - q^n z)\prod_{n=1}^{\infty}(1 - q^n z^{-1})$$

This product converges to a holomorphic function. Observe that $S(qz) = S(\frac{1}{Z}) = -z^{-1}S(z)$.

Now, out of this we construct a loxodromic function with poles $b_1, \dots, b_m$ (so $a_i \neq b_i$) with the condition that $\prod_{i=1}^m a_i = \prod_{i=1}^m b_i$.

For $z \neq b_i \mod \langle q \rangle$[3], let

$$M(z) := \frac{\prod_{i=1}^m S(z/a_i)}{\prod_{i=1}^m S(z/b_i)}$$

The class $a_i \mod \langle q \rangle$ contains the zeroes and the class $b_i \mod \langle q \rangle$ contains the poles of M. Furthermore, from our previous observations, we can conclude $M(qz) = M(z)$ and $M\left(\frac{1}{z}\right) = \frac{\prod_{i=1}^m S(a_i z)}{\prod_{i=1}^m S(b_i z)}$. In other words, M is a loxodromic function of multiplicator q, and we were able to specify zeroes and poles to a certain extent.

Theorem 4.3. *Let ϵ be such that $\partial C_q(\varepsilon)$ does not contain zeroes or poles of $f \in \mathcal{L}_q$. Let $a_1, \dots, a_m$ be the zeroes of f and $b_1, \dots, b_m$ be the poles of f in $C_1(\varepsilon)$. We have*

$$\prod_{i=1}^m \frac{a_i}{b_i} \in \langle q \rangle$$

Proof. Let M be as defined before without the condition that $\prod_{i=1}^m a_i = \prod_{i=1}^m b_i$, and let $\prod_{i=1}^m \frac{a_i}{b_i} = \lambda$. So our previous observation becomes $M(qz) = \lambda M(z)$.

Define $g(z) = \frac{f(z)}{M(z)}$. The zeroes of f are the zeroes of M, and the poles of f are the poles of M. These all cancel, which makes g and $\frac{1}{g}$ entire on $\mathbb{C}^*$.

Since holomorphicity is equivalent to analyticity (as a result of complex analysis), we can write out the Laurent expansion of g, which is of the form $\sum_{-\infty}^{\infty} c_n z^n$. Observe that

$$\lambda g(qz) = \frac{\lambda f(qz)}{\lambda M(z)} = \frac{f(z)}{M(z)} = g(z)$$

by our observations about M and because $f \in \mathcal{L}_q$ by assumption. g is nonzero, so $c_n \neq 0$ for some n. Fiddling with the terms in the Laurent expansion of $\lambda g(qz) - g(z) = 0$ gives us

$$(\lambda q^n - 1)c_n = 0$$

We can conclude $\lambda = q^{-n} \in \langle q \rangle$. ☐

From this theorem, it is possible to derive the following result:

Theorem 4.4. *If $\lambda = q^n$, the loxodromic function f of Theorem 4.3 is of the form*

$$f(z) = C\frac{\prod_{i=1}^m S(z/a_i)}{S(q^n z/b_1)\prod_{j=2}^m S(z/b_i)}$$

5 The Function ρ

Here, we study a special loxodromic function which we will call ρ.

Let $\mathcal{M} := \{f : \mathbb{C}^* \to \mathbb{C}^* \text{ meromorphic}\}$. This is a complex vector space. Define $\lambda : \mathcal{M} \to \mathcal{M}$ by

$$f(z) \longmapsto z\frac{f'(z)}{f(z)}$$

[3] $\langle q \rangle$ represents the cyclic subgroup generated by q, i.e. $\{q^i, \ i \in \mathbb{Z}\}$

We observe $\lambda(\mathcal{L}_q) \subseteq \mathcal{L}_q$. If we let $\mathcal{H} = \{h : h(qz) = -z^{-1}h(z)\}$ (taken from our earlier exapmle), then we see that

$$h'(qz) = \frac{1}{q}(z^{-2}h(z) - z^{-1}h'(z))$$

$$\implies [\lambda(h)](qz) = qz\frac{q^{-1}(z^{-2}h(z) - z^{-1}h'(z))}{-z^{-1}h(z)} = \frac{zh'(z)}{h(z)} - 1 = [\lambda(h)](z) - 1$$

In other words, if $\mathcal{S}_q$ are solutions of Abel's equation $v(qz) = v(z) - 1$, then $\lambda(\mathcal{H}) \subseteq \mathcal{S}_q$.

Lemma 5.1. *Define* $\chi(z) := \frac{zS'(z)}{S(z)}$*, where* S *from before is defined as*

$$S(z) = \prod_{n=0}^{\infty}(1 - q^n z)\prod_{n=1}^{\infty}(1 - q^n z^{-1})$$

(i) $\mathcal{S}_q$ *is the affine* $\mathbb{C}$*-space* $\chi + \mathcal{L}_q$.

(ii) $v(z) \mapsto zv'(z)$ *defines a map* $D : \mathcal{S}_q \to \mathcal{L}_q$.

Expanding by product rule gives us

$$\chi(z) = \sum_{n=1}^{\infty}\frac{q^n z^{-1}}{1 - q^n z^{-1}} - \sum_{n=0}^{\infty}\frac{q^n z}{1 - q^n z}$$

Define

$$\rho = -D(\chi) = \sum_{n=-\infty}^{\infty}\frac{q^n z}{(1 - q^n z)^2}$$

Using the observation that $S\left(\frac{1}{z}\right) = z^{-1}S(z)$, we can also derive

$$\chi(z) + \chi\left(\frac{1}{z}\right) = 1$$

If we apply D to both sides above, we get that

$$\rho(z) = \rho\left(\frac{1}{z}\right)$$

As before, we want to find the Laurent Expansion of ρ in the annulus $\Gamma = \{z \in \mathbb{C} : |q| < |z| < |q^{-1}|\}$; the eventual goal will be to express ρ as a solution to an ordinary differential equation.

We observe the following:

$$n > 0 \implies \frac{q^n z}{(1 - q^n z)^2} = q^n z + \ldots + mq^{nm}z^m + \ldots$$

$$n < 0 \implies \frac{q^n z}{(1 - q^n z)^2} = q^n\left(\frac{1}{z}\right) + \ldots + mq^{nm}\left(\frac{1}{z}\right)^m + \ldots$$

The above are convergent series, so we have

$$\rho(z) - \frac{z}{(1 - z)^2} = \sum_{m=1}^{\infty}\frac{mq^m}{1 - q^m}\left(z^m + \frac{1}{z^m}\right)$$

The above is invariant under the transformation $z \to \frac{1}{z}$. A fixed point of this symmetry is 1; observing the Laurent expansion of $\rho(z) - \frac{z}{(1-z)^2}$ in a neighborhood of 1, letting $z = 1 + \zeta$, we have

$$\rho(z) - \frac{z}{(z - 1)^2} = 2\sum_{m=1}^{\infty}\frac{mq^m}{1 - q^m} + \sum_{n=2}^{\infty}\gamma_n\zeta^n \qquad \gamma_2 = \sum_{m=1}^{\infty}\frac{m^3 q^m}{1 - q^m}$$

Now, if we let $\rho_1(z) = \rho(z) + c$ for some constant c, the above expression becomes

$$\rho(1+\zeta) = \zeta^{-1} + \zeta^{-1} + (c + \gamma_0) + \sum_{n=2}^{\infty} \gamma_n \zeta^n \qquad \gamma_0 = 2\sum_{m=1}^{\infty} \frac{mq^m}{1-q^m}$$

Applying the operator D to the equation above gives

$$(D\rho_1)(1+\zeta_= - 2\zeta^{-1} - 3\zeta^{-2} - \zeta^{-1} + \sum_{i=1}^{\infty} n\gamma_n(\zeta+1)\zeta^{n-1}$$

Remember from before that we had

$$\varrho'^2 = 4\varrho^3 + a\varrho^2 + b\varrho + c$$

Analagously, we find that

$$[D\rho_1]^2 - 4\rho_1^3 = A\zeta^{-4} + B\zeta^{-3} + C\zeta^{-2} + D\zeta^{-1} + \dots$$

$$A = 1 - 12(c+\gamma_0) \quad B = 2A \quad C = 1 - 20\gamma_2 - 12(c+\gamma_0) - 12(c+\gamma_0)^2$$

If we add the condition $A = 0$, we get $B = 0$ and $C = -\frac{1}{12} - 20\gamma_2$.

Observe the expression $[D\rho_1]^2 - 4\rho_1^3 - C\rho_1$. There is only one linear term in ρ_1, so the expression has at most one simple pole in Γ. But by Corollary 4.2.1, this implies that $[D\rho_1]^2 - 4\rho_1^3 - C\rho_1$ is constant.

Theorem 5.2. *The meromorphic function given by*

$$\rho_1(z) = \rho(z) + \frac{1}{12} - 2\sum_{m=1}^{\infty} \frac{mq^m}{1-q^m}$$

is loxodromic of multiplicator q and satisfies the differential equation

$$[z\rho_1'(z)]^2 = 4\rho_1^3 - g_4\rho_1 - g_6$$

$$g_4 = \frac{1}{12} + 20\sum_{n=1}^{\infty} \frac{m^3 q^m}{1-q^m} \quad g_6 = \frac{1}{216} - \frac{7}{3}\sum_{n=1}^{\infty} \frac{m^5 q^m}{1-q^m}$$

6 Bringing Things Full Circle

We hinted at the beginning of the previous section that there was some relation between elliptic functions and loxodromic functions.

Recall that if f is loxodromic of multiplicator $q = e^{2\pi i\omega_2/\omega_1}$ with $\mathrm{Im}(\omega_2/\omega_1) > 0$, then $f(e^{-2\pi i u/\omega_1})$ is an elliptic function of lattice $\mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$.

We can go the other way around as well. Let g be an elliptic function of lattice $\Lambda = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$ with $\mathrm{Im}(\omega_2/\omega_1) > 0$. Letting $z \in \mathbb{C}^*$, we define

$$f(z) = g\left(\frac{\omega_1}{2\pi i}\log z\right)$$

Note that if $q = e^{2i\omega_2/\omega_1}$, we get

$$f(qz) = g\left(\frac{\omega_1}{2\pi i}\log(qz)\right) = f(z)$$

We have not lost analyticity while traveling in either direction. So if we write

$$z = e^{2i\pi u/\omega_1} = 1 + \frac{2\pi i u}{\omega_1} - \frac{2\pi^2 u^2}{\omega_1^2} + \dots$$

We can write ζ from before as

$$\zeta = z - 1 = \frac{2\pi i}{\omega_1} u\left(1 + \frac{\pi i u}{\omega_1} - \frac{2\pi^2 u^2}{\omega_1^2} + \dots\right)$$

$$\zeta^2 = (z-1)^2 = -\frac{4\pi^2}{\omega_1^2}u^2\left(1 + \frac{2\pi i u}{\omega_1} - \frac{7\pi^2 u^2}{3\omega_1^2} + \dots\right)$$

Since $\rho_1(\frac{1}{z}) = \rho_1(z)$, the corresponding elliptic function of lattice Λ is even. An even elliptic function will admit a double pole at the origin; the principal part is given by

$$\zeta^{-2} + \zeta^{-1} + \frac{1}{12} = -\frac{\omega_1}{4\pi^2}u^{-2} + o(u)$$

Theorem 6.1. *The loxodromic function $\rho_1 \in \mathcal{L}_q$ corresponds to $-\omega_1/4\pi^2\varrho$, where ϱ denotes the Weierstrass function associated to Λ.*

Additionally, we have some relations between some items we computed in different sections (note that both differential equations give you similar elliptic curves, so we can translate the coefficients):

$$60G_4 = \left(\frac{2\pi i}{\omega_1}\right)^4 g_4 \quad 140G_6 = \left(\frac{2\pi i}{\omega_1}\right)^6 g_6$$

Furthermore, section 2.11 of [1] contains a computation of the discriminant Δ of the elliptic curve $Y^2 = 4X^3 + g_4X + g_6$ associated with ρ_1, which is given by

$$\Delta = q\prod_{n=1}^{\infty}(1-q^n)^{24}$$

It turns out that if Δ_1 is the discriminant of $Y^2 = 4X^3 - 60G_4X - 140G_6$, the elliptic curve associated, we get

$$\Delta_1 = \left(\frac{2\pi i}{\omega_1}\right)^{12}\Delta$$

References

[1] Yves Hellegouarch. *Invitation to the mathematics of Fermat-Wiles.* Elsevier, 2001.

[2] Elias M Stein and Rami Shakarchi. *Complex analysis.* Vol. 2. Princeton University Press, 2010.