THETA FUNCTIONS

$\boxed{1}$ EULER

Theorem (Euler's)

In the ring of formal series $\mathbb{Z}[[x,z]]$, have the identities

1) $$\prod_{m=1}^{\infty}(1+x^m z) = \sum_{n=0}^{\infty} \frac{x^{\frac{n(n+1)}{2}}}{(1-x)\cdots(1-x^n)} z^n$$ and

2) $$\prod_{m=1}^{\infty}(1-x^m z)^{-1} = \sum_{n=0}^{\infty} \frac{x^n}{(1-x)\cdots(1-x^n)} z^n$$

Proof

(1) Define $F(x,z) := LHS$.

$\Rightarrow F(x,xz) = \dfrac{F(x,z)}{1+xz}$.

Define c_n s.t. $F(x,z) = \sum_{n=0}^{\infty} c_n(x) z^n$.

$\Rightarrow c_0(x) = 1$, and if $n \geq 1$ then $(1-x^n)c_n(x) = x^n c_{n-1}(x)$

$\Rightarrow F(x,z) = \sum_{n=0}^{\infty} \dfrac{x^{1+2+\cdots+n}}{(1-x)\cdots(1-x^n)} z^n$

$\Rightarrow LHS = RHS$.

(2) Define $G(x,z) := LHS$.

$\Rightarrow G(x,xz) = (1-xz)G(x,z)$

Defines d_n s.t. $G(x,z) = \sum_{n=0}^{\infty} d_n(x) z^n$

$\Rightarrow d_0(x) = 1$, and if $n \geq 1$ then $(1-x^n)d_n(x) = x d_{n-1}(x)$

$\Rightarrow G(x,z) = \sum_{n=0}^{\infty} \dfrac{\overbrace{x \cdot x \cdot \cdots \cdot x}^{n}}{(1-x)\cdots(1-x^n)} z^n$

$\Rightarrow LHS = RHS$.

Exercise

Check that the partial products and sums in (1) and (2) are Cauchy sequences.

Corollary

(3) "$p(n)$ into parts $\le m$" is "$p\left(n+\frac{m(m+1)}{2}\right)$ without repetitions"

(4) "$p(n)$ into m parts $\le m$" is "$p(n+m)$ into m parts"

(5) "$p(n)$ into m parts" is "$p\left(n+\frac{m(m-1)}{2}\right)$ into m distinct parts".

Proof

(3) Consider the coefficient N of the term $Nx^e z^m$ in (1). If $e = n+\frac{m(m+1)}{2}$ then N is the coefficient of x^n in the expansion of $\frac{1}{(1-x)\dots(1-x^m)}$. This is by definition the generating function of $p(n)$.

(4) Likewise, if $e=n+m$, N is the coefficient of $\frac{x^m z^m}{(1-x)\cdots(1-x^m)}$, by (2). This makes N the coefficient of x^n in the expansion of $\frac{1}{(1-x)\dots(1-x^m)}$ again, the generating function of $p(n)$.

(5) Exercise, show it follows from corollaries 3 and 4

Define

$P(x) = \sum_{n=0}^{\infty} p(n)x^n = \prod_{m=1}^{\infty} \frac{1}{(1-x^m)}$, referring to lowercase $p(n)$ still as the partition function.

Define

Ramanujan's congruences: $0 \equiv p(5n+4) \bmod 5 \equiv p(7n+5) \bmod 7$
$\equiv p(11n+5) \bmod 11$.

Theorem (6)

Jacobi's Theorem: $\left(\sum_{-\infty}^{\infty} z^{(n^2)}\right)^4 = 1 + 8\sum_{m=1}^{\infty}\Bigg(\sum_{\substack{d \mid m \\ 4 \nmid d}} d\Bigg) z^m$. Consequently, the number of compositions of n into a sum of four squares equals $\begin{Bmatrix} 8 \times \text{sum of divisors of } n \text{ if } n \text{ odd} \\ 24 \times \text{sum of divisors of } n \text{ if } n \text{ even} \end{Bmatrix}$.

Proof (of the combinatoric formulation)

Define $A_4(n)$ is the number of compositions of n as a sum of four squares. Then $LHS = \sum_{n \in \mathbb{Z}} z^{n_1^2+n_2^2+n_3^2+n_4^2} = \sum_{n \in \mathbb{N}} A_4(n) z^n$. Restricting to $n>0$ gives $A_4(n) = 8\sum_{\substack{d \mid n \\ 4 \nmid d}} d$ by (6). Odd n proven. If n even, $\sum_{\substack{d \mid n \\ 4 \nmid d}} d = \sum_{\substack{d \mid n \\ d \text{ odd}}} d + 2\sum_{\substack{d \mid n \\ d \text{ odd}}} d = 3\sum_{\substack{d \mid n \\ d \text{ odd}}} d$ ✓. It remains only to prove (6), which I will spend the rest of the class doing.

[2] THE THETA FUNCTIONS

~~Define~~ Recall

From Shiv's lecture, doubly periodic entire functions are constant for a period lattice Λ, so non-constant entire functions require a weaker notion of periodicity. Call this semi-periodicity.

Define (1)

Θ being semi-periodic means $\begin{cases} \Theta(z+1) = \Theta(z) \\ \Theta(z+\tau) = F(z)\,\Theta(z) \end{cases}$ for some factor $F(z)$.

Note

If Θ is not the zero function then require $F(z+1)=F(z)$, so by convention take $F(z) = \frac{1}{ce^{2\pi i z}}$, $c \in \mathbb{C}$. Furthermore, can write the Fourier series $\Theta(z) = \sum_{-\infty}^{\infty} a_n e^{2\pi i n z}$.

Corollary

Applying the second condition of (1) gives

$$F(z)\sum_{-\infty}^{\infty} a_n e^{2\pi i n z} = \sum_{-\infty}^{\infty} a_n e^{2\pi i n (z+\tau)}, \text{ so } a_{n+1} = ce^{2\pi i n \tau} a_n$$

Define

$q = e^{i\pi\tau}$, $\forall \tau$ s.t. $0<|q|<1$.

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Corollary (2)

$$\Theta(z) = a_0 \sum_{-\infty}^{\infty} c^n e^{i\pi n(n-1)\tau + 2\pi i n z}$$

$$\Theta(z) = a_0 \sum_{-\infty}^{\infty} c^n q^{n(n-1)} e^{2\pi i n z}.$$

Define (3)

$$\Theta_3(z) = \sum_{-\infty}^{\infty} q^{(n^2)} e^{2\pi i n z} = 1 + 2 \sum_{n \in \mathbb{Z}_+} q^{(n^2)} \cos(2\pi n z)$$

obtained from (4) with $(a_0, c) = (1, q)$

$$\Theta_4(z) = \sum_{-\infty}^{\infty} (-1)^n q^{(n^2)} e^{2\pi i n z} = 1 + 2 \sum_{n \in \mathbb{Z}_+} (-1)^n q^{(n^2)} \cos(2\pi n z) = \Theta_3\left(z + \tfrac{1}{2}\right)$$

obtained from (4) with $(a_0, c) = (1, -q)$

$$\Theta_1(z) := -i q^{\frac{1}{4}} e^{\pi i z} \Theta_3\left(z + \frac{1+\tau}{2}\right)$$

$$\Theta_2(z) := q^{\frac{1}{4}} e^{\pi i z} \Theta_3\left(z + \frac{\tau}{2}\right)$$

Corollary (4)

Can summarize (3) in a table of form

	x_1	x_2
f_1	$f_1(x_1)$	$f_1(x_2)$
f_2	$f_2(x_1)$	$f_2(x_2)$

:

	$z + \frac{1}{2}$	$z + \frac{\tau}{2}$	$z + \frac{1+\tau}{2}$
Θ_1	$\Theta_2(z)$	$i\lambda\Theta_4(z)$	$\lambda\Theta_3(z)$
Θ_2	$-\Theta_1(z)$	$\lambda\Theta_3(z)$	$-i\lambda\Theta_4(z)$
Θ_3	$\Theta_4(z)$	$\lambda\Theta_2(z)$	$i\lambda\Theta_1(z)$
Θ_4	$\Theta_3(z)$	$i\lambda\Theta_1(z)$	$\lambda\Theta_2(z)$

, $\lambda := q^{-\frac{1}{4}} e^{-\pi i z}$

Prop

For two arbitrary complex numbers x and y, can prove

$$\theta_1(x,q)\theta_1(y,q) = \theta_3(x+y,q^2)\theta_2(x-y,q^2) - \theta_2(x+y,q^2)\theta_3(x-y,q^2),$$ where

$\theta_i(z,q) = \theta_i(z)$ as before

$\theta_i(z,q^2) = \theta_i(z)$ with q replaced by q^2. Proof takes too long to write but it follows immediately from plugging θ_i into (3).

Corollary (5)

By translating x,y according to (4), have

$$\theta_4(x,q)\theta_4(y,q) = \theta_3(x+y,q^2)\theta_3(x-y,q^2) - \theta_2(x+y,q^2)\theta_2(x-y,q^2)$$

and

$$\theta_1(x,q)\theta_1(y,q) = \theta_3(x+y,q^2)\theta_2(x-y,q^2) - \theta_2(x+y,q^2)\theta_3(x-y,q^2)$$

therefore, as difference of their squares,

$$\theta_4(x+y)\theta_4(x-y)\theta_4^2(0) = \theta_4^2(x)\theta_4^2(y) - \theta_1^2(x)\theta_1^2(y).$$

By translating x,y according to (6) on this now, get four more identities relating $\theta_i(x+y)\theta_i(x-y)\theta_4^2(0)$ to differences of squares of other $\theta_j(x)$, $\theta_j(y)$, $\forall i = 1,2,3,4$. Most importantly, have the Jacobi relation $\theta_3^4(0) = \theta_2^4(0) + \theta_4^4(0)$.

~~Plugging~~

Note

Expanding the Jacobi relation with (3) gives

$$\left(1+2\sum_{n\geq 1} q^{(n^2)}\right)^4=\left(2q^{\frac{1}{4}}\sum_{-\infty}^{\infty} q^{n^2+n}\right)^4+\left(1+2\sum_{n\in\mathbb{Z}_+}(-1)^n q^{(n^2)}\right)^4.$$

~~Setting~~ Setting $q=z^4$ and defining $X(z)=\Theta_2(0)$, $Y(z)=\Theta_4(0)$, $Z(z)=\Theta_3(0)$ gives the relation $X^4+Y^4=Z^4$.

Corollary

Three corollaries of (5) were

$$\begin{cases}\Theta_1'(0,q)\Theta_2(0,q)=2\Theta_1'(0,q^2)\Theta_4(0,q^2)\\ \Theta_2^2(0,q)=2\Theta_2(0,q^2)\Theta_3(0,q^2)\\ \Theta_3(0,q)\Theta_4(0,q)=\Theta_4^2(0,q^2)\end{cases},$$

the first technically being obtained by first differentiating Θ wrt x.

Corollary (6)

By substitution, since q^2 can be recursively iterated, induce

$$\frac{\Theta_1'(0,q)}{\Theta_2(0,q)\Theta_3(0,q)\Theta_4(0,q)}=\frac{\Theta_1'(0,q^{(2^n)})}{\Theta_2(0,q^{(2^n)})\Theta_3(0,q^{(2^n)})\Theta_4(0,q^{(2^n)})}.$$

Taking limit as $n\to\infty$ makes $|q|\to 0$ and

$\Theta_1'(0)\sim 2\pi q^{\frac{1}{4}}$, $\Theta_2(0)\sim 2q^{\frac{1}{4}}$, $\Theta_3(0)\sim 1$, $\Theta_4(0)\sim 1$. Plugging this into the equality above gives $\Theta_1'(0)=\pi\Theta_2(0)\Theta_3(0)\Theta_4(0)$.

Define

$$\Phi(\xi,q):=\prod_{n=1}^{\infty}(1+q^{2n-1}\xi)(1+q^{2n-1}\xi^{-1})$$

Corollary

Φ satisfies (1) with $\Phi(\xi^{-1}) = \Phi(\xi)$, $\Phi(q^2\xi) = \frac{\Phi(\xi)}{q\xi}$, $q = e^{i\pi\tau}$, $\xi = e^{2\pi i z}$, $F(z) = \frac{1}{qe^{2\pi i z}}$.

Corollary (7)

It follows from (2) that $\Theta = a_0\Theta_3$, so by translating z get $a_0\Theta_1(z) = -iq^{\frac{1}{4}}e^{\pi i z}\Phi(-qe^{2\pi i z})$,

$a_0\Theta_2(z) = q^{\frac{1}{4}}e^{\pi i z}\Phi(qe^{2\pi i z})$,

$a_0\Theta_3(z) = \Phi(e^{2\pi i z})$,

$a_0\Theta_4(z) = \Phi(-e^{2\pi i z})$.

Corollary

By (6), we can solve for a_0 and get $a_0 = \frac{1}{\prod_{n\geq 1}^{\infty}(1-q^{2n})}$.

Corollary

Plugging this back into the Fourier series at start of lecture implies all zeroes of the Θ_i are simple in $\mathbb{C}$ and the zeroes of $\Theta_1, \Theta_2, \Theta_3, \Theta_4 \equiv 0, \frac{1}{2}, \frac{1+\tau}{2}, \frac{\tau}{2} \mod \mathbb{Z}+\mathbb{Z}\tau$.

Define (8)

The heat equation: $\frac{\partial^2\Theta}{\partial z^2} = 4\pi i\frac{\partial\Theta}{\partial\tau}$

Corollary

It follows from (5) that

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$$\frac{\Theta_3^2(x)}{\Theta_1^2(x)}\Theta_2^2(0) - \frac{\Theta_2^2(x)}{\Theta_1^2(x)}\Theta_3^2(0) = \Theta_4^2(0)$$

$$\frac{\Theta_4^2(x)}{\Theta_1^2(x)}\Theta_2^2(0) - \frac{\Theta_2^2(x)}{\Theta_1^2(x)}\Theta_4^2(0) = \Theta_3^2(0)$$

$$\frac{\Theta_4^2(x)}{\Theta_1^2(x)}\Theta_3^2(0) - \frac{\Theta_3^2(x)}{\Theta_1^2(x)}\Theta_4^2(0) = \Theta_2^2(0).$$

Corollary

By (6), have
$$\begin{cases} \pi^2\Theta_3^4(0) = \frac{\Theta_4''}{\Theta_4} - \frac{\Theta_2''}{\Theta_2} \\ \pi^2\Theta_2^4(0) = \frac{\Theta_4''}{\Theta_4} - \frac{\Theta_3''}{\Theta_3} \\ \pi^2\Theta_4^4(0) = \frac{\Theta_3''}{\Theta_3} - \frac{\Theta_2''}{\Theta_2} \end{cases}$$

Corollary (9)

By (8), and since $\frac{\partial}{\partial \tau} = \frac{\partial}{\partial q}\frac{dq}{d\tau} = \pi i q \frac{\partial}{\partial q}$,

$$\Theta_3^4(0) = -4q\frac{\partial}{\partial q}\log\left(\frac{\Theta_4}{\Theta_2}(0)\right)$$

$$\Theta_2^4(0) = -4q\frac{\partial}{\partial q}\log\left(\frac{\Theta_4}{\Theta_3}(0)\right)$$

$$\Theta_4^4(0) = -4q\frac{\partial}{\partial q}\log\left(\frac{\Theta_3}{\Theta_2}(0)\right)$$

Corollary

By (7), apply the infinite product representation to (9),

getting $\Theta_3^4 = \left(\sum_{-\infty}^{\infty} q^{(n^2)}\right)^4 = 4q\left(\frac{1}{4q} - 8\sum_{m=1}^{\infty}\frac{mq^{4m-1}}{1-q^{4m}} + 2\sum_{m=1}^{\infty}\frac{mq^{m-1}}{1-q^m}\right)$,

proving Jacobi's Theorem!!

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Corollary [illegible]

For a constant C by Liouville's theorem (recall Luca's lecture, it applies since the zeroes of $\Theta_1(z,\tau)$ and $\Theta_1(\frac{z}{\tau},-\frac{1}{\tau})$ are simple) have

$$\Theta_1\left(\frac{z}{\tau},-\frac{1}{\tau}\right)=Ce^{\frac{i\pi z^2}{\tau}}\Theta_1(z,\tau)$$
$$\Theta_2\left(\frac{z}{\tau},-\frac{1}{\tau}\right)=iCe^{\frac{\pi i z^2}{2}}\Theta_4(z,\tau)$$
$$\Theta_3\left(\frac{z}{\tau},-\frac{1}{\tau}\right)=iCe^{\frac{\pi i z^2}{2}}\Theta_3(z,\tau)$$
$$\Theta_4\left(\frac{z}{\tau},-\frac{1}{\tau}\right)=iCe^{\frac{\pi i z^2}{2}}\Theta_2(z,\tau).$$

Corollary [illegible]

By (6), $C^2=i\tau$, so use $\sqrt{\frac{\tau}{i}}$ to denote the positive (on $i\mathbb{R}_+$) root of C.

Def [illegible] [illegible]

The Jacobi formulae:

$$\Theta_1\left(\frac{z}{\tau},-\frac{1}{\tau}\right)=-i\sqrt{\frac{\tau}{i}}e^{\frac{\pi i z^2}{\tau}}\Theta_1(z,\tau)$$
$$\Theta_2\left(\frac{z}{\tau},-\frac{1}{\tau}\right)=\sqrt{\frac{\tau}{i}}e^{\frac{\pi i z^2}{\tau}}\Theta_4(z,\tau)$$
$$\Theta_3\left(\frac{z}{\tau},-\frac{1}{\tau}\right)=\sqrt{\frac{\tau}{i}}e^{\frac{\pi i z^2}{\tau}}\Theta_3(z,\tau)$$
$$\Theta_4\left(\frac{z}{\tau},-\frac{1}{\tau}\right)=\sqrt{\frac{\tau}{i}}e^{\frac{\pi i z^2}{\tau}}\Theta_2(z,\tau)$$

Def [illegible]

$$\varphi(\tau)=\Theta_2^8(0,\tau)\Theta_3^8(0,\tau)\Theta_4^8(0,\tau)$$

Theorem

$$\varphi(\tau) = 2^8 q^2 \prod_{n=1}^{\infty}(1-q^{2n})^{24} = \varphi(\tau+1) = \frac{1}{\tau^{12}}\varphi\left(-\frac{1}{\tau}\right)$$

Proof

By (6), $\varphi(\tau) = \left(\frac{\theta_1'(0)}{\pi}\right)^8$.

By (9), $\frac{\theta_1'(0)}{\pi} = 2q^{\frac{1}{4}}\prod_{n=1}^{\infty}(1-q^{2n})^3$, giving the first equality.

First equality implies second equality. Evaluating (6) at $z=0$ gives the third equality.

QED