# Final project guidelines

Fermat's last theorem seminar

## 1. Timeline and requirements

The goal of the project is to 1) investigate some problem using the mathematical concepts we've studied in this class and 2) write an expository paper on the topic, i.e. explain it in detail to an audience unfamiliar with it.

This is a fairly open-ended project; you may use any resources you like, and the converse of this is that it is your job to find (and properly cite) references to understand your desired topic. That said, I am happy to help you find resources if you are having difficulty, especially on more obscure topics. (You don't have to notify me about your proposed topic, but if I don't hear from you I'll assume you are on top of finding sources etc.)

### Timeline

The deadlines are as follows:

**December 2** First draft—optional but recommended; I'll give feedback within a few days so that you have a chance to edit.

**May 1** Final deadline: submit your complete, edited project.

### Guidelines

The primary goal of this project is to understand the mathematics of your topic; nearly as important however is clearly communicating that understanding. Imagine that you are trying to explain the concepts you have studied to someone who has a similar amount of background to you, but has not necessarily studied these particular topics.

Like any paper, in addition to the main body of the exposition your paper should include a short introduction, explaining the main ideas, motivation, and background of your paper, as well as a list of sources. (The specific formatting of your sources does not matter so long as it is clear.)

All papers should be typed.[1] I encourage you to use LaTeX[2], and am happy to hold workshops on it if there is interest; however it is not required, and you may use whatever software you prefer.

There is no hard guideline for the length of your papers: they should be the length they need to be in order to concisely and clearly explain your topic in detail to the appropriate audience, but in practice probably papers should be at least around three pages (possibly less, but only if they are very well-written).

[1] If this is a particular hardship for you, we can discuss alternatives.

[2] LaTeX is a software system for creating documents, especially those involving large numbers of mathematical or scientific symbols, and is probably what virtually all mathematical documents you have encountered at least in college were written in, including this one; there are many editors available, including online ones such as `overleaf.com`.

Grading

Papers will be graded for:

- mathematical correctness;
- clarity and quality of exposition;
- depth, style, creativity.

Like other assignments in this class, they will receive a single overall grade of either E (excellent), S (satisfactory), or N (not yet satisfactory):

- a paper which is clear, mathematically correct, and interesting and of appropriate depth will receive a mark of E;
- a paper which is largely correct and readable but has meaningful errors, is sometimes unclear, or has insufficient scope will receive a mark of S;
- a paper which has essential errors, is not readable, or is meaningfully incomplete will receive a mark of N.

As you'll have an opportunity to receive feedback before turning in the final draft, I expect everyone to be able to meet the high standard for an E.

## 2. Topic suggestions

Any of these should be taken as a collection of related possible ideas around which to base your project; you do not necessarily need to cover everything mentioned, and might cover aspects not mentioned.

**More questions about elliptic curves.** Elliptic curves are very interesting objects, for many reasons beyond modularity and Fermat's last theorem. For example, they have applications to cryptography; they are the subject of the Birch–Swinnerton-Dyer conjecture and the Sato–Tate conjecture (the latter is now a theorem); they are the simplest examples of abelian varieties. Investigate one of these aspects, or another, in greater depth. Various computational projects are also possible; for example, we know that the Mordell–Weil group of an elliptic curve is finitely generated, meaning it is isomorphic to $^r \times T$ for some finite abelian group $T$ and a nonnegative integer $r$, the rank of the elliptic curve. Most curves have rank 0 or 1, but it is known that there exist elliptic curves of rank at least 29; an open question is whether there exist elliptic curves of arbitrarily high rank. You could look into methods for finding high-rank elliptic curves, or try to numerically study elliptic curves (or special types) and make empirical observations about e.g. their average rank, distribution, etc.

**More about modular forms.** There is also a lot more to say about modular forms. For example, they can be viewed as differential forms on modular curves, or more generally sections of line bundles; functions on adelic quotients, as automorphic forms for $\mathrm{SL}_2$; vectors in automorphic representations; etc. There are also more types of modular forms that we have not discussed, e.g. Maass or Bianchi forms, which are more analytic objects, as well as generalizations such as automorphic forms, e.g. Siegel modular forms, and more. Look into one of these areas, or other related topics, and write a short expository paper about it.

**The proof of the modularity theorem.** It is not reasonably possible to explain the proof of the modularity theorem in a paper of this size, but what you can do is try to understand and explain some of the ideas that go into it beyond what we've talked about in this class. Some relevant keywords include modularity lifting, the 3-5 trick, R = T, .... Note that there are many complicated ideas here; though this is very interesting, be cautious not to get in over your head.

**The Langlands program.** As we've briefly mentioned a few times, the modularity theorem is part of a more general theory: the Langlands program, which links automorphic objects (of which modular forms are an example) to "Galois" or "spectral" objects (such as Galois representations associated to elliptic curves). Give a brief introduction to the Langlands program and how it generalizes this picture. (There are many possible things to say here; even for a general overview, you will have to pick a direction to focus on.)

**Other Diophantine equations.** Wiles's method is fairly special to the Fermat equation, but similar methods can be used for some other Diophantine equations as well. Work out another example; this will probably necessitate assuming some technical facts, but try to at least make sure everything you say is true even if you can't include proofs in your paper.

**Choose your own.** Propose your own topic! It should be related to the material from this class, so elliptic curves, modular forms, Diophantine equations, Galois representations, etc. Otherwise you are free to choose any topic that interests you, using the above as a guide.