Homework 1

Fermat's last theorem seminar

Due Monday, October 14 at 1 PM

Choose *one* of the following exercises and solve it. Write up your solution carefully; it should be not only mathematically correct but also clear and easy to read and follow.

Problem 1. Exercise 1.5 from the textbook: we will show that the equation $x^4 + y^4 = z^2$ has no nontrivial integer solutions (i.e. with $xyz \neq 0$).

(a) Show that we can reduce to the case $\gcd(x, y) = 1$ and y is even, i.e. if we knew that there were no solutions with this property then we'd know there were none at all. (Hint: all squares (and thus fourth powers) are congruent to 0 or 1 modulo 4).

(b) Assuming there exists a nontrivial integer solution (x, y, z) with $\gcd(x, y) = 1$ and y even, show that there exist integers a and b such that

$$x^2 = a^2 - b^2, \qquad y^2 = 2ab.$$

(c) Deduce that there exist integers c and d such that

$$y^2 = 4cd(c^2 + d^2)$$

with $\gcd(c, d) = 1$ and at least one of c and d divisible by 2.

(d) Deduce that there exist integers u, v, w such that

$$c = u^2, \qquad d = v^2, \qquad c^2 + d^2 = w^2.$$

(e) Conclude by infinite descent that $x^4 + y^4 = z^2$ can have no nontrivial integer solutions.

Problem 2. Exercise 2.3 from the textbook: using the results of §2.6, show that

$$1 + \sum_{k\geq 2} \frac{(2k-1)(2k-2)(2k-3)}{6} G_{2k} z^{2k} = \left(1 + \sum_{k\geq 2} (2k-1) G_{2k} z^{2k}\right)^2 - 5G_4 z^4.$$

Deduce that

$$G_8 = \frac{3}{7} G_4^2, \qquad G_{10} = \frac{5}{11} G_4 G_6.$$

Problem 3. The Eisenstein series G_{2k} depend on a lattice Λ in $\mathbb{C}$, and are given by

$$G_{2k}(\Lambda) = G_{2k}(\mathbb{Z} \cdot 1 + \mathbb{Z} \cdot \tau) = \sum_{\omega \in \Lambda \setminus \{0\}} \frac{1}{\omega^{2k}}.$$

If we take the lattice $\mathbb{Z} \cdot 1 + \mathbb{Z} \cdot \tau$ for some complex number τ with imaginary part greater than 0, we can view G_{2k} as a function of the complex variable τ:

$$G_{2k}(\tau) = \sum_{(m,n)\in\mathbb{Z}^2\setminus\{(0,0)\}} \frac{1}{(m+n\tau)^{2k}}.$$

These are modular forms of weight $2k$, which for our purposes means that they satisfy the relations

$$G_{2k}(\tau+1) = G_{2k}(\tau)$$

and

$$G_{2k}(-1/\tau) = \tau^{2k} G_{2k}(\tau).$$

Justify these relations, using as needed the transformation law for Eisenstein series of lattices

$$G_{2k}(\alpha\Lambda) = \alpha^{-2k} G_{2k}(\Lambda).$$