# Lecture 3: introduction to limits

Calculus I, section 10
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## 1. Introduction to limits

Now that we've finished our lightning review of precalculus and functions, it's time for our first really calculus-based notion: the limit. This is really a very intuitive concept, but it's also kind of miraculous and lets us do some very powerful things.

I'll eventually write down a formal definition for a limit, but it's not really important and I won't ask you to use it: the important thing is that you understand the fundamental idea. Let's start with examples.

The first example is a little bit silly: consider the expression $\frac{x}{x}$, where $x$ is a real number. What can we say about this?

Well, for almost any $x$ we pick, this is just 1: for example $\frac{3}{3} = 1$, or $\frac{-2}{-2} = 1$. Why almost? Because if $x = 0$, this is $\frac{0}{0}$ which is not defined.

Even so, it seems reasonable to say $\frac{x}{x} = 1$ in general, even though technically this doesn't make sense at $x = 0$. One justification for this is that if we take $x$ very close to 0, the ratio doesn't change: it's always 1. We formalize this by saying that as we approach 0 (from either the left or the right), $\frac{x}{x} = 1$, i.e.

$$\lim_{x \to 0} \frac{x}{x} = 1.$$

This does *not* mean that we can say $\frac{0}{0} = 1$; it is still undefined, and we'll see examples where if we took that convention we would get something that didn't make sense. However, in this context it suggests that 1 is the right value to assign to $\frac{x}{x}$ at $x = 0$, instead of the "naive" undefined value $\frac{0}{0}$.

Another similar example is $\frac{x^2}{x}$. For nonzero $x$, we can similarly simplify this to $x$; again at $x = 0$, this doesn't make sense and we again get $\frac{0}{0}$. We could look at $x$ very close to zero, in which case we again get $x$; this is no longer constant, but since we assume that $x$ is very close to zero it follows that so is $\frac{x^2}{x}$, so in this case we can say

$$\lim_{x \to 0} \frac{x^2}{x} = 0.$$

The general principle of what's going on here is that we have some expression $f(x)$ and some fixed number $a$ (in the above examples, $a = 0$, but this does not have to be true). If $f(x)$ is defined at $x = a$, then we could just plug $a$ into $f$ to get out a number; but sometimes, as in the above examples, $f(a)$ doesn't make sense, we instead take the *limit* of $f(x)$ as $x$ approaches $a$, written

$$\lim_{x \to a} f(x).$$

This is the value that $f(x)$ approaches as $x$ gets closer and closer to $a$.

Let me recall the examples from the survey:

$$\lim_{x\to 0} \frac{x+1}{x+2}$$

and

$$\lim_{x\to 1} \frac{x^2-1}{x-1}.$$

In the first case, $\frac{x+1}{x+2}$ makes perfect sense at $x = 0$: it is just $\frac{0+1}{0+2} = \frac{1}{2}$.

In the second case, things are trickier: if we plug $x = 1$ into the expression, we get

$$\frac{1^2-1}{1-1} = \frac{0}{0}$$

which doesn't make sense. However, we can rewrite this expression as

$$\frac{x^2-1}{x-1} = \frac{(x+1)(x-1)}{x-1},$$

and then for all $x \neq 1$ we can cancel the factors of $x - 1$. Therefore for all $x \neq 1$ the expression we're taking the limit of is just $x + 1$, and so

$$\lim_{x\to 1} \frac{x^2-1}{x-1} = \lim_{x\to 1} x+1 = 1+1 = 2.$$

Many limits can be evaluated through this sort of algebraic rewriting and canceling method, but not all! An interesting example, which we'll come back to, is

$$\lim_{x\to 0} \frac{\sin(x)}{x}.$$

We have $\sin(0) = 0$, so at $x = 0$ this expression is undefined; you can try plugging in different small numbers and see what you get, but we don't yet have tools to abstractly compute this limit; we'll come back to it next class.

So far we've seen the following kinds of limits: either $f(x)$ is defined at $x = a$, in which case we can just plug in $a$ and get

$$\lim_{x\to a} f(x) = f(a);$$

it's undefined at $a$ (e.g. something like $\frac{0}{0}$, though there are other possibilities too), but we can figure out a number $L$ which $f(x)$ approaches as $x \to a$, in which case we say that

$$\lim_{x\to a} f(x) = L;$$

or complicated cases where it's undefined at $a$, and we aren't sure how to evaluate it. There's another possibility, too: not only is $f(x)$ undefined at $a$, but even the limit doesn't exist. An example is

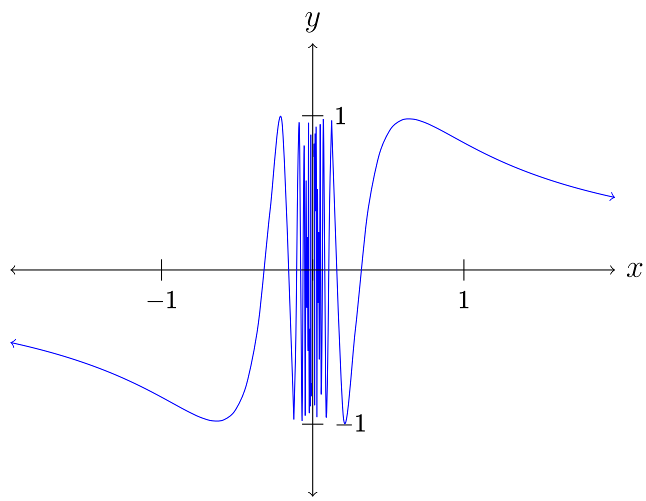
$$\lim_{x\to 0} \frac{1}{x}.$$

If we plug in $x = 0$, we get $\frac{1}{0}$, which is undefined; if we look at some very small $x$, like $x = 0.001$, we'd get 1000, and as we take $x$ smaller and smaller this blows up towards infinity. But if we were to take $x$ very small and *negative*, i.e. approaching zero from the other direction, we'd have $\frac{1}{x}$ going to $-\infty$, e.g. at $x = -0.001$ we'd get $-1000$. Thus in any sense the limit doesn't exist: it doesn't converge to any real number, and taking the limit in the two different directions gives two different things (each of which is already not a number in any usual sense).

If we took $\lim_{x\to 0} \frac{1}{x^2}$, then the behavior is a little better: it doesn't matter which direction we go from, we get $+\infty$ either way. Nevertheless since this isn't a real number we say that the limit does not exist.

You might complain that this is only for functions which blow up to infinity. Here's an example which doesn't:

$$\lim_{x\to 0} \sin\left(\frac{1}{x}\right).$$




We can't plug in 0, since $\sin(1/0)$ doesn't make sense; but although the function never goes to infinity in either direction (it's always bounded between $-1$ and 1), the limit still does not exist.

Let's go back to some more examples we can evaluate. Consider

$$\lim_{x\to -2} \frac{x+2}{x^2+x-2}.$$

If we plug in $-2$, we get

$$\frac{-2+2}{4-2-2} = \frac{0}{0},$$

so we need to do some more work. We can simplify by factoring the denominator: $x^2+x-2 = (x+2)(x-1)$, so for all $x \neq -2$ we have

$$\frac{x+2}{x^2+x-2} = \frac{x+2}{(x+2)(x-1)} = \frac{1}{x-1}.$$

Therefore

$$\lim_{x\to -2} \frac{x+2}{x^2+x-2} = \lim_{x\to -2} \frac{1}{x-1} = \frac{1}{-2-1} = -\frac{1}{3}.$$

How about $\lim_{x\to 1} \frac{1}{\log_2 x}$?

One can also take, instead of $x \to a$ for some fixed number $a$, $x \to \infty$ or $x \to -\infty$. For example,

$$\lim_{x\to\infty} \frac{1}{x} = 0,$$

since $\frac{1}{x}$ gets smaller and smaller as $x \to \infty$.

We have to be very careful with this sort of thing: it is often tempting to "plug in $\infty$," as with this example, and say something like the limit is $\frac{1}{\infty} = 0$. Formally, this statement doesn't mean anything; we can't treat infinity like a number in most ways. This can be a useful mnemonic device, though, so long as we're careful with it. For example, saying $\frac{1}{\infty} = 0$ is, more or less, okay; but applying that logic, we might end up saying something like

$$\lim_{x\to\infty} \frac{x}{x} = \lim_{x\to\infty} x \cdot \frac{1}{\infty} = \lim_{x\to\infty} x \cdot 0 = 0,$$

when of course the answer is instead 1. We'll talk more about this kind of mistake when we talk about limit laws; for now just keep in mind that this sort of thing can be deceptive.

Let me now give you the formal definition of a limit. We won't generally use this and if you feel you have a good understanding and want to tune out you can, but I think it's good for you to at least see the definition. It is this: given the setup as before, where we have an expression $f(x)$ and a number $a$ such that $f(x)$ is defined near $a$ but not necessarily at $a$,[1] we say that $\lim_{x\to a} f(x) = L$ if for every $\epsilon > 0$, no matter how small, we can find some $\delta > 0$ (generally also very small) such that for any $x$ with $|x-a| < \delta$, we have $|f(x) - L| < \epsilon$. In other words, no matter how close we want $f(x)$ to be to $L$, so long as we require $x$ to be close enough to $L$ we can make this be true.

For example, in the $\lim_{x\to 0} \frac{x^2}{x}$ example, the claim is that this is equal to $L = 0$. For every $\epsilon > 0$, we want to find $x$ with $|\frac{x^2}{x} - L| = |\frac{x^2}{x}| < \epsilon$; for $x$ nonzero, we have $\frac{x^2}{x} = x$, so this is true so long as we choose $|x| < \epsilon$. Thus in this case we can choose $\delta = \epsilon$.

When we're taking an infinite limit, we make the same definition but this time instead of choosing $x$ sufficiently close to $a$, we choose it sufficiently large; you can try and work out what exactly the right definition should be as an exercise if you're interested.[2] An example is $\lim_{x\to\infty} \frac{1}{x}$; for any $\epsilon > 0$, by choosing $x$ sufficiently large (indeed, larger than $\frac{1}{\epsilon}$) we can guarantee that $|\frac{1}{x}| < \epsilon$, and so $\frac{1}{x} \to 0$ as $x \to \infty$.

## 2. Limit laws

Now that we have some intuitive understanding of limits as well as a formal definition, we want to be able to compute them. Some we can do directly, as above, but for more

[1]There is a way of formalizing this, too: what it means is that there is some interval $(A, B)$ containing $a$ such that $f(x)$ is defined everywhere on this interval except possibly at $a$.

[2]This is completely optional and will not be graded, but I'm happy to talk about it with you if you want.

complicated ones it's useful to have some more powerful tools. The first tools we'll use are the limit laws, which tell us how we can break down limits into simpler ones.

First, we could have a limit which is the sum of two terms:

$$\lim_{x\to a}(f(x)+g(x)).$$

Generally, we'd expect that we could split this up:

$$\lim_{x\to a}(f(x)+g(x)) = \lim_{x\to a} f(x) + \lim_{x\to a} g(x).$$

The exception is that we have to make sure that both of the limits on the right-hand side exist: you could imagine a situation like

$$\lim_{x\to 0}\left(\frac{-1}{x}+\frac{1}{x}\right) = \lim_{x\to 0} 0 = 0,$$

but if we were to split it up $\lim_{x\to 0}\frac{-1}{x}$ and $\lim_{x\to 0}\frac{1}{x}$ don't exist. The limit law says that other than this restriction, what we hope is true: so long as $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ both exist,

$$\lim_{x\to a}(f(x)+g(x)) = \lim_{x\to a} f(x) + \lim_{x\to a} g(x).$$

The same thing is true for subtraction: if both limits exist separately, then

$$\lim_{x\to a}(f(x)-g(x)) = \lim_{x\to a} f(x) - \lim_{x\to a} g(x).$$

It's also true for multiplication: under the same assumption,

$$\lim_{x\to a} f(x)g(x) = \left(\lim_{x\to a} f(x)\right)\left(\lim_{x\to a} g(x)\right).$$

A special case is when one of these is constant: since $\lim_{x\to a} c = c$ always exists,

$$\lim_{x\to a} cf(x) = c\cdot\lim_{x\to a} f(x).$$

The same rule is true for division, but we have to be slightly more careful: not only do the limits of $f(x)$ and $g(x)$ have to exist to make sure this is true, but we also need to make sure that $\lim_{x\to a} g(x)\neq 0$, to make sure we don't get a divide-by-zero error. Subject to that restriction, the same rule holds:

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}.$$

For example, consider

$$\lim_{x\to 1}\frac{\frac{x^2-1}{x-1}-2x}{(x+3)^2}.$$

We can't directly substitute in $x = 1$, due to the denominator $x - 1$ in the larger numerator. We could expand everything algebraically and simplify, as we've done before. However, much easier is to apply the limit laws. First, we observe that $\lim_{x\to 1}(x+3)^2 = 16$ exists and is nonzero, and so by the limit law for division dividing by $(x+3)^2$ inside the limit is the same as computing the rest of the limit and then dividing by 16. Similarly, $\lim_{x\to 1} 2x = 2$ exists and so subtracting $2x$ inside the limit is the same as subtracting 2 outside. Finally, we've already computed $\lim_{x\to 1} \frac{x^2-1}{x-1}$ to be 2, so by the limit laws we're done: the result is

$$\frac{2-2}{16} = 0.$$

Next time we'll see some more examples, and at least one additional tool that will let us make some harder computations.