Lecture 4: limit laws and the squeeze theorem

Calculus I, section 10
September 15, 2022

Last time, we introduced limits and saw a formal definition, as well as the limit laws. Today we'll review limit laws and look at some one-sided limits, and introduce the squeeze theorem.

The limit laws essentially say that subject to reasonable conditions, you can split up limits in a variety of ways: addition, multiplication, subtraction, division. These "reasonable conditions" are that, in order to say that e.g.

$$\lim_{x\to a} f(x)g(x) = \lim_{x\to a} f(x) \cdot \lim_{x\to a} g(x),$$

we need both limits on the right to exist. In the case of division, to have

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)},$$

we also need the denominator to be nonzero, or else the right-hand side can fail to exist even if the left-hand side is reasonable.

We could combine limit laws in certain ways to get similar statements. For example,

$$\lim_{x\to a} f(x)^2 = \left(\lim_{x\to a} f(x)\right)^2$$

so long as $\lim_{x\to a} f(x)$ exists, by the multiplication limit law. We could keep going like this to see that

$$\lim_{x\to a} f(x)^n = \left(\lim_{x\to a} f(x)\right)^n$$

for any n.

One very powerful limit law we haven't talked about before is about function composition. If $\lim_{x\to a} g(x)$ exists and is equal to some number L, then

$$\lim_{x\to a} f(g(x)) = \lim_{y\to L} f(y).$$

For example, to compute

$$\lim_{x\to 2} \log_2\left(\frac{x^2-4}{x-2}\right),$$

this limit law tells us that we can first compute

$$\lim_{x\to 2} \frac{x^2-4}{x-2} = \lim_{x\to 2} \frac{(x-2)(x+2)}{x-2} = \lim_{x\to 2} x+2 = 4,$$

and then our limit is

$$\lim_{y\to 4} \log_2(y) = \log_2(4) = 2.$$

This lets us think about complicated limits piece-by-piece, which is very useful, but we have to be careful. For example, we might be tempted to say that we can use it to compute limits of the form

$$\lim_{x\to a} x \cdot f(x),$$

by first computing $\lim_{x\to a} f(x) = L$ (if it exists) and then $\lim_{x\to L} xL = L^2$. But this is generally not true: for example, if $f(x) = 1$, so $L = 1$, then

$$\lim_{x\to 0} xf(x) = \lim_{x\to 0} x = 0 \neq L^2 = 1.$$

What went wrong? The expression $xf(x)$ might look like a function composition, since we're feeding $f(x)$ into a machine to produce a new number. But actually this expression doesn't only depend on $f(x)$: it also depends on the original x! (This is now a multi-variable function, which might come up if you take through calculus 3, but definitely not in this class, and it won't have this nice behavior that composition of one-variable functions does here.) So the important thing to keep in mind when using the composition rule is to make sure that your expression is actually a composition of functions!

Even if the inside limit doesn't exist, we can still take advantage of this law if it goes to ∞ or $-\infty$. In this case we do treat ∞ as a number: if $\lim_{x\to a} g(x) = \infty$, then $\lim_{x\to a} f(g(x)) = \lim_{y\to\infty} f(y)$, and similarly for $-\infty$. For example, to compute

$$\lim_{x\to 0} 2^{-\frac{1}{x^2}},$$

we first compute

$$\lim_{x\to 0} -\frac{1}{x^2} = -\infty.$$

We then plug it in:

$$\lim_{x\to 0} 2^{-\frac{1}{x^2}} = \lim_{y\to -\infty} 2^y = 0.$$

You might complain at this point that last time I told you you don't have to worry about saying a limit goes to ∞ or $-\infty$, any such limit can just be said not to exist. That's true, and you could evaluate this limit without writing these symbols: just observe that as x goes to 0, $\frac{1}{x^2}$ gets larger and larger, so $2^{-\frac{1}{x^2}} = \frac{1}{2^{1/x^2}}$ gets smaller, since 2 to the power of a large number is big and the inverse of a big number is small. This notation with the ∞ symbols is just a way of keeping track of this sort of calculation, which you may find makes it easier for you; if it doesn't make it easier, feel free not to use it.

Next, let's come back to another concept we touched on last class: one-sided limits. We discussed before how if $\lim_{x\to a} f(x) = L$, this means that $f(x)$ approaches L as x goes to a from either direction, i.e. whether x is slightly less than a or slightly more than a we should have $f(x)$ close to L. We could instead weaken this requirement to only needing it to be true as x goes to a from one side or the other. We write

$$\lim_{x\to a^+} f(x)$$

for the limit as x goes to a from above, and

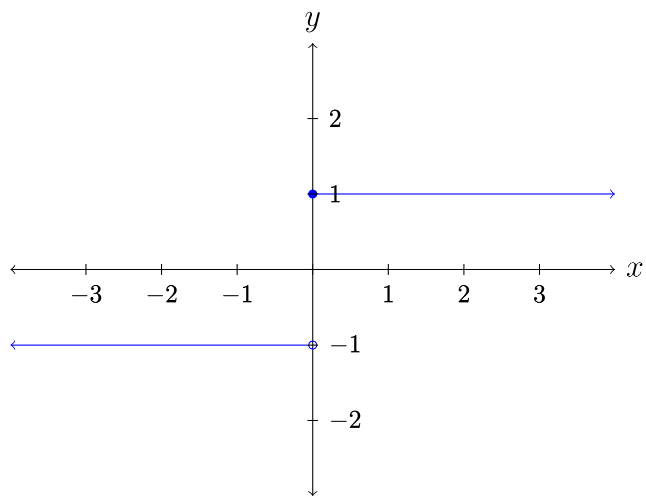
$$\lim_{x \to a^-} f(x)$$

for the limit as x goes to a from below. If $\lim_{x\to a} f(x)$ exists, then these one-sided limits must both exist and be the same; but it's possible that even if the total limit fails to exist, one or both of the one-sided limits may still exist (and if they both do, they may be different).

For example, consider the function

$$f(x) = \begin{cases} 1 & x \geq 0 \\ -1 & x < 0 \end{cases},$$

whose graph looks like this.



As $x \to 0$ from above, the function is always 1, and so $\lim_{x\to 0^+} f(x) = 1$. But as $x \to 0$ from below, the function is -1, so $\lim_{x\to 0^-} f(x) = -1$.

Another common application of one-sided limits is to functions which do not exist on the whole domain and so can only be evaluated from one side. We saw an example last time involving logarithms; another example is

$$\lim_{x\to 0} \sqrt{x}.$$

Strictly speaking, even though we can plug in $x = 0$ to get $\sqrt{0} = 0$, this limit does not exist! This is because we can't approach it from below, only above, since $\sqrt{x}$ doesn't make sense for negative numbers.[1] If we replace the limit with a one-sided limit, $\lim_{x\to 0^+} \sqrt{x}$, then everything is as expected: this exists and is equal to 0.

A more complicated example is

$$\lim_{x\to 0^+} \frac{\sqrt{x}}{\sqrt{x+1}-1}.$$

[1] Again, if you allow complex numbers, you solve this problem at the price of introducing new ones.

Similarly, we need to require that the limit is only from above, since we can't plug in negative values to $\sqrt{x}$. Does this make the limit exist?

Well, the first thing to do is to get the square root out from the bottom, which we can do by conjugation:

$$\frac{\sqrt{x}}{\sqrt{x+1}-1} = \frac{\sqrt{x}}{\sqrt{x+1}-1} \cdot \frac{\sqrt{x+1}+1}{\sqrt{x+1}+1} = \frac{\sqrt{x}\sqrt{x+1}+\sqrt{x}}{x}.$$

Canceling a factor of $\sqrt{x}$, this is

$$\frac{\sqrt{x+1}+1}{\sqrt{x}},$$

and now as we take the limit as $x \to 0$ from above we see that this will blow up: the numerator goes to $\sqrt{1}+1=2$ while the denominator goes to 0.

Our final idea for the day is the squeeze theorem. This is based on the idea the limits respect inequalities: if $f(x) \leq g(x) \leq h(x)$, then (assuming all the limits exist)

$$\lim_{x\to a} f(x) \leq \lim_{x\to a} g(x) \leq \lim_{x\to a} h(x).$$

In particular, suppose that we know that $\lim_{x\to a} f(x) = \lim_{x\to a} h(x) = L$. Then

$$L \leq \lim_{x\to a} g(x) \leq L$$

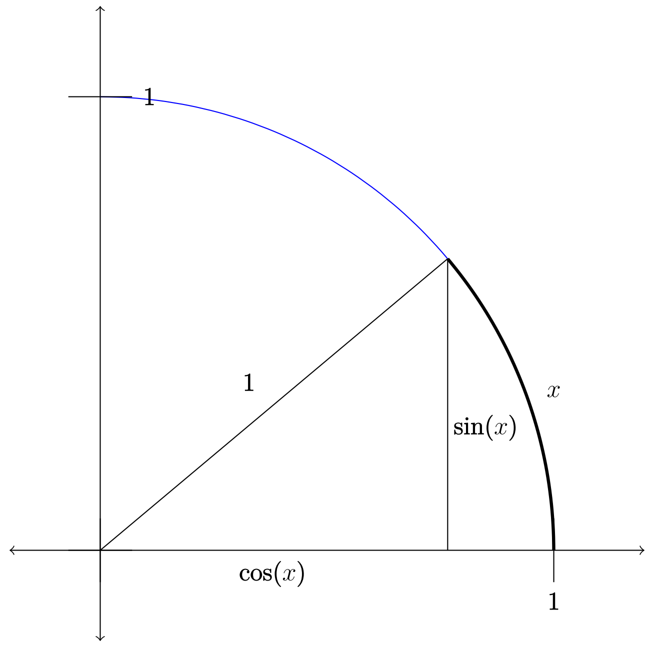
and so $\lim_{x\to a} g(x)$ must also be equal to L.

In fact, the squeeze theorem is a little stronger: we don't need to assume that the inner limit exists. If we have $f(x) \leq g(x) \leq h(x)$, at least for x sufficiently close to a, and $\lim_{x\to a} f(x) = \lim_{x\to a} h(x) = L$, then it follows that $\lim_{x\to a} g(x) = L$. (The same thing works for one-sided limits.)

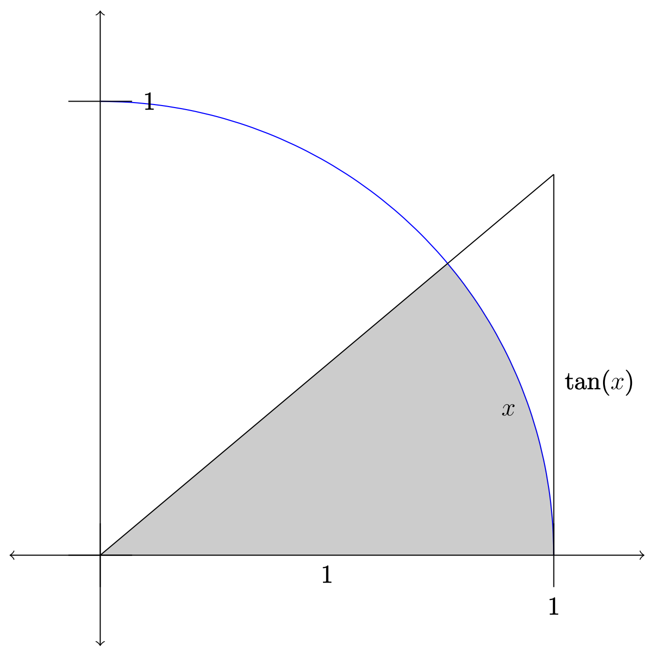
To see why such a thing might be useful, let's go back to the example I mentioned last class,

$$f(x) = \frac{\sin(x)}{x}.$$

I'm going to make two claims: at least for x small, $\frac{\sin(x)}{x} \geq \cos(x)$ and $\frac{\sin(x)}{x} \leq 1$. Assuming everything is positive for simplicity, these are the same thing as $\tan x \geq x$ and $\sin(x) \leq x$. To check that these are actually true, we can look at the unit circle:



The length of the vertical line ($\sin x$) must be less than the length of the curved line (x), so $\sin(x) \leq x$.



If we instead look at a larger triangle, the area of the whole triangle is $\frac{1}{2}\tan(x)$, while the area of the wedge is $\frac{x}{2\pi}$ of the area of the whole unit circle, which is π, and so the area of the wedge is $\frac{x}{2}$. Since the triangle contains the wedge, it follows that $\frac{x}{2} \leq \frac{1}{2}\tan(x)$, and so $\tan(x) \geq x$.

We could do the same thing for negative values (and take one-sided limits each way to see that they agree), or just add on absolute value signs.

Now that we know these bounds, so $\cos(x) \leq \frac{\sin x}{x} \leq 1$, we can apply the squeeze theorem: taking the limit as $x \to 0$, we have

$$\lim_{x\to 0} \cos(x) = \cos(0) = 1$$

and

$$\lim_{x\to 0} 1 = 1,$$

so without doing essentially any real limit work we get for free

$$\lim_{x\to 0} \frac{\sin x}{x} = 1.$$

This is, a priori, a very difficult statement!